PMATH 450 (Spring 2013 - 1135)

Lebesgue Integration and Fourier Analysis

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These notes are currently a work in progress, and as such may be incomplete or contain errors.

Acknowledgments:

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Also thanks to John Liu who provided notes for some of my missed classes.

Abstract

The purpose of these notes is to provide a primary reference to the material covered in PMATH 450. The official prerequisite to this course is PMATH 351, which this author believes is sufficient for the level of difficulty of this course. That being said, this course, itself, is known to be one of the most difficult PMATH (or otherwise) courses at the University of Waterloo and is comparable to taking MATH 145 in the first year of undergrad at Waterloo.

The author strongly recommends to the students taking this course that they review and completely understand the content in PMATH 351 because almost 30-40% of the material in this course follows from the results in that course.

Financial applications are scarce in this course, but because it leads into PMATH 451, it is highly recommended that Mathematical Finance majors take this course very seriously.

Errata

Midterm on Thursday, June 20th @ 5:30pm-7:00pm. Double classes?

Make-up class 1 (5:10pm-6:00pm MC 5045)

1 Riemann Integration

Recall that if $a, b \in \mathbb{R}$ with $a < b$ then $[a, b]$ is compact with $f : [a, b] \mapsto \mathbb{R}$ bounded. Let $P = \{t_i | t_0 = a < t_1 < ... < t_{n-1} < t_n = b\} \subseteq [a, b]$ be a partition of $[a, b]$. For each $1 \leq i \leq$we put

$$M_i = \sup\{f(t) : t \in [t_{i-1,} t_i]\}$$

and

$$m_i = \inf\{f(t) : t \in [t_{i-1,} t_i]\}$$

and these exist because f is bounded since it is defined on a compact domain.

Note that f is continuous M_i and m_i are attained by f (i.e. they are in the image of f).

Definition 1.1. We define the *lower* and *upper Riemann sums* over the partition P as

$$\begin{aligned} U(f, P) &= \sum_{i=1}^{n} M_i \underbrace{(t_i - t_{i-1})}_{\triangle t_i} \\ L(f, P) &= \sum_{i=1}^{n} m_i \underbrace{(t_i - t_{i-1})}_{\triangle t_i} \end{aligned}$$

We also put $\|P\| = \max_{1 \leq i \leq n} \triangle t_i = \max_{1 \leq i \leq n}(t_i - t_{i-1})$. If $P \subseteq Q$ then Q is a refinement of P. Finally, a Riemann sum over a partition P is denoted by

$$S(f, P) = \sum_{i=1}^{n} f(t_i^*)(t_i - t_{i-1}), t_i \in [t_{i-1}, t_i]$$

Then we define the *lower Riemann integral* as

$$\underline{\int_a^b} f = \sup\{L(f, P) : P \text{ a partition of } [a, b]\}$$

and similarly the *upper Riemann integral* as

$$\overline{\int_a^b} f = \sup\{L(f, P) : P \text{ a partition of } [a, b]\}$$

Definition 1.2. Let $[a, b] \subseteq \mathbb{R}$ compact and $f : [a, b] \mapsto \mathbb{R}$ be bounded. We say f is Riemann integrable if

$$\underline{\int_a^b} f = \overline{\int_a^b} f$$

and we denote this as $\int_a^b f$. Note that constant and continuous functions are Riemann integrable.

1.1 Riemann Sums on Vector Valued Functions

Definition 1.3. A real or complex vector space X is called a *Banach space* if it is a *complete normed linear space*, where completeness is when all Cauchy sequences in X converge.

Note 1. Recall the properties of a norm $\|\cdot\|$:

1) $\|x\| = 0 \iff x = 0$

2) $\|x+y\| \leq \|x\| + \|y\|$

3) $\|\alpha x\| = |\alpha|\|x\|$

Example 1.1. Here are some examples of Banach spaces from various analysis courses:

1) $\mathbb{R}$ with $|\cdot|$

2) $\mathbb{R}^n$ with $\|\cdot\|_2$

3) $\mathcal{C}([a,b])$ with $\|f\|_\infty$

Definition 1.4. For a given Banach space X, partition $P_r = \{t_i | t_0 = a < t_1 < ... < t_{n-1} < t_n = b, \max_i(t_i - t_{i-1}) \leq r\} \subseteq [a,b]$ and $f : [a,b] \mapsto X$, we define the Riemann sum over P_r for this Banach space valued function f as

$$S(f, P_r) = \sum_{i=1}^{n} \underbrace{f(t_i^*)}_{\in X} \underbrace{(t_i - t_{i-1})}_{\in \mathbb{R}} \in X$$

Definition 1.5. Let $f : [a,b] \mapsto X$ where X is a Banach space. We say that f is Riemann integrable if there is $x \in X$ such that $\forall \epsilon > 0$ there is P_ϵ with for any $P \supseteq P_\epsilon$ we have

$$\|S(f,P) - x\| < \epsilon$$

for any Riemann sum over P, independent of the t_i^*s.

Remark 1.1. Suppose $x, y \in X$ which satisfies the above the definition, with $x \neq y \implies x - y \neq 0 \implies \|x - y\| \neq 0$. Let

$$\epsilon = \frac{\|x-y\|}{2} > 0$$

We then apply the definition of x and y to get $P_\epsilon^X$and P_ϵ^Y. Put $P = P_\epsilon^X \cup P_\epsilon^Y \implies P$ is a refinement of P_ϵ^Y and P_ϵ^X which is a contradiction of the above definition. Therefore if x exists, it is unique. Hence, we define $\int_a^b f = x \in X$ and call this the Riemann integral of f.

Note 2. Given $f : [a,b] \mapsto \mathbb{R}$ we have 2 definitions of $\mathbb{R}-$integrals, one from upper and lower sums and the one that comes from Riemann sums over Banach spaces. We will see that these definitions are equivalent.

Theorem 1.1. *(Cauchy Criterion) LEt χ be a Banach space. A function $f : [a,b] \mapsto \chi$ is Riemann integrable $\iff \forall \epsilon, \exists$ partition Q_ϵ such that for any $P, Q \supseteq Q_\epsilon$ and any Riemann sums over P, Q we have*

$$\|S(f,P) - S(f,Q)\| < \epsilon$$

Proof. ($\implies$) Exercise. Hint: For given $\frac{\epsilon}{2} > 0$, apply the definition of Riemann integrability to get $P_{\frac{\epsilon}{2}}$. Then $Q_\epsilon = P_{\frac{\epsilon}{2}}$ and the result follows from the triangle inequality.

($\impliedby$) Assume that the Cauchy Criterion holds. For each $n \in \mathbb{P}$ let Q_n be a partition of $[a,b]$ such that

$$\|S(f,P) - S(f,Q)\| < \frac{1}{2^n}$$

If $P, Q \supseteq Q_n$ and $S(f,Q)$ and $S(f,P)$ are any Riemann sums over P and Q. Let

$$\begin{aligned} P_1 &= Q_1 \\ P_2 &= Q_1 \cup Q_2 \supset P_1 \\ &\vdots \\ P_n &= \bigcup_{k=1}^{n} Q_k \supset P_{n-1} \supset \ldots \supset P_1 \end{aligned}$$

and for each n fix $x_n = S_n(f, P_n)$ for some Riemann sum over P_n.

Consider $\{x_n\}_{n=1}^{\infty} \subseteq \chi$ Then if $n > m$ we observe that

$$\|x_n - x_m\| = \|S_n(f,P_n) - S_m(f,P_m)\| \leq \frac{1}{2^n}$$

with $P_n \supseteq P_m$. Note that $\{x_n\}_{n=1}^{\infty}$ is Cauchy in χ and since χ is complete, there is a limit point $x = \lim_{n\to\infty} x_n \in \chi$. We claim that $\int_a^b f = x$. Let $\epsilon > 0$ and choose n large enough such that $\frac{1}{2^{n-1}} < \frac{\epsilon}{2}$ and $\|x_n - x\| < \frac{\epsilon}{2}$. Let P_n be as above and $P \supseteq P_n = P_\epsilon$ together with $S(f,P)$, any Riemann sum over P.

Then we have

$$\begin{aligned} \|S(f,P) - x\| &\leq \|S(f,P) - S_n(f,P_n)\| + \|S_n(f,P_n) - x\| \\ &\leq \frac{1}{2^n} + \frac{1}{2^{n-1}} = \frac{3}{2^n} \\ &< 2\epsilon \end{aligned}$$

□

Lemma 1.1. *Assume that $f : [a,b] \mapsto \chi$ is continuous. Let $\epsilon > 0$. Then $\exists \delta > 0$ such that if P is any partition with $\|P\| < \delta$ then for any $P_1 \supseteq P$ and any $S(f,P)$, $S(f,P_1)$ we have*

$$\underbrace{\|S(f,P) - S(f,P_1)\|}_{\text{norm in } \chi} < \epsilon$$

Proof. Exercise. Hint: Note that f is uniformly continuous. For $\frac{\epsilon}{(b-a)}$, uniform continuity gives us some $\delta > 0$. The result follows for this δ. □

Theorem 1.2. *Assume that $f : [a,b] \mapsto \chi$ is continuous. Then f is Riemann integrable.*

Proof. Follows from the above Lemma and triangle inequality. Left as an exercise. Make sure to verify that the Cauchy Criterion works. □

Example 1.2. Consider the function $\chi_{[0,\frac{1}{2})} : [0,1] \mapsto \mathbb{R}$ where χ_A is the characteristic/indicator function on some set A. Observe that $\int_0^1 \chi_{[0,\frac{1}{2})} = \frac{1}{2}$. Note that for any $[a,b] \subseteq [c,d]$ we have $\int_c^d \chi_{[a,b]} = b - c$.

Example 1.3. Consider the function $\chi_{\mathbb{Q}\cap[0,1]} : [0,1] \mapsto \mathbb{R}$. Let $P = \{x_i | 0 = x_0 < \ldots < x_n = 1\}$ be a any partition of $[0,1]$. Then for each $1 \leq i \leq n$,

$$\begin{aligned} M_i &= \sup\{\chi_{\mathbb{Q}\cap[0,1]}(t) : t \in [x_{i-1}, x_i]\} = 1 \\ m_i &= \inf\{\chi_{\mathbb{Q}\cap[0,1]}(t) : t \in [x_{i-1}, x_i]\} = 0 \end{aligned}$$

and so upper and lower Riemann sums will never converge ($1 = U(\chi_{\mathbb{Q}\cap[0,1]}, P) \neq L(\chi_{\mathbb{Q}\cap[0,1]}, P) = 0$) and the Riemann integral does not exist.

2 General Measures and Measure Spaces

Definition 2.1. Given a set X, we denote the *power set* of X as $\mathcal{P}(X)$. By definition, this is the set of all subsets of X.

Definition 2.2. Let X be a non-empty set. An *algebra of subsets* of X is a collection $A \subseteq \mathcal{P}(X)$ such that

1) $\emptyset$ and $X \in A$

2) If $E_1, E_2 \in A$ then $E_1 \cup E_2 \in A$

3) If $E \in A$ then $E^c = X \backslash E \in A$

Definition 2.3. A *σ-algebra of subsets* of $X$is a collection $A \subseteq P(X)$ such that

1) $\emptyset$ and $X \in A$

2) If $E_1, E_2, ... \in A$ then $\bigcup_{n=1}^{\infty} E_n \in A$

3) If $E \in A$ then $E^c = X \backslash E \in A$

Remark 2.1. All σ-algebras are algebras.

Note 3. Note that $E_1 \cap E_2 = (E_1^c \cup E_2^c)^c$ and so algebras are closed under finite intersections and σ-algebras are closed under countable intersections.

Example 2.1. Let X be an infinite set and let A be the collection of subsets $\{E_n\}_{n \in I}$ of X such that either E or E^C is finite. Then A is an algebra but not always a σ-algebra. This is due to the fact that the countable unions of sets may produce a set whose complement and itself is not finite.

Example 2.2. If $\{A_\alpha\}_{\alpha \in I}$ a family of algebras (σ-algebra) then $\bigcap_{\alpha \in I} A_\alpha$ is an algebra (σ-algebra).

Note 4. Given $S \subseteq \mathcal{P}(X)$, there exists a smallest algebra (σ-algebra) containing S which follows from the above example.

Notation 1. Let $S \subseteq \mathcal{P}(X)$. We denote:

$A(S)$: the algebra generated by S which is defined to be the smallest algebra containing S.

$\sigma(S)$: the σ-algebra generated by S which is the smallest σ-algebra containing S

Definition 2.4. Let $\mathcal{G}$ =$\{U \subseteq \mathbb{R} | U$ is open$\}$. The σ-algebra generated by $\mathcal{G}$, $\sigma(\mathcal{G})$, will be called the Borel σ-algebra of $\mathbb{R}$ and will also be denoted by $\mathcal{B}(\mathbb{R})$.

Remark 2.2. More generally, we may consider the Borel σ-algebra on any *topological space*. We will examine this shortly.

Given any set X and $M \subseteq \mathcal{P}(X)$, let

$$\begin{aligned} M_\delta &= \left\{ A \in \mathcal{P}(X) : A = \bigcap_{i=1}^{\infty} M_i, M_i \in M \right\} \\ M_\sigma &= \left\{ A \in \mathcal{P}(X) : A = \bigcup_{i=1}^{\infty} M_i, M_i \in M \right\} \end{aligned}$$

and G be the set of all open subsets of $\mathbb{R}$ and F be the set of closed subsets of $\mathbb{R}$

Then we have

$$\begin{aligned} \mathcal{G}_\delta &= \{\text{countable intersections of open sets of } \mathbb{R}\} \\ \mathcal{F}_\sigma &= \{\text{countable unions of closed sets of } \mathbb{R}\} \end{aligned}$$

and $\mathcal{G}_\sigma = G$, $\mathcal{F}_\sigma = F$. Therefore,

$$\begin{aligned} G &\subset \mathcal{G}_\delta \subset \mathcal{G}_{\delta\sigma} \subset \mathcal{G}_{\delta\sigma\delta} \subset ... \subset \mathcal{B}(\mathbb{R}) \\ F &\subset \mathcal{F}_\sigma \subset \mathcal{F}_{\sigma\delta} \subset \mathcal{F}_{\sigma\delta\sigma} \subset ... \subset \mathcal{B}(\mathbb{R}) \end{aligned}$$

and note that $\mathcal{G}_\delta$ sets are exactly the complements of $\mathcal{F}_\sigma$-sets. Note that none of these sets are equal.

Example 2.3. $\mathbb{Q}$ is $\mathcal{F}_\sigma$ but $\mathbb{Q} \notin F$. Similarly $\mathbb{R}\backslash\mathbb{Q}$ is G_δ (why?) but $\mathbb{R}\backslash\mathbb{Q} \notin G$.

Proposition 2.1. *$F \subset \mathcal{G}_\delta$ and $G \subset \mathcal{F}_\sigma$.*

Proof. Suppose that $f \in F$ a closed set. For each $n \in \mathbb{P}$, we define

$$U_n = \left\{ x \middle| |x-y| < \frac{1}{n}, y \in f \right\}$$

Then U_n are open and $f \subset U_n \implies f \subset \bigcap_{n=1}^{\infty} U_n$. Note that $f = \emptyset \iff U_n = \emptyset$.

To prove the reverse inclusion, we observe that f is closed and any $x \in \bigcap_{n=1}^{\infty} U_n$ is a limit point of f. So $x \in f \implies f = \bigcap_{n=1}^{\infty} U_n \in \mathcal{G}_\delta$. If $U \in G$ is open, then U_c is closed $\implies U^c = \bigcap_{n=1}^{\infty} U_n$ where $U_n's$ are open $\implies U_n^c$ is closed and $U = \left(\bigcap_{n=1}^{\infty} U_n\right)^c = \bigcap_{n=1}^{\infty} U_n^c \in \mathcal{F}_\sigma$. □

Note 5. About the Borel $\sigma-$algebra:

$$\begin{aligned}
\mathcal{B}(\mathbb{R}) &= \sigma(G) \\
&\subseteq \sigma\{(a,b)|a,b \in \mathbb{R}\} \\
&\subseteq \sigma\{(a,b]|a,b \in \mathbb{R}\} \\
&= \sigma\{[a,b)|a,b \in \mathbb{R}\} \\
&\subseteq \sigma\{[a,b]|a,b \in \mathbb{R}\}
\end{aligned}$$

Proof. The first inclusion follows from A1 where we will see that any $U \subseteq \mathbb{R}$ open can be written as $U = \bigcup_{i=1}^{\infty} (a_i, b_i)$. For the second inclusion we note that

$$(a,b) = \bigcup_{n=1}^{\infty} \left(a, b - \frac{k}{n}\right]$$

where $k = \frac{a-b}{2}$. □

Remark 2.3. $\mathcal{G}_\delta = \mathcal{G}_{\delta\delta}$ and $\mathcal{F}_\sigma = \mathcal{F}_{\sigma\sigma}$ because the countable union and intersection of countable sets is countable.

2.1 Measures

Definition 2.5. The set $\mathbb{R}$ together with σ-algebra A, $(\mathbb{R}, A)$ is a called a measurable space. A (countably additive) *measure* on A is a function $\mu : A \mapsto \mathbb{R}^* := \mathbb{R} \cup \{\pm\infty\}$ with the properties:

1) $\mu(\emptyset) = 0$

2) $\mu(E) \geq 0$ for all $E \in A$

3) If $\{E_n\}_{n=1}^{\infty} \subset A$ is sequence of disjoint sets, then $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$

Definition 2.6. If we replace 3) by

3') If $\{E_n\}_{n=1}^{N} \subseteq A$ is a finite sequence of disjoint sets then $\mu\left(\bigcup_{n=1}^{N} E_n\right) = \sum_{n=1}^{N} \mu(E_n)$ where $N \in \mathbb{N}$.

then such a μ is called a finitely additive measure. Usually, we will assume a measure is countably additive unless otherwise specified.

Definition 2.7. We will call a measure μ finite if $\mu(\mathbb{R}) < \infty$ and call it σ-finite if there exists $\{E_n\}_{n=1}^{\infty} \subset A$ such that $\bigcup_{n=1}^{\infty} E_n = \mathbb{R}$ and each $\mu(E_n) < \infty$.

Definition 2.8. A triple $(\mathbb{R}, A, \mu)$ is called a *measure space* where A is a σ-algebra and μ is a measure on A. We also say that such a triple is complete if for any $E \in A$ with $\mu(E) = 0$ and $S \subset E$ we have $S \in A$. For $E \in A$ we call E a measurable set.

Proposition 2.2. *(Monotonicity) Let $(\mathbb{R}, A, \mu)$ be a measure space. If $E \subset F$ and $E, F \in A$ then $\mu(E) \leq \mu(F)$.*

Proof. Let $E, F \in A$ with $E \subset F$. Note that E and $F \backslash E$ are disjoint and so by property 3) we have

$$\mu(F) = \mu(E \cup F \backslash E) = \mu(E) + \underbrace{\mu(F \backslash E)}_{\geq 0} \implies \mu(F) \geq \mu(E)$$

□

Corollary 2.1. *If $\mu(E) < \infty$ then $\mu(F \backslash E) = \mu(F) - \mu(E)$.*

Proof. Since $\mu(E) < \infty$ we can subtract it in the previous proof to get our result. □

Note 6. If $\mu(E) = \infty$ then $\mu(F) = \infty$ and the difference $\mu(F) - \mu(E)$ is undetermined.

Proposition 2.3. *(Countable Subadditivity) Let $(\mathbb{R}, A, \mu)$ be a measurable space. Let $\{E_n\}_{n=1}^{\infty} \subset A$. Then $\mu(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu(E_n)$*

Proof. Let $F_1 = E_1$, $F_2 = E_2 \backslash F_1$ and in general

$$F_n = E_n \backslash \underbrace{\bigcup_{i=1}^{n-1} F_i}_{\in A} \in A$$

for $n \in \mathbb{N}$. Then for all $k \in \mathbb{N}$ we have $\bigcup_{i=1}^{k} F_i = \bigcup_{i=1}^{k} E_i$, $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$ and $\{F_i\}_{i=1}^{\infty}$ are pairwise disjoint. Hence

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} \mu(F_i) \leq \sum_{i=1}^{\infty} \mu(E_i)$$

by monotonicity. □

2.2 Lebesgue Outer Measure

Problem 2.1. We want to define a measure λ on $\mathcal{P}(\mathbb{R})$ such that

(1) $\lambda : \mathcal{P}(\mathbb{R}) \mapsto \mathbb{R}^{\geq 0} \cup \{\infty\} = [0, \infty]$

(2) If $I = (a, b)$ then $\lambda(I) = \lambda((a, b)) = b - a$

(3) λ is countably additive

(4) $\lambda(E + x) = \lambda(E)$, $E \subseteq \mathbb{R}$, $x \in \mathbb{R}$ (translation invariance)

Unfortunately, this is note possible. Thus, we relax our conditions by restricting our domain to a σ-algebra which is a proper subset of $\mathcal{P}(\mathbb{R})$. Still, we want to have $\mathcal{B}(\mathbb{R})$ to be contained in that σ-algebra.

Definition 2.9. A function $\mu^* : \mathcal{P}(\mathbb{R}) \to \mathbb{R}^*$ is a called an *outer measure* if

1) $\mu^*(\emptyset) = 0$

2) $\mu^*(A) \leq \mu^*(B)$ if $A \subseteq B \subseteq \mathbb{R}$

3) If $\{E_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathbb{R})$ then $\mu^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$

Definition 2.10. μ^* is finite if $\mu^*(\mathbb{R}) < \infty$ and is called σ-finite if $\mathbb{R} = \bigcup_{n=1}^{\infty}$ and $|\mu^*(E_n)| < \infty$.

Definition 2.11. *(*Caratheodory Criterion) A set $E \in \mathcal{P}(\mathbb{R})$ is μ^*-measurable (measurable) if for any $A \subset \mathbb{R}$

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Note 7. By definition,

$$\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

so to prove measurability of E, it is enough to show that

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

for every $A \subset \mathbb{R}$. Furthermore, if $\mu^*(A) = \infty$ then the above trivially holds. So be only need to consider finite cases ($\mu^*(A) < \infty$).

Definition 2.12. Let $I = (a, b)$ and $l(I) = b - a$ with $l((a, \infty)) = +\infty$ and $l((-\infty, b)) = +\infty$. For any $E \subset \mathbb{R}$,

$$\lambda^*(E) = \inf \left\{ \sum_{n=1}^{\infty} l(I_n) : E \subset \bigcup_{n=1}^{\infty} I_n, I_n's \text{ are open intervals} \right\}$$

Remark 2.4. $\lambda^*(E) \geq 0$.

Proposition 2.4. *λ^* is an outer measure on $\mathbb{R}$.*

Proof. We go through each of the properties

1) ($\lambda^*(\emptyset) = 0$) For $\epsilon > 0$, $\emptyset \subseteq \left(-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right) \implies \lambda^*(\emptyset) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ and since ϵ is arbitrary, $0 \leq \lambda^*(\emptyset) \leq 0 \implies \lambda^*(\emptyset) = 0$

2) (Monotonicity) Let $F \subset E \subset \mathbb{R}$. Then

$$\lambda^*(F) = \inf \underbrace{\left\{ \sum_{n=1}^{\infty} l(I_n) : F \subset \bigcup_{n=1}^{\infty} I_n, I_n's \text{ are open intervals} \right\}}_{V} = \inf V$$

$$\lambda^*(E) = \inf \underbrace{\left\{ \sum_{n=1}^{\infty} l(I_n) : E \subset \bigcup_{n=1}^{\infty} J_n, J_n's \text{ are open intervals} \right\}}_{U} = \inf U$$

and any sequence $\{J_n\}_{n=1}^{\infty}$ also "appears" in V and $U \subseteq V \implies \lambda^*(F) \subseteq \lambda^*(E)$.

3) (Countable Subadditivity) Let $\{E_n\}_{n=1}^{\infty} \subset P(\mathbb{R})$. If $\sum_{n=1}^{\infty} \lambda^*(E_n) = +\infty$ the result trivially holds. So suppose the previous sum is finite. Then each $\lambda^*(E_n)$ is finite. Let $\epsilon > 0$ and for each n we can find $\{I_{n,i}\}_{i=1}^{\infty}$ such that $E_n \subset \bigcup_{i=1}^{\infty} I_{n,i}$ and $\lambda^*(E_n) + \frac{\epsilon}{2^n} > \sum_{i=1}^{\infty} l(I_{n,i})$. Then $\{\{I_{n,i}\}_{i=1}^{\infty}\}_{n=1}^{\infty}$ covers $E = \bigcup_{n=1}^{\infty} E_n$ by open intervals

$$\begin{aligned}
\lambda^*(E) &\leq \sum_{i,n=1}^{\infty} l(I_{n,i}) = \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} l(I_{n,i}) \\
&< \sum_{n=1}^{\infty}\left(\lambda^*(E_n) + \frac{\epsilon}{2^n}\right) = \sum_{n=1}^{\infty} \lambda^*(E_n) + \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} \\
&= \sum_{n=1}^{\infty} \lambda^*(E_n) + \epsilon
\end{aligned}$$

and since ϵ was arbitrary we get

$$\lambda^*(E) = \lambda^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \lambda^*(E_n)$$

□

2.3 Lebesgue Measure

Definition 2.13. λ^* is called the *Lebesgue outer measure* on $\mathbb{R}$. We denote the σ-algebra of λ^*-measurable sets by $\mathcal{L}(\mathbb{R})$. Elements of $\mathcal{L}(\mathbb{R})$ are called *Lebesgue measurable*. $\lambda = \lambda^*\Big|_{\mathcal{L}(\mathbb{R})}$ is called the Lebesgue measure of $\mathbb{R}$.

Proposition 2.5. *If $a < b$ and are both in $\mathbb{R}$ and J is an interval of the form $(a,b),[a,b],(a,b],[a,b)$ then $\lambda^*(J) = b-a$.*

Proof. We will consider $J=(a,b)$ and leave the rest as exercises. First, (a,b) covers itself. $\lambda^*(J) \le l((a,b)) = b-a$.

Let $\{I_n\}_{n=1}^{\infty}$ be any cover of J by open intervals. Let $0 < \epsilon < \frac{b-a}{2}$. The $\{I_n\}_{n=1}^{\infty}$ is also a cover of $[a+\epsilon, b-\epsilon]$ which is compact. Hence, there is a finite cover $\{I_{n_k}\}_{k=1}^{N}$ of $[a+\epsilon, b-\epsilon]$. For each $1 \le k \le N$ let $I_{n_k} = (a_k, b_k)$.

Without loss of generality (WLOG) we can assume that $b_{k+1} < a_k$. for each k by getting rid of some of them. We also assume by reindexing $a_1 < a+\epsilon$ and $b-\epsilon < b_N$. Thus we have

$$\begin{aligned}
\sum_{n=1}^{\infty} l(I_n) &\ge \sum_{k=1}^{N} l(I_k) \\
&= \sum_{k=1}^{N} l((a_k,b_k)) \\
&= b_1 - a_1 + b_2 - a_2 + \ldots + b_N - a_N \\
&= -a_1 + \underbrace{(b_1 - a_2)}_{\ge 0} + \ldots + \underbrace{(b_{N-1} - a_N)}_{\ge 0} + b_N \\
&\ge b_N - a_1 \ge b - \epsilon - (a+\epsilon) = b - a - 2\epsilon
\end{aligned}$$

and so $\sum_{n=1}^{\infty} l(I_n) \ge b-a$ by letting $\epsilon \to 0$. Since ϵ was arbitrary, we get

$$\lambda^*(J) \ge b-a$$

□

Theorem 2.1. *(Caratheodory's Theorem) The set $\mathcal{L}(\mathbb{R})$ of Lebesgue measurable sets is a σ-algebra and $\lambda^*\Big|_{\mathcal{L}(\mathbb{R})} = \lambda$ is a complete measure.*

Proof. We will first show that $\mathcal{L}(\mathbb{R})$ is a σ-algebra.

(1) $\emptyset, \mathbb{R} \in \mathcal{L}(\mathbb{R})$. Let $A \subseteq \mathbb{R}$ be arbitrary. Then

$$\lambda^*(A \cap \emptyset) + \lambda^*(A \backslash \emptyset) = \lambda^*(\emptyset) + \lambda^*(A) = \lambda^*(A)$$

and

$$\lambda^*(A \cap \mathbb{R}) + \lambda^*(A \backslash \mathbb{R}) = \lambda^*(A) + \lambda^*(\emptyset) = \lambda^*(A)$$

and hence $\emptyset$ and $\mathbb{R}$ are in $\mathcal{L}(\mathbb{R})$.

(2) Let $A \subseteq \mathbb{R}$ be arbitrary and suppose $E \in \mathcal{L}(\mathbb{R})$. Then

$$\lambda^*(A \cap E^c) + \lambda^*(A \cap (E^c)^c) = \lambda^*(A \cap E^c) + \lambda^*(A \cap E) = \lambda^*(A)$$

since E satisfies the Caratheodory criterion. We need to prove that $\mathcal{L}(\mathbb{R})$ is closed under taking countable unions. First, we will show that if $E_1, E_2 \in \mathcal{L}(\mathbb{R})$ then $E_1 \cup E_2 \in \mathcal{L}(\mathbb{R})$. Observe that

$$\begin{aligned}
\lambda^*(A \cap (E_1 \cup E_2)) + \lambda^*(A \cap (E_1 \cup E_2)^c) &= \lambda^*(A \cap (E_1 \cup E_2) \cap E_1) + \lambda^*(A \cap (E_1 \cup E_2) \cap E_1^c) + \lambda^*(A \cap (E_1 \cup E_2)^c) \\
&= \lambda^*(A \cap E_1) + \lambda^*(A \cap E_1^c \cap E_2) + \lambda^*(A \cap E_1^c \cap E_2^c) \\
&= \lambda^*(A \cap E_1) + \lambda^*(A \cap E_1^c) \\
&= \lambda^*(A)
\end{aligned}$$

and hence $E_1 \cup E_2 \in \mathcal{L}(\mathbb{R})$. Thus, $\mathcal{L}(\mathbb{R})$ is at least an algebra. Next, consider $\{E_n\}_{n=1}^\infty \subset \mathcal{L}(\mathbb{R})$ a disjoint sequence of λ^*-measurable sets.

First, we will prove by induction that

$$(1)\ \lambda^*(A) = \sum_{i=1}^n \lambda^*(A \cap E_i) + \lambda^*\left(A \cap \left(\bigcap_{i=1}^n E_i^c\right)\right)$$

for all $A \subseteq \mathbb{R}$ and $n \in \mathbb{N}$. In the case of $n = 1$, we use the λ^* measurability of E_1 and use our previous result. Now suppose that (1) holds for some n. We want to show the case for $n+1$. Since E_{n+1} is measurable,

$$\begin{aligned}\lambda^*\left(A \cap \left(\bigcap_{i=1}^n E_i^c\right)\right) &= \lambda^*\left(A \cap \underbrace{\left(\bigcap_{i=1}^n E_i^c\right) \cap E_{n+1}}_{E_{n+1}}\right) + \lambda^*\left(A \cap \left(\bigcap_{i=1}^n E_i^c\right) \cap E_{n+1}^c\right) \\ &= \lambda^*\left(A \cap E_{n+1}\right) + \lambda^*\left(A \cap \left(\bigcap_{i=1}^{n+1} E_i^c\right)\right)\end{aligned}$$

and since (1) works for n we have

$$\begin{aligned}\lambda^*(A) &= \sum_{i=1}^n \lambda^*(A \cap E_i) + \lambda^*\left(A \cap E_{n+1}\right) + \lambda^*\left(A \cap \left(\bigcap_{i=1}^{n+1} E_i^c\right)\right) \\ &= \sum_{i=1}^{n+1} \lambda^*(A \cap E_i) + \lambda^*\left(A \cap \left(\bigcap_{i=1}^{n+1} E_i^c\right)\right)\end{aligned}$$

and so (1) works for $n+1$ and by induction it work for all $n \in \mathbb{N}$. Since

$$A \cap \left(\bigcap_{i=1}^\infty E_i^c\right) \subseteq A \cap \left(\bigcap_{i=1}^n E_i^c\right)$$

we have

$$\lambda^*(A) \geq \sum_{i=1}^n \lambda^*(A \cap E_i) + \lambda^*\left(A \cap \left(\bigcap_{i=1}^\infty E_i^c\right)\right)$$

by monotonicity. Taking $n \to \infty$, we get

$$\begin{aligned}(2)\ \lambda^*(A) &\geq \sum_{i=1}^\infty \lambda^*(A \cap E_i) + \lambda^*\left(A \cap \left(\bigcap_{i=1}^\infty E_i^c\right)\right) \\ &\geq \lambda^*\left(A \cap \left(\bigcup_{i=1}^\infty E_i\right)\right) + \lambda^*\left(A \cap \left(\bigcap_{i=1}^\infty E_i^c\right)\right) \geq \lambda^*(A)\end{aligned}$$

and so $\bigcup_{i=1}^\infty E_i \in \mathcal{L}(\mathbb{R})$. Therefore $\{E_n\}_{n=1}^\infty \subset \mathcal{L}(\mathbb{R})$ are disjoint implies that $\bigcup_{n=1}^\infty E_n \in \mathcal{L}(\mathbb{R})$. Finally, consider $\{F_n\}_{n=1}^\infty \subset \mathcal{L}(\mathbb{R})$. Then we can write $\{F_n\}_{n=1}^\infty$ as a union of disjoint sets in $\mathcal{L}(\mathbb{R})$ (from our assignment) from which $\bigcup_{n=1}^\infty F_n \in \mathcal{L}(\mathbb{R})$. Therefore $\mathcal{L}(\mathbb{R})$ is a σ-algebra.

(3) Trivial. □

Proposition 2.6. *λ is a measure.*

Proof. (1) $\lambda^*(\emptyset) = 0 = \lambda(\emptyset)$

(2) $\lambda^*(E) \geq 0$ follows from the definition of λ^*

(3) We need to prove that λ is countably additive. Let $\{E_i\}_{i=1}^{\infty}$ be a sequence of disjoint sets. In equation (2) above, we replace the set A with $\bigcup_{i=1}^{\infty} E_i$ to get

$$\begin{aligned}\lambda^*\left(\bigcup_{i=1}^{\infty} E_i\right) &\geq \sum_{i=1}^{\infty}\lambda^*\left(\underbrace{\left(\bigcup_{i=1}^{\infty} E_i\right)\cap E_j}_{E_j}\right)+\lambda^*\left(\left(\bigcup_{i=1}^{\infty} E_i\right)\cap\left(\bigcap_{j=1}^{\infty} E_j^c\right)\right)\\ &= \sum_{i=1}^{\infty}\lambda^*\left(E_j\right)+\lambda^*\left(\emptyset\right)\\ &= \sum_{i=1}^{\infty}\lambda^*\left(E_j\right)\end{aligned}$$

and since the reverse inequality always works, we get the result that λ is a measure on $\mathcal{L}(\mathbb{R})$. □

Proposition 2.7. *λ is complete.*

Proof. Let $E \in \mathcal{L}(\mathbb{R})$ with $\lambda(E) = 0$. We consider $F \subset E$. We then note that for arbitrary $A \subset \mathbb{R}$ we have

$$\begin{aligned}\lambda^*(A) &\geq \lambda^*(A\cap F^c)\\ &= \lambda^*(A\cap F^c)+\underbrace{\lambda(A\cap F)}_{\leq\lambda^*(A\cap E)\leq\lambda^*(E)=0}\\ &\geq \lambda^*(A)\end{aligned}$$

and hence $F \in \mathcal{L}(\mathbb{R})$ with $\lambda(F) \leq \lambda(E) = 0$. so $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is a complete measure space. □

Theorem 2.2. *Let μ^* be a non-negative outer measure on $\mathbb{R}$. Let $\mathcal{M}_{\mu^*}$ denote the μ^* measurable subsets of $\mathbb{R}$. Then $\mathcal{M}_{\mu^*}$ is a σ-algebra and $\mu^*\Big|_{\mathcal{M}_{\mu^*}} = \mu$ is a measure on $\mathcal{M}_{\mu^*}$ with the associated space $(\mathbb{R}, \mathcal{M}_{\mu}, \mu)$ being complete.*

Lemma 2.1. *Every bounded open interval $(a, b) \subset \mathbb{R}$ is in $\mathcal{L}(\mathbb{R})$*

Proof. Let $(a, b) \subset \mathbb{R}$ be a bounded open interval. Let $A \subseteq \mathbb{R}$ with $\lambda^*(A) < \infty$. It is enough to prove

$$\lambda^*(A) \geq \lambda^*(A\cap(a,b))+\lambda^*(A\cap(a,b)^c)$$

Let $\epsilon > 0$ be arbitrary. Since $\lambda^*(A) < \infty$, we can find $\{I_n\}_{n=1}^{\infty}$ open intervals such that

$$A \subseteq \bigcup_{n=1}^{\infty} I_n$$

and

$$\lambda^*(A)+\frac{\epsilon}{2} > \sum_{n=1}^{\infty} l(I_n)$$

For each n define

$$\begin{aligned}J_n &= I_n\cap(a,b)\\ L_n &= I_n\cap(-\infty,a)\\ R_n &= I_n\cap(b,\infty)\end{aligned}$$

Then $\{J_n\}_{n=1}^{\infty}$ covers $A \cap (a, b)$. Next, note $\{L_n, R_n\}_{n=1}^{\infty} \cup \left\{\left(a-\frac{\epsilon}{8}, a+\frac{\epsilon}{8}\right), \left(b-\frac{\epsilon}{8}, b+\frac{\epsilon}{8}\right)\right\}$ cover $A \cap (a, b)^c$. We relabel this sequence as $\{K_n\}_{n=1}^{\infty}$. Observe that

$$\sum_{n=1}^{\infty}(l(J_n)+l(L_n)+l(R_n)) = \sum_{n=1}^{\infty} l(I_n)$$

and hence

$$\begin{aligned}\sum_{n=1}^{\infty}(l(J_n)+l(K_n)) &= \sum_{n=1}^{\infty} l(I_n)+l\left(\left(a-\frac{\epsilon}{8},a+\frac{\epsilon}{8}\right)\right)+l\left(\left(b-\frac{\epsilon}{8},b+\frac{\epsilon}{8}\right)\right)\\ &= \sum_{n=1}^{\infty} l(I_n)+\frac{\epsilon}{2}\end{aligned}$$

and so

$$\begin{aligned}\lambda^*(A\cap(a,b))+\lambda^*(A\cap(a,b)^c) &\leq \sum_{n=1}^{\infty} l(J_n)+\sum_{n=1}^{\infty} l(K_n)\\ &= \sum_{n=1}^{\infty} l(I_n)+\frac{\epsilon}{2}\\ &< \lambda^*(A)+\frac{\epsilon}{2}+\frac{\epsilon}{2}\end{aligned}$$

and since $\epsilon > 0$ is arbitrary, $(a,b)\in\mathcal{L}(\mathbb{R})$. □

Corollary 2.2. *$\mathcal{B}(\mathbb{R})=\sigma(\{(a,b):a,b\in\mathbb{R}\})\subset\mathcal{L}(\mathbb{R})$ since $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra that is generated by open sets ($\mathcal{L}(\mathbb{R})$ is a larger σ-algebra that contains open sets).*

Remark 2.5. For $x\in\mathbb{R}$, $\{x\}$ is closed $\implies$ $\{x\}\in\mathcal{L}(\mathbb{R})$. We have

(i) $\lambda(\{x\})=0$

(ii) $\lambda(E)=0$ for countable E

Proof. (i) $\{x\}\subseteq\left(x-\frac{1}{n},x+\frac{1}{n}\right),\forall n\in\mathbb{N}$. By monotonicity, $\lambda(\{x\})\leq\frac{2}{n},\forall n\in\mathbb{N}$ so $\lambda(\{x\})=0$.

(ii) Follows from countable subadditivity □

Problem 2.2. If $\lambda(E)=0$ do we need $|E|\leq\aleph_0$? The answer is no!

Example 2.4. (Cantor set) Let $C_0=[0,1],C_1=\left[0,\frac{1}{3}\right]\cup\left[\frac{2}{3},1\right],...,C_n=C_{n-1}\backslash(I_{n,1}\cup...\cup I_{n,2^{n-1}})$ where $I_{n,k}$ is the open middle third of the k^{th} set from C_{n-1} and let

$$C=\bigcap_{n=1}^{\infty}C_n$$

where we call C the Cantor set.

Remark 2.6. $C\neq\emptyset$ due to the Cantor Intersection Theorem ($\{C_n\}$ has the finite intersection property).

Proposition 2.8. *(i) C is closed*

(ii) C is nowhere dense

(iii) $\lambda(C)=0$

Proof. This is part of Assignment 2. □

Proposition 2.9. *$|C|=c$ where c is the cardinality of the continuum.*

Proof. If $x\in[0,1]$, write it in its ternary expansion $x=0.\epsilon_1\epsilon_2,...=\sum_{i=1}^{\infty}\frac{\epsilon_i}{3^i}$ where $\epsilon_i\in\{0,1,2\}$ where this expansion is not necessarily unique. It can be shown that numbers in the Cantor set in base 3 only have ϵ_i with digits in the set $\{0,2\}$ and the set of all sequences that can be constructed with these elements is

$$2^{\aleph_0}=c$$

□

Definition 2.14. Let $E \subseteq \mathbb{R}, x \in \mathbb{R}$. We define the translate of E by x as

$$E + x = \{y + x : y \in E\}$$

Proposition 2.10. *(Translation Invariance of the Lebesgue Measure)*

(i) If $E \subseteq \mathbb{R}, x \in \mathbb{R}$ then $\lambda^*(x + E) = \lambda^*(E)$

(ii) If $E \in \mathcal{L}(\mathbb{R}), x \in \mathbb{R}$ then $x + E \in \mathcal{L}(\mathbb{R})$

(iii) If $E \subseteq \mathbb{R}, x \in \mathbb{R}$ then $\lambda(x + E) = \lambda(E)$

Proof. (i) Let $E \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Let $\epsilon > 0$ be given. Let $\{I_n\}_{n=1}^\infty$ be a cover of E by open intervals such that

$$\lambda^*(E) + \epsilon > \sum_{n=1}^\infty l(I_n)$$

and for each n, we define $I_n' = I_n + x$. Note that each I_n' is an open interval. Also for each n,

$$l(I_n') = l(I_n) \implies \sum_{n=1}^\infty l(I_n) = \sum_{n=1}^\infty l(I_n')$$

Now since the sequence $\{I_n'\}_{n=1}^\infty$ covers $E + x$ we have

$$\lambda^*(E) + \epsilon \geq \sum_{n=1}^\infty l(I_n) = \sum_{n=1}^\infty l(I_n') \geq \lambda^*(E + x)$$

and since ϵ is arbitrary, we have

$$\lambda^*(E) \geq \lambda^*(E + x)$$

Conversely, we write $E = (E + x) + (-x)$. Then by above

$$\lambda^*(E + x) \geq \lambda^*((E + x) + (-x)) = \lambda^*(E) \implies \lambda^*(E) = \lambda^*(E + x)$$

(ii) Let $E \in \mathcal{L}(\mathbb{R}), x \in \mathbb{R}, A \subseteq \mathbb{R}$ for arbitrary A. Consider

$$\begin{aligned}
\lambda^*(A \cap (E + x)) + \lambda^*(A \cap (E + x)^c) &\overset{\text{by (i)}}{=} \lambda^*(\underbrace{[A \cap (E + x)] - x}_{=(A-x)\cap E}) + \lambda^*(\underbrace{[A \cap (E + x)^c] - x}_{=(A-x)\cap E^c}) \\
&= \lambda^*((A - x) \cap E) + \lambda^*((A - x) \cap E^c) \\
&= \lambda^*(A - x) \\
&= \lambda^*(A)
\end{aligned}$$

and so $E + x \in \mathcal{L}(\mathbb{R})$.

(iii) This follows from (i) and (ii). □

2.4 Non-Measurable Sets

Theorem 2.3. *There exist non-measurable subsets (called Vitali sets) of $\mathbb{R}$. That is $\mathcal{P}(\mathbb{R}) \backslash \mathcal{L}(\mathbb{R}) \neq \emptyset$. (Note that the proof will depend on the Axiom of Choice (AoC). Without it, it is possible to show $\mathcal{P}(\mathbb{R}) \backslash \mathcal{L}(\mathbb{R}) = \emptyset$ (c.f. R.M. Solovay, 1970, Ann. of Math)).*

Proof. We consider a single counterexample. Let $a > 0$ be fixed and consider $(-a, a)$. We define an equivalence relation for $x, y \in (a, b)$ where we say

$$x \sim y \iff x - y \in \mathbb{Q}$$

and $\sim$ is an equivalence relation because $\mathbb{Q}$ is a group (Exercise). We denote the equivalence class of x as

$$[x] \quad = \quad \{y \in (-a, a) : y \sim x\} = \{y \in (-a, a) : x - y \in \mathbb{Q}\} = (x + \mathbb{Q}) \cap (-a, a)$$

Let E be a subset of $(-a, a)$ such that

(i) If $x, y \in E$, $x \neq y$ then $x \nsim y$

(ii) The union of the equivalence classes of elements in E generate $(-a, a)$:

$$\bigcup_{x \in E} [x] = (-a, a)$$

The existence of E depends on AoC. E is called a transversal of $\sim$. Note that if $r \in \mathbb{Q}$ then $(r + E) \cap E = \emptyset$ if $r \neq 0$. Let $\{r_k\}_{k=1}^{\infty}$ be an enumeration of $\mathbb{Q} \cap (-2a, 2a)$. Then,

$$(1)\ (-a, a) \subset \bigcup_{k=1}^{\infty} (r_k + E) \subset (-3a, 3a)$$

(First inclusion) If $x \in (-a, a)$ then there is a unique $x_E \in E$ such that $x_E \sim x$ ($x_E \in E \cap [x]$). Now $x \sim x_E \implies$ there is r_k such that

$$x - x_E = r_k, k \in \mathbb{N} \implies x = x_E + r_k \in r_k + E$$

Furthermore, $x, x_E \in (-a, a) \implies x - x_E = r_k \in (-2a, 2a)$. Hence $x \in \bigcup_{k=1}^{\infty} r_k + E$.

(Second inclusion) Let $y \in \bigcup_{k=1}^{\infty} r_k + E \implies y = r_k + e$ for some $k \in \mathbb{N}, e \in E$. Then $r_k \in (-2a, 2a)$ and $e \in E \subset (-a, a)$. So $r_k + e \in (-3a, 3a)$.

We claim that $E \notin \mathcal{L}(\mathbb{R})$. Suppose otherwise, that is $E \in \mathcal{L}(\mathbb{R}) \implies \lambda(E) \geq 0$.

<u>Case 1</u>: Suppose that $\lambda(E) = 0$. Then from equation (1) above,

$$2a = \lambda((-a, a)) \leq \lambda \left(\underbrace{\bigcup_{k=1}^{\infty} \underbrace{r_k + \underbrace{E}_{meas.}}_{meas}}_{meas.\ +\ disjoint} \right) = \sum_{k=1}^{\infty} \lambda(r_k + E) = \sum_{k=1}^{\infty} \lambda(E) = 0 \implies 0 < 2a \leq 0$$

which is clearly not possible.

<u>Case 2</u>: Suppose $\lambda(E) > 0$. Since $(r_k + E) \cap (r_l \cap E) = \emptyset$ if $k \neq l$. We have for each n

$$\lambda \left(\bigcup_{k=1}^{n} (r_k + E) \right) = \sum_{k=1}^{n} \lambda(r_k + E) = \sum_{k=1}^{n} \lambda(E) = n\lambda(E)$$

but by equation (1) above,

$$n\lambda(E) \leq \lambda((-3a, 3a)) = 6a$$

However, the left side diverges and the left side doesn't which is clearly a contradiction. Thus, $E \notin \mathcal{L}(\mathbb{R})$. □

3 Measurable Functions

Definition 3.1. A function $f : \mathbb{R} \mapsto \mathbb{R}$ is called measurable if for every $\alpha \in \mathbb{R}$ we have

$$f^{-1}((\alpha, +\infty)) = \{x \in \mathbb{R} : f(x) > \alpha\}$$

is λ-measurable. f is called Borel measurable if $f^{-1}((\alpha, +\infty)) \in \mathcal{B}(\mathbb{R})$ for all $\alpha \in \mathbb{R}$.

Example 3.1. If $f : \mathbb{R} \mapsto \mathbb{R}$ is continuous, then $f^{-1}((\alpha, +\infty))$ is open and f is λ-measurable and Borel measurable.

Example 3.2. Let $A \subseteq \mathbb{R}$. Consider the characteristic function

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

We claim that χ_A is measurable. That is, $\chi_A \in \mathcal{M}(\mathbb{R}) \iff A \in \mathcal{L}(\mathbb{R})$. To prove this, let $\alpha \in \mathbb{R}$ and note that

$$\chi_A^{-1}((\alpha, \infty)) = \begin{cases} \emptyset & \alpha \geq 1 \\ A & 0 < \alpha \leq 1 \\ \mathbb{R} & \alpha \leq 0 \end{cases}$$

So χ_A is measurable if $A \in \mathcal{L}(\mathbb{R})$.

Proposition 3.1. *Let $f : \mathbb{R} \mapsto \mathbb{R}$. TFAE.*

(i) f is measurable (Borel measurability)

(ii) $\forall \alpha \in \mathbb{R}$, $f^{-1}((-\infty, \alpha])$ ($\in \mathcal{B}(\mathbb{R})$)

(iii) $\forall \alpha \in \mathbb{R}$, $f^{-1}((-\infty, \alpha))$ ($\in \mathcal{B}(\mathbb{R})$)

(iv) $\forall \alpha \in \mathbb{R}$, $f^{-1}([\alpha, \infty))$ ($\in \mathcal{B}(\mathbb{R})$)

Proof. $(i) \implies (ii)$ Let $\alpha \in \mathbb{R}$ and consider

$$\begin{aligned} f^{-1}((-\infty, \alpha]) &= \{x \in \mathbb{R} | f(x) \leq \alpha\} \\ &= \mathbb{R} \backslash \{x \in \mathbb{R} | f(x) > \alpha\} \\ &= \mathbb{R} \backslash \underbrace{f^{-1}((\alpha, \infty))}_{\in \mathcal{L}(\mathbb{R}) \text{ by (i)}} \in \mathcal{L}(\mathbb{R}) \end{aligned}$$

since $\mathcal{L}(\mathbb{R})$ is a σ-algebra.

$(ii) \implies (iii)$ Let $\alpha \in \mathbb{R}$. Consider

$$\begin{aligned} f^{-1}((-\infty, \alpha)) &= f^{-1}\left(\bigcup_{n=1}^{\infty} \left(-\infty, \alpha - \frac{1}{n} \right] \right) \\ &= \bigcup_{n=1}^{\infty} \underbrace{f^{-1}\left(\left(-\infty, \alpha - \frac{1}{n} \right] \right)}_{\in \mathcal{L}(\mathbb{R})} \end{aligned}$$

and so $f^{-1}((-\infty, \alpha)) \in \mathcal{L}(\mathbb{R})$.

$(iii) \implies (iv)$ is similar to $(i) \implies (ii)$.

$(iv) \implies (i)$ Let $\alpha \in \mathbb{R}$. Consider

$$\begin{aligned} f^{-1}((-\infty,\alpha)) &= f^{-1}\left(\bigcup_{n=1}^{\infty}\left[\alpha+\frac{1}{n},\infty\right)\right) \\ &= \bigcup_{n=1}^{\infty}\underbrace{f^{-1}\left(\left[\alpha+\frac{1}{n},\infty\right)\right)}_{\in\mathcal{L}(\mathbb{R})} \in \mathcal{L}(\mathbb{R}) \end{aligned}$$

□

Proposition 3.2. *A function $f:\mathbb{R}\mapsto\mathbb{R}$ is (Borel) measurable if and only if $f^{-1}(A)$ is (Borel) measurable for each Borel set A ($A \in \mathcal{B}(\mathbb{R})$)*

Proof. We will consider the measurability of $f:\mathbb{R}\mapsto\mathbb{R}$.

$(\Longleftarrow)$ Trivial since $(\alpha,\infty)\in\mathcal{B}(\mathbb{R})$ for any $\alpha\in\mathbb{R}$.

$(\implies)$ Assume that f is measurable. First, we will consider $(a,b)\in\mathbb{R}$. We write $(a,b)=(-\infty,b)\cap(a,\infty)$. So, \iff

$$f^{-1}((a,b)) = \underbrace{f^{-1}((-\infty,b))}_{\in\mathcal{L}(\mathbb{R})}\cap\underbrace{f^{-1}((a,\infty)}_{\in\mathcal{L}(\mathbb{R})}\in\mathcal{L}(\mathbb{R})$$

Next, let $G\subseteq\mathbb{R}$ be an open set with

$$G=\bigcup_{n=1}^{\infty}(a_i,b_i)$$

and hence

$$f^{-1}(G)=\bigcup_{n=1}^{\infty}\underbrace{f^{-1}((a_i,b_i))}_{\text{for each } i \text{is in } \mathcal{L}(\mathbb{R})}\in\mathcal{L}(\mathbb{R})$$

Let $\mathcal{M}_f=\{A\subseteq\mathbb{R}|f^{-1}(A)\in\mathcal{L}(\mathbb{R})\}$. By the above, any open subset of $\mathbb{R}$ is an element of $\mathcal{M}_f$. We want to show that $\mathcal{B}(\mathbb{R})\subset\mathcal{M}_f$, using the fact that $\mathcal{B}(\mathbb{R})$ if the small σ-algebra that contains the open sets. We claim that $\mathcal{M}_f$ is a σ-algebra.

(i) $\emptyset$ is open $\implies \emptyset\in\mathcal{M}_f$

(ii) Let $A\in\mathcal{M}_f \implies f^{-1}(A)\in\mathcal{L}(\mathbb{R})$ and so $\mathbb{R}\backslash f^{-1}(A)=f^{-1}(\mathbb{R}\backslash A)\in\mathcal{L}(\mathbb{R})$; thus, $A^c=\mathbb{R}\backslash A\in\mathcal{M}_f$

(iii) Let $A_1,A_2,...\in\mathcal{M}_f$ then for each i, $f^{-1}(A_i)\in\mathcal{L}(\mathbb{R})$ and

$$f^{-1}\left(\bigcup_{i=1}^{\infty}A_i\right)=\bigcup_{i=1}^{\infty}f^{-1}(A_i)\in\mathcal{L}(\mathbb{R})$$

and hence $\bigcup_{n=1}^{\infty}A_i\in\mathcal{M}_f$

Thus, $\mathcal{M}_f$ is a σ-algebra containing all open sets and $\mathcal{B}(\mathbb{R})\subset\mathcal{M}_f$. □

Proposition 3.3. *Let $f,g:\mathbb{R}\mapsto\mathbb{R}$ be measurable, $c\in\mathbb{R}$ and $\phi:\mathbb{R}\mapsto\mathbb{R}$ be continuous. Then*

(i) cf is measurable

(ii) $f+g$ is measurable

(iii) $\phi\circ f$ is measurable, ϕ continuous

(iv) fg is measurable

Note that (i), (ii), and (iv), as a corollary, tells us that $\mathcal{M}(\mathbb{R})$ is an algebra.

Proof. (i) Fix $\alpha \in \mathbb{R}$. Then

$$cf^{-1}((\alpha,+\infty)) = \begin{cases} f^{-1}\left(\frac{\alpha}{c},\infty\right) & c>0 \\ \mathbb{R} & c=0,\alpha<0 \\ \emptyset & c=,\alpha\geq 0 \\ f^{-1}\left((-\infty,\frac{\alpha}{c}\right) & c<0 \end{cases}$$

and so $cf \in \mathcal{M}(\mathbb{R})$.

(ii) Let $Q = \{q_k\}_{k=1}^{\infty}$ be an enumeration. If $\alpha \in \mathbb{R}$, then we have

$$\begin{aligned} (f+g)^{-1}((\alpha,+\infty)) &= \{x\in\mathbb{R}|f(x)+g(x)>\alpha\} \\ &= \{x\in\mathbb{R}|f(x)>\alpha-g(x)\} \\ &= \{x\in\mathbb{R}|f(x)>q>\alpha-g(x),\text{some } q\in Q\} \\ &= \{x\in\mathbb{R}|f(x)>q,q>\alpha-g(x),\text{some } q\in Q\} \\ &= \bigcup_{k=1}^{\infty}\left(\{x\in\mathbb{R}|f(x)>r_k\}\cap\{x\in\mathbb{R}|r_k>\alpha-g(x)\}\right) \\ &= \bigcup_{k=1}^{\infty}\left(f^{-1}((r_k,\infty))\cap g^{-1}(-\infty,\alpha-r_k)\right)\in\mathcal{M}(\mathbb{R}) \end{aligned}$$

(iii) Let $\alpha \in \mathbb{R}$.

$$(\phi\circ f)^{-1}(\alpha,\infty) = f^{-1}(\underbrace{\phi^{-1}((\alpha,\infty))}_{open}) \in \mathcal{L}(\mathbb{R})$$

(iv) Note that $fg = \frac{(f+g)^2-(f-g)^2}{4}$, $\phi(x)=x^2$ and use the above. □

Corollary 3.1. *If $f:\mathbb{R}\mapsto\mathbb{R}$ is measurable, then so are $|f|$, f^+, f^- where*

$$f^+(x) = \max\{f(x),0\}, f^-(x) = -\min\{f(x),0\}$$

Proof. Consider $\phi:\mathbb{R}\mapsto\mathbb{R}$ given by $\phi(x)=|x|$. Then $\phi\circ f$ is measurable. Next, note that $f^+ = \frac{1}{2}(|f|+f)$ and $f^- = \frac{1}{2}(|f|-f)$ which are measurable because their components are measurable. □

3.1 The Extended Reals

Definition 3.2. Define the *extended real line* $\mathbb{R}^*$ as

$$\mathbb{R}^* = \mathbb{R}\cup\{\pm\infty\} = [-\infty,\infty]$$

(1) A function f on $\mathbb{R}$ is called extended real valued if $f:\mathbb{R}\mapsto\mathbb{R}^*$

(2) An extended real valued function is called measurable if $\forall\alpha\in\mathbb{R}$,

$$f^{-1}((\alpha,\infty])\in\mathcal{L}(\mathbb{R})$$

Proposition 3.4. *An extended real valued function $f:\mathbb{R}\mapsto\mathbb{R}^*$ is measurable if and only if the following conditions are satisfied.*

1) $f^{-1}(\{-\infty\})$ and $f^{-1}(\{\infty\})$ are in $\mathcal{L}(\mathbb{R})$

2) The real valued function f_0 defined by

$$f_0(x) = \begin{cases} f(x) & f(x)\in\mathbb{R} \\ 0 & f(x)\in\{\pm\infty\} \end{cases}$$

is measurable (i.e. $f_0\in\mathcal{L}(\mathbb{R})$)

Proof. (Exercise) □

Notation 2. The set of measurable extended $\mathbb{R}^*$ valued function are denoted by $\mathcal{M}^*(\mathbb{R})$.

Remark 3.1. Note that if $f, g \in \mathcal{M}^*(\mathbb{R})$ we could have that $f + g$ is indeterminate ($\infty - \infty$) and so $\mathcal{M}^*(\mathbb{R})$ is not necessarily an algebra. Also, if $\phi : \mathbb{R} \mapsto \mathbb{R}$ is continuous, then $\phi \circ f$ may fail to make sense.

Proposition 3.5. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{M}^*(\mathbb{R})$. Then the following functions are also measurable:*

(i) $\sup_{n\in\mathbb{N}} f_n$ *(pointwise infimum)*

(ii) $\inf_{n\in\mathbb{N}} f_n$ *(pointwise supremum)*

(iii) $\limsup_{n\to\infty} f_n$ *where* $(\limsup_{n\to\infty} f_n)(x) = \inf_n \left(\sup_{k\geq n} f_k(x)\right)$

(iv) $\liminf_{n\to\infty} f_n$ *where* $(\liminf_{n\to\infty} f_n)(x) = \sup_n (\inf_{k\geq n} f_k(x))$

Proof. (i) Consider for any $\alpha \in \mathbb{R}$,

$$\begin{aligned}\left(\sup_{n\in\mathbb{N}} f_n\right)^{-1}([-\infty,\infty]) &= \left\{x \in \mathbb{R} | \sup_{n\in\mathbb{N}} f_n(x) \leq \alpha\right\} \\ &= \bigcap_{n=1}^{\infty} \{x \in \mathbb{R} | f_n(x) \leq \alpha\} \\ &= \bigcap_{n=1}^{\infty} \underbrace{f_n^{-1}([-\infty,\infty])}_{\in\mathcal{L}(\mathbb{R})} \in \mathcal{L}(\mathbb{R})\end{aligned}$$

(ii) Note that

$$\inf_{n\in\mathbb{N}} f_n = -\sup_{n\in\mathbb{N}}(\underbrace{-f_n}_{\in\mathcal{M}^*(\mathbb{R})}) \in \mathcal{M}^*(\mathbb{R})$$

(iii) Let $g_n = \sup_{k\geq n}\{f_k(x)\}$. Then by (i) $g_n \in \mathcal{M}^*(\mathbb{R})$. From (ii) $\limsup_{n\in\mathbb{R}} = \inf_{n\in\mathbb{N}} g_n \in \mathcal{M}^*(\mathbb{R})$.

(iv) This is similar to the above (iii). □

Corollary 3.2. *If $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{M}^*(\mathbb{R})$ with pointwise limit $f(x)$ then $f \in \mathcal{M}^*$.*

Proof. If f exists, then

$$f = \limsup_{n\in\mathbb{N}} f_n = \liminf_{n\in\mathbb{N}} f_n$$

□

4 Lebesgue Integration

Instead of partitioning the domain of a function, like in Riemann integration, we instead partition in the range. That is, we divide the range of f into a partition

$$y_0 < y_1 < \ldots < y_n$$

and define

$$E_i = \{t \in A : y_{i-1} \leq f(t) < y_i\}$$

We then find the sized of $E_i = \lambda(E_i)$ and we will estimate $\int f$ by sums

$$\sum_{k=1}^{n} y_{i-1}\lambda(E_i)$$

4.1 Simple Functions

Definition 4.1. Let $A \in \mathcal{L}(\mathbb{R})$, a function $\phi : A \mapsto \mathbb{R}$ is called a *simple function* if $\phi(A) = \{\phi(x) : x \in A\}$ is a finite set.

Remark 4.1. If $\phi(A) = \{\alpha_1 < ... < \alpha_n\}$, define the preimage of α_i as $E_i = \phi^{-1}(\{\alpha_i\})$ for $1 \leq i \leq n$. Note that $E_i \cap E_j = \emptyset$ if $i \neq j$. So we have

$$\phi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$$

and we call it the *standard representation of the simple function* ϕ.

Proposition 4.1. *Let A be a measurable set and $\phi : A \mapsto \mathbb{R}$ be a simple function with $\phi(A) = \{\alpha_1 < ... < \alpha_n\}$. Then ϕ is measurable iff each $1 \leq i \leq n$ we have that the $E_i = \phi^{-1}(\{a_i\})$ are measurable.*

Proof. ($\implies$) Observe that $\{a_i\}$ is closed $\implies$ $\{a_i\}$ is Borel so $E_i = \phi^{-1}(\{a_i\}) \in \mathcal{L}(\mathbb{R})$.

($\Longleftarrow$) Suppose that for each $1 \leq i \leq n, E_i \in \mathcal{L}(\mathbb{R})$. Then $\chi_{E_i} \in \mathcal{M}(\mathbb{R})$ so

$$\phi = \sum_{i=1}^{n} \alpha_i \chi_{E_i} \in \mathcal{M}(\mathbb{R})$$

□

Definition 4.2. Let

$$\begin{aligned} S(A) &= \{\phi : A \mapsto \mathbb{R} : \phi \text{ is simple and measurable}\} \\ S^+(A) &= \{\phi \in S(A) : \phi(x) \geq 0\} \end{aligned}$$

for $A \in \mathcal{L}(\mathbb{R})$.

Proposition 4.2. *If $\phi, \psi \in S(A)$, $\alpha \in \mathbb{R}$ then $\alpha\phi$,$\phi + \psi$ and $\phi \cdot \psi$ are all in $S(A)$.*

Proof. Measurability follows from our previous results. Let

$$\begin{aligned} \phi(A) &= \{\alpha_1 < ... < \alpha_n\} \\ \psi(A) &= \{\beta_1 < ... < \beta_m\} \end{aligned}$$

then

$$\begin{aligned} \alpha\phi(A) &= \{\alpha\alpha_1 < ... < \alpha\alpha_n\} \\ (\phi + \psi)(A) &\subseteq \{\alpha_i + \beta_j : 1 \leq i \leq n, 1 \leq i \leq m\} \\ (\phi \cdot \psi)(A) &\subseteq \{\alpha_i\beta_j : 1 \leq i \leq n, 1 \leq i \leq m\} \end{aligned}$$

□

Definition 4.3. If $\phi \in S^+(A)$ for $A \in \mathcal{L}(\mathbb{R})$ with $\phi(A) = \{\alpha_1 < ... < \alpha_n\}$ and for $1 \leq i \leq n$, $E_i = \phi^{-1}(\{a_i\})$ define

$$I_A(\phi) = \sum_{i=1}^{n} \underbrace{\alpha_i}_{\in\mathbb{R}} \underbrace{\lambda(E_i)}_{\in[0,\infty]} \in [0, \infty]$$

and if $\alpha_i > 0$ and $\lambda(E_i) = \infty$ then will define $\alpha_i\lambda(E_i) = \infty$. Also if $\alpha_i = 0$ then will set $\alpha_i\lambda(E_i) = 0$.

Proposition 4.3. *Let $A \in \mathcal{L}(\mathbb{R})$ and $\phi, \psi \in S^+(A)$, $c \geq 0$ then*

(i) $I_A(c\phi) = cI_A(\phi)$

(ii) $I_A(\phi + \psi) = I_A(\phi) + I_A(\psi)$

(iii) If $\phi \leq \psi$ then $I_A(\phi) \leq I_A(\psi)$

Proof. (i) Trivial from the definition

(ii) Let $\phi(A) = \{\alpha_1 < ... < \alpha_n\}$, $E_i = \phi^{-1}(\{\alpha_i\})$ for $1 \le i \le n$ and $\psi(A) = \{\beta_1 < ... < \beta_n\}$, $F_i = \psi^{-1}(\{\beta_j\})$ for $1 \le j \le m$. Then let

$$\{\gamma_1 < ... < \gamma_{l=mn}\} = \underbrace{\{\alpha_i + \beta_j : 1 \le i \le n, 1 \le j \le m\}}_{\text{not necessarily distinct}} \supseteq (\phi + \psi)(A)$$

and observe that

$$\begin{aligned}
\phi + \psi &= \sum_{i=1}^{n} \alpha_i \chi_{E_i} + \sum_{j=1}^{m} \beta_i \chi_{F_i} \\
&= \sum_{i=1}^{n} \alpha_i \sum_{j=i}^{m} \chi_{E_i \cap F_j} + \sum_{j=1}^{m} \beta_i \sum_{i=1}^{n} \chi_{E_i \cap F_j} \\
&= \sum_{j=1}^{m} \sum_{i=1}^{n} \underbrace{(\alpha_i + \beta_j)}_{\gamma_k \text{ for some } 1 \le k \le l = mn} \chi_{E_i \cap F_j} \\
&= \sum_{k=1}^{l} \gamma_k \chi_{D_k}
\end{aligned}$$

since

$$E_i \subseteq A = \bigsqcup_{j=1}^{m} F_j \implies E_i = \bigsqcup_{j=1}^{m} F_j \cap E_i \implies \chi_{E_i} = \sum_{j=i}^{m} \chi_{E_i \cap F_j} \implies \chi_{F_j} = \sum_{j=i}^{n} \chi_{E_i \cap F_j}$$

where $\bigsqcup$ denotes a disjoint union of sets and

$$D_k = \bigsqcup_{\{i,j : \alpha_i + \beta_j = \gamma_k\}} E_i \cap F_j \implies \chi_{D_k} = \sum_{\{i,j : \alpha_i + \beta_j = \gamma_k\}} \chi_{E_i \cap F_j}$$

where some of the D_k's may be $\emptyset \implies \chi_{D_k} = 0$. Note that if $1 \le k_1 \ne k_2 \le l$ then $D_{k_1} \cap D_{k_2} = \emptyset$ and $\gamma_{k_1} \ne \gamma_{k_2}$. So the above, $\sum_{k=1}^{l} \gamma_k \chi_{D_k}$ is the standard representation of $\phi + \psi$. Therefore

$$\begin{aligned}
I_A(\phi + \psi) &= \sum_{k=1}^{l} \gamma_k \lambda(D_k) \\
&= \sum_{k=1}^{l} \gamma_k \sum_{\{i,j : \alpha_i + \beta_j = \gamma_k\}} \lambda(E_i \cap F_j) \\
&= \sum_{k=1}^{l} \sum_{\{i,j : \alpha_i + \beta_j = \gamma_k\}} \gamma_k \lambda(E_i \cap F_j) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha_i + \beta_i) \lambda(E_i \cap F_j) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} [\alpha_i \lambda(E_i \cap F_j) + \beta_i \lambda(E_i \cap F_j)] \\
&= \sum_{i=1}^{n} \alpha_i \lambda(E_i) + \sum_{j=1}^{m} \beta_i \lambda(F_j) \\
&= I_A(\phi) + I_A(\psi)
\end{aligned}$$

(iii) $\phi \le \phi$ (pointwise order) then $(\psi - \phi)(x) \ge 0$ for all $x \in A$. Clearly $\psi - \phi$ is measurable and simple. So $\psi - \phi \in S^+(A)$ and

$$I_A(\psi) = I_A(\underbrace{\phi}_{\ge 0} + \underbrace{(\phi - \psi)}_{\ge 0}) = \underbrace{I_A(\psi - \phi)}_{\ge 0} \ge I_A(\phi)$$

□

Notation 3. Let $A \in \mathcal{L}(\mathbb{R})$, $A \neq \emptyset$. We put

$$(\mathcal{M}^*)^+(A) = \{f : A \mapsto \mathbb{R} : f \text{ measurable}, f \geq 0\}$$

For $f \in (\mathcal{M}^*)^+(A)$ we define

$$S_f^+(A) = \{\phi \in S^+(A) : \phi \leq f\}$$

4.2 The Lebesgue Integral

Definition 4.4. Let $A \in \mathcal{L}(\mathbb{R}), A \neq \emptyset$ and $f \in (M^*)^+(A)$. The *Lebesgue integral* of f is defined by

$$\int_A f = \sup_{\phi \in S_f^+(A)} \underbrace{I_A(\phi)}_{\in[0,\infty]} \in [0, \infty]$$

Exercise 4.1. If $f : \mathbb{R} \mapsto \mathbb{R}^*$ is measurable, then $f\Big|_A$ is measurable as a function on $A \subseteq \mathbb{R}$.

Proposition 4.4. *Let $A \subseteq \mathcal{L}(\mathbb{R})\backslash\{\emptyset\}$ and $f, g \in (M^*)^+(A)$. Then*

(i) If $f \leq g$ then $\int_A f \leq \int_A g$

(ii) If $\emptyset \neq B \subset A$, $B \in \mathcal{L}(\mathbb{R})$ then $\int_B f = \int_A f\chi_B$

(iii) If $\phi \in S^+(A)$ then $I_A(\phi) = \int_A \phi$

Proof. (i) Suppose that $f \leq g$ on A. Then

$$S_f^+(A) \subseteq S_g^+(A) \implies \int_A f = \sup_{\phi \in S_f^+(A)} I_A(\phi) \leq \sup_{\psi \in S_g^+(A)} I_A(\psi) = \int_A g$$

(ii) Let $\phi \in S_f^+(B)$, that is $\phi : B \mapsto \mathbb{R}$ is measurable and simple on B with $\phi \leq f$. Then we define

$$\tilde{\phi} = \begin{cases} \phi & B \\ 0 & A\backslash B \end{cases}$$

where $\tilde{\phi}$ is simple and measurable (check) $\implies \tilde{\phi} \in S^+(A)$. Also, $\tilde{\phi} = \phi \leq f$ on B, $\tilde{\phi} = 0 \leq f\chi_B = 0$ on $A\backslash B$, and so $\tilde{\phi} \leq f\chi_B \implies \tilde{\phi} \in S_{f\chi_B}^+(A)$. Also note that

$$I_A(\tilde{\phi}) = I_B(\tilde{\phi}) + 0\chi_{A\backslash B} = I_B(\tilde{\phi})$$

and since $\phi \in S_f^+(B)$ was arbitrary, we get that

$$I_B(\phi) = I_A(\tilde{\phi}) \leq \int_A f\chi_B \implies \int_B f \leq \int_A f\chi_B$$

To prove the reverse, let $\psi \in S_{f\chi_B}^+(A)$. Then on B, $\psi \leq f\chi_B = f$ and since on $A\backslash B$ we have

$$0 \leq \psi \leq f\chi_B = 0 \implies \psi = 0$$

on $A\backslash B$, then $I_A(\psi) = I_B(\psi) + 0\lambda(A\backslash B) = I_B(\psi) \leq \int_B f$. Therefore,

$$\int_A f\chi_B \leq \int_B f \implies \int_A f\chi_B = \int_B f$$

(iii) First we note that

$$\phi \in S_\phi^+(A) \implies I_A(\phi) \le \int_A \phi$$

and on the other hand, for any $\psi \in S_\phi^+(A)$, $\psi \le \phi \implies I_A(\psi) \le I_A(\phi)$. Taking the the $\sup$ over ψ,

$$\int_A \phi \le I_A(\phi) \implies \int_A \phi = I_A(\phi)$$

□

Problem 4.1. If $\{f_n\}_{n=1}^\infty \subset (\mathcal{M}^*)^+(A)$ and $f_n \to f$ pointwise, then $f \in (\mathcal{M}^*)^+(A)$. Can we have $\lim_{n\to\infty} \int_A f_n = \int_A f$? The answer is unfortunately no. We do have some theorems that allow convergence.

4.3 Monotone Convergence Theorem

Theorem 4.1. *(Monotone Convergence Theorem (MCT)) Let $A \in \mathcal{L}(\mathbb{R})\backslash\{\emptyset\}$ and $\{f_n\}_{n=1}^\infty \subset (\mathcal{M}^*)^+(A)$. Suppose that*

$$0 \le f_1 \le ... \le f_n < ...$$

and

$$f = \lim_{n\to\infty} f_n$$

(pointwise). Then $f \in (\mathcal{M}^*)^+(A)$ with

$$\int_A f = \lim_{n\to\infty} \int_A f_n \in [0, \infty]$$

Lemma 4.1. *(Continuity of λ) If $A_1 \subset A_2 \subset A_3 \subset ... \in \mathcal{L}(\mathbb{R})$ then*

$$\lambda\left(\bigcup_{i=1}^\infty A_i\right) = \lim_{n\to\infty} \lambda(A_n)$$

Proof. Let $C_1 = A_1$ and $C_n = A_n\backslash A_{n-1}$ if $n \ge 2$. Then for each n

$$A_n = \bigcup_{i=1}^n A_i = \bigsqcup_{i=1}^n C_i \implies \bigcup_{i=1}^\infty A_i = \bigsqcup_{i=1}^\infty C_i$$

Then

$$\lambda\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty \lambda(C_i) = \lim_{n\to\infty} \sum_{i=1}^n \lambda(C_i) = \lim_{n\to\infty} \lambda\left(\bigsqcup_{i=1}^\infty C_i\right) = \lim_{n\to\infty} \lambda\left(\bigcup_{i=1}^n A_i\right) = \lim_{n\to\infty} \lambda(A_n)$$

□

Proof. (Of Monotone Convergence Theorem) We note first that as a limit of measurable functions, $f \in (\mathcal{M}^*)^+(A)$, and for each n

$$\int_A f_n \le \int_A f_{n+1} \le \int_A f$$

and hence $\lim_{n\to\infty} \int_A f_n \le \int_A f$. To prove the converse inequality, let $\phi \in S_f^+(A)$ and $0 < \alpha < 1$. We claim that

$$\lim_{n\to\infty} \int_A f_n \ge \alpha \int_A \phi$$

To see this, define

$$A_n = \{x \in A | f_n(x) \ge \alpha\phi(x)\}$$

and then observe

(1) If $x \in A_n$ for some n,

$$f_{n+1}(x) \geq f_n(x) \geq \alpha\phi(x) \implies f_{n+1}(x) \geq \alpha\phi(x) \implies x \in A_{n+1}$$

That is, $A_1 \subseteq A_2 \subseteq ...$

(2) For $x \in A$, $\lim_{n\to\infty} f_n(x) = f(x) \geq \phi(x) > \alpha\phi(x)$ since $\alpha < 1$. So there is N large enough such that $f_N(x) > \alpha\phi(x) \implies x \in A_N$ and hence $A = \bigcup_{n=1}^{\infty} A_n$. Consider the simple function $\alpha\phi = \{\alpha_1 < ... < \alpha_m\}$ and for each $1 \leq i \leq m$ put $E_i = (\alpha\phi)^{-1}(\{\alpha_i\})$. For each $n \in \mathbb{N}$ we have

$$\int_A n \geq \int_A f_n \underbrace{\geq}_{\text{defn of } A_n} \int_{A_n} \alpha\phi = \sum_{i=1}^{m} \alpha_i \lambda(E_i \cap A_n)$$

and taking $n \to \infty$ we have that each $\lambda(E_i \cap A_n) \to \lambda(E_i)$. Thus,

$$\lim_{n\to\infty} \int_A f_n \geq \sum_{i=1}^{m} \alpha_i \lambda(E_i) = \alpha \int_A \phi$$

Since the claim works for arbitrary $0 < \alpha < 1$, let $\alpha \to 1^-$ to get

$$\lim_{n\to\infty} \int_A f_n \geq \lim_{\alpha\to 1^-} \alpha \int_A \phi = \int_A \phi$$

and since $\phi \in S_f^+(A)$ was arbitrary, we get

$$\lim_{n\to\infty} \int_A f_n \geq \sup_{\phi\in S_f^+(A)} \int_A \phi = \int_A f \implies \lim_{n\to\infty} \int_A f_n = \int_A f$$

□

Corollary 4.1. *If* $\sup_{n\in\mathbb{N}} \int_A f_n < \infty$ *then* $\int_A f < \infty$.

Lemma 4.2. *Let* $f : A \mapsto [0,\infty]$ *where* $A \in \mathcal{L}(\mathbb{R})\setminus\{\emptyset\}$. *Then* $f \in (\mathcal{M}^*)^+(A)$ *if and only if there is a sequence* $\{\phi_n\}_{n=1}^{\infty} \subset S^+(A)$ *such that*

$$\lim_{n\to\infty} \phi_n = f$$

Moreover, we can choose $\phi_1 \leq \phi_2 \leq ... \leq f$ *pointwise.*

Proof. ($\Longleftarrow$) Pointwise, limits of measurable functions are measurable.

($\implies$) Suppose that f is measurable. Let $k \in \mathbb{N}$ be fixed. Let $F_k = f^{-1}([k,\infty]) \in \mathcal{L}(\mathbb{R})$ and $1 \leq i \leq k2^k$ with

$$E_{k,i} = f^{-1}\left(\left[\frac{i-1}{2^k}, \frac{i}{2^k}\right]\right) \in \mathcal{L}(\mathbb{R})$$

Then the $E_{k,i}$ and F_k are disjoint and

$$A = F_k \cup \bigsqcup_{i=1}^{2^k k} E_{k,i}$$

Define

$$\phi_k = k\chi_{F_k} + \sum_{i=1}^{k2^k} \frac{i-1}{2^k} \chi_{E_{k,i}}$$

where ϕ_k is simple, measurable, in $S^+(A)$ for each $k \in \mathbb{N}$. Consider $\{\phi_k\}_{k=1}^{\infty}$ where $\phi_k \stackrel{k\to\infty}{\longrightarrow} f$ pointwise and

$$\phi_1 \leq \phi_2 \leq ... \leq f$$

□

Corollary 4.2. *Let $A \in \mathcal{L}(\mathbb{R})\backslash\{\emptyset\}$. Then we have*

(i) If $f, g \in (\mathcal{M}^*)^+(A)$, $c \geq 0$ then

$$\int_A cf = c\int f \text{ and } \int_A (f+g) = \int_A f + \int_A g$$

(ii) If $\{f_n\}_{n=1}^{\infty} \subset (\mathcal{M}^*)^+(A)$ then

$$\int_A \sum_{i=1}^{\infty} f_i = \sum_{i=1}^{\infty} \int_A f_i$$

(iii) If $A_1, A_2, ... \subseteq A$ are measurable disjoint sets such that $\bigsqcup_{n=1}^{\infty} A_n = A$ and

$$\int_A f = \sum_{i=1}^{\infty} \int_{A_i} f$$

where $f \in (\mathcal{M}^*)^+(A)$.

Proof. (i) f, g are measurable by the above lemma and so there are $\{\phi_n\}_{n=1}^{\infty}, \{\psi_n\}_{n=1}^{\infty}$ such that

$$\begin{array}{ccc} \phi_1 \leq \phi_2 \leq ... \leq f & \text{and} & \lim_{n\to\infty} \phi_n = f \\ \psi_1 \leq \psi_2 \leq ... \leq f & \text{and} & \lim_{n\to\infty} \psi_n = g \end{array}$$

where ψ_n and ϕ_n are simple functions. My MCT and properties of I_A we get

$$\begin{aligned} \int_A (f+g) &= \int_A \lim_{n\to\infty} (\phi_n + \psi_n) \\ &= \lim_{n\to\infty} \int_A (\phi_n + \psi_n) \\ &= \lim_{n\to\infty} I_A(\phi_n + \psi_n) \\ &= \lim_{n\to\infty} I_A(\phi_n) + I_A(\psi_n) \\ &= \lim_{n\to\infty} I_A(\phi_n) + \lim_{n\to\infty} I_A(\psi_n) \end{aligned}$$

and using the fact that $\{\psi_n + \phi_n\}$ is also an increasing sequence, we get that

$$\begin{aligned} \int_A (f+g) &= \lim_{n\to\infty} \int_A \phi_n + \lim_{n\to\infty} \int_A \psi_n \\ &= \int_A f + \int_A g \end{aligned}$$

Similarly, using properties of I_A,

$$\int_A cf = \lim_{n\to\infty} (c\phi_n) \overset{MCT}{=} \lim_{n\to\infty} \int_A c\phi_n = c \lim_{n\to\infty} \int_A \phi_n = c\int_A f$$

(ii) Let for each n, $g_n = \sum_{i=1}^{n} f_i$ and $\int_A g_n = \sum_{i=1}^{\infty} \int_A f_i$ from (i). But $f_i \geq 0 \implies g_1 \leq g_2 \leq ...$ and $\lim_{n\to\infty} g_n = \sum_{i=1}^{\infty} f_i$. Apply MCT to $\{g_n\}_{n=1}^{\infty}$. □

(iii) Let $f \in (\mathcal{M}^*)^+(A)$ and $f_n \sum_{i=1}^{n} f\chi_{A_i}$. Then $f_1 \leq f_2 \leq ...$ and $\lim_{n\to\infty} f_n = f$. Apply part (ii) to get the result.

Notation 4. Let $f \in \mathcal{M}^*(A) = \{f : A \to \mathbb{R}^* = [-\infty, \infty] : f \text{ is measurable}\}$ where $A \in \mathcal{L}(\mathbb{R})\backslash\{\emptyset\}$. We have

$$\begin{aligned} f^+ &= \max\{f, 0\} \geq 0 \\ f^- &= \max\{-f, 0\} = -\min\{f, 0\} \geq 0 \end{aligned}$$

and $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

Definition 4.5. Let $A \in \mathcal{L}(\mathbb{R})\backslash\{\emptyset\}$. We say $f : A \mapsto \mathbb{R}^*$ is (Lebesgue) integrable if $f \in \mathcal{M}^*(A)$ and $\left|\int_A f^+ - \int_A f^-\right| < \infty$. In this case, we define the (Lebesgue) integral of f as

$$\int_A f = \int_A f^+ - \int_A f^- \in \mathbb{R}$$

We define the set of $\mathbb{R}^*-$valued integrable functions by $L^*(A)$.

Lemma 4.3. *(i) $f \in L^*(A)$ implies $\lambda(f^{-1}(\{\pm\infty\}) = 0$.*

(ii) If $f \in \mathcal{M}^*(A)$ then $\int_A |f| = 0$ if and only if

$$\lambda\left(\{x \in A | f(x) \neq 0\}\right) = \lambda\left(f^{-1}([-\infty, 0)) \cup f^{-1}((0, \infty])\right) = 0$$

Proof. (i) Let $f \in L^*(A)$. Then $f : A \mapsto \mathbb{R}^*$ and $\int_A f^+, \int_A f^- < \infty$. Define $E^+ = f^{-1}(\{+\infty\})$. Then $n\chi_{E^+} \leq f^+, \forall n \in \mathbb{N}$ and thus

$$n\lambda(E^+) = \int_A n\chi_{E^+} \leq \int_A f^+ < \infty$$

for each $n \in \mathbb{N}$. Hence $\lambda(E^+) = 0$. Similarly if $E^- = f^{-1}(\{-\infty\})$ then $\lambda(E^-)$. Therefore,

$$\lambda\left(\{x \in A | f(x) \in \{\pm\infty\}\right) = \lambda\left(E^+\right) + \lambda\left(E^-\right) = 0$$

(ii) ($\Longrightarrow$) Let $n \in \mathbb{N}$ and put $E_n = \{x \in A : |f(x)| \geq \frac{1}{n}\}$ and then

$$\frac{1}{n}\chi_{E_n} \leq |f| \implies 0 \leq \frac{1}{n}\lambda(E_n) = \int_A \frac{1}{n}\chi_{E_n} \leq \int_A |f| = 0 \implies \lambda(E_n) = 0$$

So

$$\{x \in A : f(x) > 0\} = \bigcup_{n=1}^{\infty} E_n \implies \lambda\left(\{x \in A : |f(x)| > 0\}\right) \leq \sum_{i=1}^{\infty} \lambda(E_n) = 0$$

($\Longleftarrow$) Let $\phi \in S^+_{|f|}(A)$ and write $\phi = \sum_{i=1}^n a_i\chi_{E_i}$ with disjoint and measurable E_i. If $a_i > 0$ for some i then $a_i\chi_{E_i} \leq \phi \leq |f|$ and so

$$E_i \subset \{x \in A : |f(x)| \geq a_i > 0\} \subset \underbrace{\{x \in A : f(x) \neq 0\}}_{\text{null set}} \implies \lambda(E_i) = 0$$

Then $\int_A \phi = \sum_{i=1}^n a_i\lambda(E_i) = 0$ and taking the sup over all such ϕ, $\int_A |f| = 0$. □

Definition 4.6. If $f, g \in \mathcal{M}^*(A)$ we say f and g are equal almost everywhere (a.e.) on A, written as $f = g$ a.e. (on A) if

$$\lambda\left(\{x \in A : f(x) \neq g(x)\}\right) = 0$$

Corollary 4.3. *(of Lemma (ii)) If $f, g \in \mathcal{M}^*(A)$ such that $f = g$ a.e. on A then*

$$\int_A |f - g| = 0$$

whenever $f - g$ makes sense

Notation 5. Let

$$\begin{aligned} L(A) &= \{f \in L^*(A) : f \text{ is real valued}\} \\ &= \{f : A \mapsto \mathbb{R} : f \text{ measurable and integrable}\} \end{aligned}$$

Corollary 4.4. *(of Lemma (i)) If $f \in L^*(A)$, there is $f_0 \in L(A)$ such that $f = f_0$ a.e. on A. So,*

$$\int_A |f - f_0| = 0$$

The proof is done by considering

$$f_0(x) = \begin{cases} f(x) & f(x) \in \mathbb{R} \\ 0 & \textit{otherwise} \end{cases}$$

Theorem 4.2. *If $f, g \in L(A)$ and $c \in \mathbb{R}$, then*

(i) $cf \in L(A)$ and $\int_A cf = c \int_A f$

(ii) $f + g \in L(A)$ and $\int_A (f+g) = \int_A f + \int_A g$ (*)

(iii) $|f| \in L(A)$ and $\left|\int_A f\right| \le \int_A |f|$

In fact, $f \in L(A) \iff f$ is measurable and $|f|$ is integrable.

Proof. (i) Straightforward (consider $c \ge 0$ and $c < 0$ separately)

(ii) $f, g \in L(A) \implies f + g$ is measurable. Observe that

$$\begin{aligned} (f+g)^+ &\le f^+ + g^+ \implies \int_A (f+g)^+ \le \int_A (f^+ + g^+) = \int_A f^+ + \int_A g^+ < \infty \\ (f+g)^- &\le f^- + g^- \implies \int_A (f+g)^- \le \int_A (f^- + g^-) = \int_A f^- + \int_A g^- < \infty \end{aligned}$$

Hence $f + g \in L(A)$. To prove (*) we need first to prove the claim: if $h, k, \phi, \psi \in \mathcal{L}^+(A)$ such that $h - k = \phi - \psi$ then

$$\int_A h - \int_A k = \int_A \phi - \int_A \psi$$

To prove this, note that since $h + \psi = \phi + k$, by the corollary of the MCT, we have

$$\int_A h + \int_A \psi = \int_A (h + \psi) = \int_A (\phi + k) = \int_A \phi + \int_A k$$

and the claim follows by re-ordering. To prove (*), note that

$$\begin{aligned} \underbrace{(f+g)^+}_{h} - \underbrace{(f+g)^-}_{k} = f + g &= f^+ - f^- + g^+ - g^- \\ &= \underbrace{(f^+ + g^+)}_{\phi} - \underbrace{(f^- + g^-)}_{\psi} \end{aligned}$$

and when we apply our previous claim,

$$\begin{aligned} \int_A (f+g) &= \int_A (f+g)^+ - \int_A (f+g)^- \\ &= \int_A (f^+ + g^+) + \int_A (f^- + g^-) = \int_A f^+ + \int_A g^+ - \left(\int_A f^- + \int_A g^- \right) = \int_A f + \int_A g \end{aligned}$$

(iii) Since $|f| = f^+ + f^-$ we have

$$\begin{aligned} \left|\int_A f\right| &= \left|\int_A f^+ - \int_A f^-\right| \le \left|\int_A f^+\right| + \left|\int_A f^-\right| = \int_A f^+ + \int_A f^- < \infty \\ &= \int_A \left(f^+ + f^-\right) = \int_A |f| \end{aligned}$$

so $|f|$ is integrable. Why is $|f|$ measurable? $f : A \mapsto \mathbb{R}$ is measurable and $\phi(x) = |x|$ is continuous on $\mathbb{R}$.

The last statement in the ($\Longrightarrow$) direction follows from (ii). The other direction ($\Longleftarrow$) follows from the fact that

$$\int_A f^+, \int_A f^- \leq \int_A |f|$$

□

Example 4.1. Let $E \in \mathcal{P}(\mathbb{R})\backslash\mathcal{L}(\mathbb{R})$ bounded, say $E \subset (a,b)$. Define $f = \chi_{((a,b)\backslash E)} - \chi_E$ and clearly f is not measurable. However, $|f| = \chi_{((a,b))}$ is measurable and integrable.

Lemma 4.4. *(Fatou's Lemma) If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $(\mathcal{M}^*)^+(A)$ then*

$$\int_A \liminf_{n\in\mathbb{N}} f_n \leq \liminf_{n\in\mathbb{N}} \int_A f_n$$

Proof. For each n, let $g_n = \inf_{k\geq n} f_k$ so $g_1 \leq g_2 \leq \ldots$ and $\lim_{n\to\infty} g_n = \liminf_{n\in\mathbb{N}} f_n$. So by the MCT,

$$(f) \quad \int_A \liminf_{n\in\mathbb{N}} g_n = \lim_{n\to\infty} \int_A g_n$$

For each $k \geq n$, $g_n \leq f_k$ so $\int_A g_n \leq \int_A f_n$ and hence for each n,

$$(ff) \quad \int_A g_n \leq \liminf_{k\to\infty} \int_A f_k$$

and the result follows if we put (f) and (ff) together. □

Definition 4.7. A sequence of $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}^*(A)$, $f_n : A \mapsto \mathbb{R}^*$ is said to converge to $f : A \mapsto \mathbb{R}^* \in \mathcal{M}^*(A)$ almost everywhere (on A), written $f_n \to f$ a.e. (on A) if

$$\lambda(\underbrace{\{x \in A : \lim_{n\to\infty} f_n(x) \neq f(x)\}}_{N}) = 0$$

Exercise. Why is $N \in \mathcal{L}(\mathbb{R})$?

Note 8. (1) If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{M}^*(A)$, $f = \lim_{n\to\infty} f_n$ a.e. on A then f is measurable on A. (Proof as an exercise)

(2) The MCT and Fatou's Lemma remain valid if pointwise convergence is replaced by a.e. convergence.

(3) Pointwise convergence $\Longrightarrow$ a.e. convergence but the converge may fail.

(4) If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{M}(A)$ and $f = \lim_{n\to\infty} f_n \in \mathcal{M}^*(A)$. Furthermore, suppose that f is integrable ($f \in L^*(A)$). Then we replace f by $f_0 : A \mapsto \mathbb{R}$ such that $f = f_0$ a.e. on A. Then $f_0 \in L(A)$ and $f_n \to f_0$ a.e. on A.

4.4 Lebesgue Dominated Convergence Theorem

Theorem 4.3. *(Lebesgue Dominated Convergence Theorem (LDCT)): If $\{f_n\}_{n=1}^{\infty} \subset L(A)$, $f : A \mapsto \mathbb{R}$ and $g \in L^+(A)$ are such that*

(i) $f = \lim_{n\to\infty} f_n$ pointwise a.e. on A

(ii) $|f_n| \leq g$ a.e. on A for all $n \in \mathbb{N}$ (g is called an integrable majorant for $\{f_n\}_{n\in\mathbb{N}}$)

Then $f \in L(A)$. That is, f is measurable and integrable with

$$\int_A f = \lim_{n\to\infty} \int_A f_n$$

Proof. Let

$$N = \underbrace{\{x \in A : \lim_{n\to\infty} f_n(x) \neq f(x)\}}_{\text{null set}} \cup \bigcup_{n\in\mathbb{N}} \underbrace{\{x \in A : |f_n|(x) > g(x)\}}_{\text{null sets}}$$

which is a null set since a countable union of null sets is a null set. Hence $\lambda(N) = 0$. Consider $A\backslash N$. On $A\backslash N$ all our assumptions hold pointwise. That is, $f_n \to f$ pointwise and $|f_n| \leq g$ for each n. Then f is measurable (exercise) and $|f| = \lim_{n\to\infty} |f_n| \leq g$. So

$$\int_A |f| \leq \int_A g < \infty \implies |f| \text{ is integrable}$$

Then f is integrable. Since $g + f_n \geq 0$ for each n, $g + f = \lim_{n\to\infty} g + f_n = \liminf_{n\in\mathbb{N}}(g + f_n)$ because the limit exists (recall that if a limit $\lim_{n\to\infty} a_n$ exists, then $\lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$). We have

$$\int_A g + \int_A f = \int_A g + f = \int_A \liminf_{n\to\infty}(g + f_n) \overset{\text{Fatou}}{\leq} \liminf_{n\in\mathbb{N}} \int_A (g + f_n) = \liminf_{n\in\mathbb{N}} \left(\int_A g + \int_A f_n\right)$$
$$= \underbrace{\int_A g}_{\in\mathbb{R},\geq 0} + \liminf_{n\in\mathbb{N}} \int_A f_n$$

and hence, taking away $\int_A g$ on both sides gives us

$$(*) \quad \int_A f \leq \liminf_{n\in\mathbb{N}} \int f_n$$

On the other hand $g - f_n \geq 0$ for each n and $g - f = \liminf_{n\in\mathbb{N}}(g - f_n)$ so

$$\int_A g - \int_A f = \int_A g - f = \int_A \liminf_{n\to\infty}(g - f_n) \overset{\text{Fatou}}{\leq} \liminf_{n\in\mathbb{N}} \int_A (g - f_n) = \liminf_{n\in\mathbb{N}} \left(\underbrace{\int_A g}_{\in\mathbb{R}} - \int_A f_n\right)$$
$$= \underbrace{\int_A g}_{\in\mathbb{R},\geq 0} - \limsup_{n\in\mathbb{N}} \int_A f_n$$

and hence $\limsup_{n\in\mathbb{N}} \int_A f_n \leq \int_A f \leq \liminf_{n\in\mathbb{N}} \int_A f_n$. Therefore $\int_A f = \lim_{n\to\infty} \int_A f_n$. □

Example 4.2. (Of necessary of existence of integrable majorant in LDCT) Let

$$f_n(x) = \begin{cases} n & x \in (0, \frac{1}{n}] \\ 0 & x \in (\frac{1}{n}, 1] \end{cases}, A = [0, 1]$$

Then if g is an integrable majorant of f_n we have for any m,

$$\int_A g \geq \int_{[\frac{1}{m},1]} g = \sum_{n=1}^{m-1} \int_{(\frac{1}{n+1},\frac{1}{n}]} g \geq \sum_{n=1}^{m-1} \int_{(\frac{1}{n+1},\frac{1}{n}]} n = \sum_{n=1}^{m-1} \frac{1}{n+1}$$

and taking $n \to \infty$, this is the harmonic series and g cannot be integrable. Remark that $\int_0^1 \liminf f_n = 0$ and $\lim_{n\to\infty} \int_A f_n = \lim_{n\to\infty} 1 = 1$.

5 L_p−Spaces

Let $A \in \mathcal{L}(\mathbb{R})\backslash\{\emptyset\}$ (usually $A = \mathbb{R}$ or $A = [a, b]$). Here are the cases for different values of p.

Summary 1. p=1: The space $L_1(A)$.

For $f \in L(A)$, define $\|f\|_1 = \int_A |f| \in \mathbb{R}^{\geq 0}$ and $\|\cdot\|_1 : L(A) \mapsto [0,\infty)$ is a seminorm, that is for any $f, g \in L(A)$, $c \in \mathbb{R}$,

(i) $\|cf\|_1 = |c|\|f\|_1$ (homogeneity)

(ii) $\|f+g\|_1 \leq \|f\|_1 + \|g\|_1$ (subadditivity)

The proof of this is straightforward. Note that we are lacking non-degeneracy. We say earlier that $\|f\|_1 = \int_A |f| = 0 \iff f = 0$ a.e. on A.

Remark 5.1. On $L(A)$ we define an *equivalence relation* $\sim$ as

$$f \sim g \iff f = g \text{ a.e. on } A \iff \|f-g\|_1 = 0$$

(proving that $\sim$ is an equivalence relation will be left as an exercise) We put $L_1(A) = L(A)/\sim$ and will think of $L_1(A)$ as the space of integrable functions and agree that $f = g$ in $L_1(A) \iff f = g$ a.e. on A. So $\|\cdot\|_1$ is a norm on $L_1(A)$.

Note 9. Since $\{x\}$ is a null set for $x \in A$, the value of '$f(x)$' is meaningless. That is, we lose the notion of pointwise convergence.

Fact 5.1. *(Convergence in $(L_1(A), \|\cdot\|_1)$)*

1) If $\{f_n\}_{n=1}^{\infty} \subset L_1(A)$ and $f \in L_1(A)$ such that $\lim_{n\to\infty} f_n = f$ a.e. on A and there is $g \in L_1^+(A)$ such that $|f_n| \leq g$ then we can conclude that $\lim_{n\to\infty} \|f_n - f\|_1 = 0$.

2) If $\{f_n\}_{n=1}^{\infty} \subset L_1^+(A)$ and $f \in L_1^+(A)$ such that $\lim_{n\to\infty} f_n = f$ a.e. and $f_1 \leq f_2 \leq \ldots$, then by the MCT we get

$$\lim_{n\to\infty} \|f_n - f\|_1 = 0$$

3) In general, a.e. convergence or pointwise convergence does not imply convergence w.r.t (with respect to) $\|\cdot\|_1$.

4) Can convergence w.r.t. $\|\cdot\|_1 \implies$ a.e. convergence or pointwise convergence? (Ans: No)

Proof. 1) First, $|f| = \lim_{n\to\infty} |f_n|$ a.e. $\leq g$ a.e. on A. So $|f_n - f| \leq |f_n| + |f| \leq 2g$ is also in $\mathcal{L}_1^+(A)$. Then by LDCT

$$\|f_n - f\|_1 = \int_A |f_n - f| \to \int_A 0 = 0$$

4) Let $A = [0,1]$ and consider $f_1 = \chi_{[0,\frac{1}{2}]}$, $f_2 = \chi_{[\frac{1}{2},1]}$, $f_3 = \chi_{[0,\frac{1}{3}]}$, $f_4 = \chi_{[\frac{1}{3},\frac{2}{3}]}$ $f_5 = \chi_{[\frac{2}{3},1]}$, $f_6 = \chi_{[0,\frac{1}{4}]}$, ... Let $f = 0$ on $[0,1]$. Then

$$\|f_n - f\|_1 = \int_{[0,1]} |f_n - 0| = \int_{[0,1]} f_n \to 0$$

But $\liminf_{n\in\mathbb{N}} f_n(x) = 0$ and $\limsup_{n\in\mathbb{N}} f_n(x) = 1$ so $\lim_{n\to\infty} f_n(x)$ does not exist for any $x \in [0,1]$ and f_n does not converge to f a.e. on $[0,1]$. □

5.1 $0 < p < 1$: The Spaces $L_p(A)$

Definition 5.1. Let $0 < p < \infty$ and define the conjugate to p as the number q such that $\frac{1}{p} + \frac{1}{q} = 1 \implies q = \frac{p}{1-p}$. Note that if $p = 1$ then $q = +\infty$ and if $p = +\infty$ we put $q = 1$.

Definition 5.2. Let $1 \leq p < \infty$ and $f \in \mathcal{M}(A)$. Define $\|f\|_p = \left(\int_A |f|^p\right)^{\frac{1}{p}}$.

Definition 5.3. Let $1 \leq p < \infty$ and $\sim$ denote the almost everywhere equivalence relation. Define

$$L_p(A) = \{f \in \mathcal{M}(A) : |f|^p \in L(A)\}/\sim$$

Hence we think of $L_p(A)$ as the space of p-integrable functions on A and agree that

$$f = g \text{ in } L_p(A) \iff f = g \text{ a.e. on } A$$

We want to show that $\|\cdot\|_p : L_p(A) \mapsto [0,\infty)$ is a norm on $L_p(A)$.

Lemma 5.1. *If $1 < p < \infty$ and q is the conjugate to p. Suppose that $a, b \in [0,\infty)$. Then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

and equality holds if $a^p = b^q$.

Proof. If $ab = 0$, we are done. Hence, we assume that $a, b \in (0,\infty)$. Let $0 < \alpha < 1$ and $\phi : [0,\infty) \mapsto \mathbb{R}$ by

$$\phi(t) = \alpha t - t^\alpha$$

Then $\phi'(t) = \alpha - \alpha t^{\alpha-1} = \alpha\left(1 - \frac{1}{t^{1-\alpha}}\right)$ and $\phi'(t) < 0$ for $0 < t < 1$, $\phi'(t) > 0$ for $t > 1$, $\phi'(t) = 0$ or $t = 1$. Thus by the Mean Value Theorem (MVT)

$$\alpha t - t^\alpha = \phi(t) \geq \phi(1) = \alpha - 1, \forall t \in [0,\infty)$$

and hence for all $t \geq 0$, $\alpha t - t^\alpha \geq \alpha - 1 \implies t^\alpha \leq at + (1-\alpha)$. Now set $t = \frac{a^p}{b^q}$ and get

$$\begin{aligned}
\left(\frac{a^p}{b^q}\right)^\alpha \leq \alpha\left(\frac{a^p}{b^q}\right) + (1-\alpha) \quad &\implies \quad a^{p\alpha} \leq \alpha a^p b^{q(\alpha-1)} + (1-\alpha)b^{q\alpha} \\
&\implies \quad a^{p\alpha} b^{q-q\alpha} \leq \alpha a^p + b^{q\alpha}(1-\alpha)b^{q-q\alpha} \\
&\implies \quad a^{p\alpha} b^{q(1-\alpha)} \leq \alpha a^p + b^q(1-\alpha)
\end{aligned}$$

Finally, set $\alpha = \frac{1}{p} \implies 1 - \alpha = \frac{1}{q}$ to get $ab = a^{p\cdot\frac{1}{p}} b^{q\frac{1}{q}} \leq \frac{a^p}{p} + \frac{b^q}{q}$. □

5.2 Norm Inequalities

Proposition 5.1. *(Hölder's Inequality) If $f \in L_p(A)$ and $g \in L_q(A)$ where $1 < p < \infty$ and q is conjugate to p then fg is integrable and*

$$\|fg\|_1 = \int_A |fg| \leq \|f\|_p \|g\|_q$$

(that is, $fg \in L_1(A)$). Moreover, equality holds when

$$\|g\|_q^q |f|^p = \|f\|_p^p |g|^q \text{ a.e. on } A$$

Proof. If $\|f\|_p = 0$ or $\|g\|_q = 0$ then $\leq$ follows trivially. Suppose that $\|f\|_p > 0$ and $\|g\|_q > 0$. For almost every $x \in A$ we define

$$a(x) = \frac{|f(x)|}{\|f\|_p}, b(x) = \frac{|g(x)|}{\|g\|_q}$$

and apply the previous lemma to get

$$\frac{|f(x)g(x)|}{\|f\|_p\|g\|_q} = a(x)b(x) \leq \frac{a(x)^p}{p} + \frac{b(x)^q}{q} = \frac{|f(x)|^p}{p\|f\|_p} + \frac{|b(x)|^q}{q\|g\|_q}$$

Note that f, g are measurable $\implies fg$ is measurable. So by monotonicity of $\int_A$,

$$\frac{1}{\|f\|_p\|g\|_q}\int_A |fg| \leq \int_A \left(\frac{|f(x)|^p}{p\|f\|_p} + \frac{|b(x)|^q}{q\|g\|_q}\right) = \frac{\int_A |f(x)|^p}{p\|f\|_p} + \frac{\int_A |b(x)|^q}{q\|g\|_q} < \infty$$

and $fg \in L(A) \implies fg \in L_1(A)$. Using definition of the norm,

$$\frac{1}{\|f\|_p\|g\|_q}\int_A |fg| \le \frac{1}{p}+\frac{1}{q} \quad \implies \quad \frac{1}{\|f\|_p\|g\|_q}\int_A |fg| \le 1$$
$$\implies \quad \|fg\| \le \|f\|_p\|g\|_q$$

From the statement of the Lemma, we know that equality holds when $a(x)^p = b(x)^q$ a.e. on A if and only if $\|g\|_q^q|f|^p = \|f\|_p^p|g|^q$. □

Proposition 5.2. *(Minkowski's Inequality) If $1 < p < \infty$, $f, g \in L_p(A)$ ($A \in \mathcal{L}(\mathbb{R})\backslash\{\emptyset\}$) then $f+g \in L_p(A)$ and*

$$\|f+g\|_p \le \|f\|_p + \|g\|_p$$

Moreover, the equality will hold only if there are $c_1c_2 \ge 0$, $c_1, c_2 \ne 0$ such that $c_1 f = c_2 g$ a.e. on A.

Proof. Let $f, g \in L_p(A)$. Then $|f+g|^p \le (2\{\max|f|,|g|\})^p = 2^p\left(\{\max|f|,|g|\}\right)^p \le 2^p\left(\{|f|+|g|\}\right)^p$and so

$$0 \le \int_A |f+g|^p \le \int_A 2^p(|f|^p+|g|^p) = 2^p\int_A(|f|^p+|g|^p) < \infty$$

and so $|f+g| \in L(A)$ and $|f+g| \in L_p(A)$. Next, we want to prove subadditivity. First observe hat

$$(*)\ |f+g|^p = |f+g||f+g|^{p-1} = |f||f+g|^{p-1} + |g||f+g|^{p-1}$$

and letting q denote the conjugate of p (i.e. $q = \frac{p}{1-p}$) then we see that

$$\int_A \left(|f+g|^{p-1}\right)^q = \underbrace{\int_A |f+g|^p}_{f+g\in L_p} < \infty$$

because $p = (p-1)\frac{p}{p-1} = (p-1)q$ and hence $|f+g|^{p-1}$ is q integrable and by Hölder's inequality,

$$(**)\ \int_A |f||f+g|^{p-1} \le \|f\|_p\||f+g|^{p-1}\|_q = \|f\|_p\left(\int_A |f+g|^{q(p-1)}\right)^{\frac{1}{q}} = \|f\|_p\left(\int_A |f+g|^p\right)^{\frac{1}{q}} = \|f\|_p\|f+g\|_p^{\frac{p}{q}}$$

and similarly,

$$(***)\ \int_A |g||f+g|^{p-1} \le \|g\|_p\|f+g\|_p^{\frac{p}{q}}$$

Hence from above, we get that

$$\|f+g\|_p^p = \int_A |f+g|^p \le \left(\|f\|_p+\|g\|_p\right)\|f+g\|_p^{\frac{p}{q}}$$

If $\|f+g\|_p = 0$ there is nothing to prove (it follows trivially by the definition). So assume that $\|f+g\|_p > 0$ and hence we divide both sides of the above equation by $\|f+g\|_p^{\frac{p}{q}}$ to get

$$\|f+g\|_p^{p-\frac{p}{q}} \le \|f\|_p + \|g\|_p$$

and since $p - \frac{p}{q} = p - p\left(\frac{p-1}{p}\right) = 1$ we have

$$\|f+g\|_p \le \|f\|_p + \|g\|_p$$

as desired. Finally to obtain equality, we need equality in (*), (**), and (***). In (*) $\equiv |f+g| = |f|+|g|$ we need the condition that $sgn(f) = sgn(g)$ a.e. on A. (**) uses Hölder's inequality and so requires

$$\underbrace{\frac{\|f+g\|_p^q}{\|f\|_p^p}}_{c_1}|f|^p = \underbrace{\frac{\|f+g\|_p^q}{\|g\|_p^p}}_{c_2}|g|^p = |f+g|^{(p-1)q}$$

when $\|f+g\|_p \neq 0$. Both of these conditions only hold when we have $c_1, c_2 \in [0,\infty)$ such that $c_1 + c_2 > 0$ such that $c_1 f = c_2 g$ a.e. on A. □

Corollary 5.1. *$\|\cdot\|_p$ is a norm on $L_p(A)$ where $1 < p < \infty$.*

Proof. Homogeneity: $\|cf\|_p = |c|\|f\|_p$, $c \in \mathbb{R}$ by the properties of $\int_A$

Non-degeneracy: $\|f\|_p = 0 \iff \int_A |f|^p = 0 \iff |f|^p = 0$ a.e. on $A \iff f = 0$ a.e. on $A \iff f = 0$ in $L_p(A)$.

Triangle inequality: Follows from Minkowski's inequality. □

Goal. For $A \in \mathcal{L}(\mathbb{R})$ and $\lambda(A) > 0$ we want to show that $(L_p(A), \|\cdot\|_p)$ is a Banach space (complete normed linear space) where $1 \leq p < \infty$.

5.3 Completeness

Lemma 5.2. *Let $(X, \|\cdot\|)$ be a normed vector space. Then X is complete w.r.t. $\|\cdot\| \iff$ for every sequence $\{x_n\}_{n=1}^\infty \subset X$ with $\sum_{n=1}^\infty \|x_z\| < \infty$ we have $\sum_{n=1}^\infty x_n = \lim_{n\to\infty} \sum_{k=1}^n x_n$ converges.*

Proof. ($\implies$) Suppose that X is complete and let $\{x_n\}_{n=1}^\infty \subset X$ such that $\sum_{n=1}^\infty \|x_n\| < \infty$. Put $s_n = \sum_{i=1}^n x_i$ for each $n \in \mathbb{N}$. Then $\{s_n\}_{n=1}^\infty = \{\sum_{i=1}^n x_i\}_{n=1}^\infty$. Let $n > m$ in $\mathbb{N}$ and observe that

$$\|s_n - s_m\| = \left\| \sum_{k=m+1}^n x_k \right\| \leq \sum_{k=m+1}^n \|x_k\|$$

and since $\sum_{n=1}^\infty \|x_n\|$ converges, by choosing n and m large enough, $\|s_n - s_m\|$ can be made small. Therefore $\{s_n\}$ is Cauchy in X. Since X is complete, there is $x \in X$ such that $x = \lim_{n\to\infty} s_n$. Then $x = \lim_{n\to\infty} \sum_{k=1}^n x_k = \sum_{k=1}^\infty x_k$.

($\impliedby$) Assume that every absolutely convergent series converges. To prove that X is complete, let $\{x_n\}_{n=1}^\infty$ be a Cauchy sequence. Pick $n_1 \in \mathbb{N}$ such that if $n, m \geq n_1$ then $\|x_n - x_m\| < \frac{1}{2}$, pick $n_2 \in \mathbb{N}$ such that if $n, m \geq n_2$ then $\|x_n - x_m\| < \frac{1}{2^2}$, and in general pick $n_k \in \mathbb{N}$ such that if $n, m \geq n_k$ then $\|x_n - x_m\| < \frac{1}{2^k}$. For each $k \in \mathbb{N}$, define $y_k = x_{n_{k+1}} - x_{n_k}$. Then

$$\sum_{j=1}^k \|y_j\| = \sum_{j=1}^k \|x_{n_{j+1}} - x_{n_j}\| < \sum_{j=1}^k \frac{1}{2^j} \implies \sum_{j=1}^\infty \|y_j\| \leq \sum_{j=1}^\infty \frac{1}{2^j} = 1$$

so $\sum_{j=1}^\infty y_j$ is absolutely convergent. By our assumption, $\sum_{j=1}^\infty y_j$ converges in X to say $x \in X$. Observe that

$$x_{n_{k+1}} - x_{n_1} = \sum_{j=1}^k (x_{n_{j+1}} - x_{n_j}) = \sum_{j=1}^k y_j \to x \implies x_{n_1} + x = \lim_{k\to\infty} x_{n_k}$$

So the subsequence $\{x_{n_k}\}_{k=1}^\infty$ is convergent. Since $\{x_n\}_{n=1}^\infty$ is Cauchy, $x_k \to x + x_{n_1}$ also. Hence X is complete. □

Theorem 5.1. *Let $A \in \mathcal{L}(\mathbb{R})$ and $\lambda(A) > 0$. Then $(L_p(A), \|\cdot\|_p)$ is a complete space where $1 \leq p < \infty$.*

Proof. We will apply the Lemma. Consider $\{f_n\}_{n=1}^\infty \subset L_p(A)$ with $\sum_{n=1}^\infty \|f_n\| < \infty$. We will consider each f_n as a $p-$integrable, measurable function on $A \implies$ for each n, $0 \leq \int_A |f_n|^p < \infty$. Let $g_n = \sum_{k=1}^n |f_k|$. Then $g_1 \leq g_2 \leq \ldots$ and we define $g = \lim_{n\to\infty} g_n$ (pointwise). Observe that for each n,

$$\|g_n\|_p \leq \sum_{k=1}^n \||f_k|\|_p = \sum_{k=1}^n \|f_k\|_p \leq \underbrace{\sum_{k=1}^\infty \|f_k\|_p}_{M} < \infty$$

Hence by MCT, let $n \to \infty$ and so

$$\int_A |g|^p = \int_A g^p \overset{MCT}{=} \lim_{n\to\infty} \int_A g_n^p = \lim_{n\to\infty} \|g_n\|_p^p \le M^p < \infty$$

So g^p is integrable $\implies g \in L_p(A)$ and $g^p(x) \in \mathbb{R}$ a.e. on $A \implies g(x) \in \mathbb{R}$ a.e. on A. We then observe that

$$\sum_{k=1}^{n} |f_k(x)| = g_n \le g(x)$$

for any n. Let $n \to \infty$ and see that $\sum_{k=1}^{\infty} |f_k(x)| \le g(x) < \infty$ a.e. on A. So, consider $\sum_{k=1}^{\infty} f_k(x)$ in $\mathbb{R}$. This series is absolutely convergent i $\mathbb{R}$ for a.e. $x \in A$. $\mathbb{R}$ is a complete normed vector space with $|\cdot|$. By the above Lemma, for a.e. $x \in A$, $\sum_{k=1}^{\infty} f_k(x)$ converges in $\mathbb{R}$. Define $f(x) = \sum_{k=1}^{\infty} f_k(x)$ a.e. on A which since it is a pointwise limit of measurable functions, f is measurable. Moreover,

$$|f|^p = \lim_{n\to\infty} \left|\sum_{k=1}^{n} f_k\right|^p \le \lim_{n\to\infty} \left(\sum_{k=1}^{n} |f_k|\right)^p = \lim_{n\to\infty} g_n^p = g^p$$

so $\int_A |f|^p \le \int_A g^p < \infty$ and hence f defines an element of $L_p(A)$. It remains to show that $\|f - \sum_{k=1}^{n} f_k\|_p \to 0$ as $n \to \infty$. We first observe that for each n,

$$\left|f - \sum_{k=1}^{n} f_k\right|^p \le \left(\underbrace{|f|}_{\le g} + \underbrace{\sum_{k=1}^{n} |f_k|}_{\le g}\right)^p \le 2^p g^p$$

and note that $2^p g^p$ is integrable, since g is $p-$integrable. So $2^p g^p$ is an integrable majorant for $\{|f - \sum_{k=1}^{n} f_k|\}_{n=1}^{\infty}$ a.e. on A. Therefore by LDCT,

$$\lim_{n\to\infty} \left\|f - \sum_{k=1}^{n} f_k\right\|_p^p = \lim_{n\to\infty} \int_A \left|f - \sum_{k=1}^{n} f_k\right|^p = \int_A \lim_{n\to\infty} \left|f - \sum_{k=1}^{n} f_k\right|^p = \int_A 0 = 0$$

and so $L_p(A)$ is complete by the Lemma. □

Corollary 5.2. *$A \in \mathcal{L}(\mathbb{R})$ with $\lambda(A) > 0$ and $1 \le p \le \infty$, $(L_p(A), \|\cdot\|_p)$ is a Banach space.*

5.4 The Space $L_\infty(A)$

Definition 5.4. If $f \in \mathcal{M}(A)$, let $\|f\|_\infty = \text{ess sup}_{x\in A}|f(x)| = \inf(\{c > 0, \lambda(\{x \in A : |f(x)| > c\}) = 0\})$ where we call each c an *essential upper bound* for f.

Let $L_\infty(A) = \{f \in \mathcal{M}(A) : \|f\|_\infty < \infty\}$ where $\sim$ is the a.e. equivalence relation. Hence, $L_\infty(A)$ is the space of "essentially bounded functions" on A where $f = g$ in $L_\infty(A)$ iff $f = g$ a.e. on A.

Proposition 5.3. *$\|\cdot\|_\infty$ is a norm on $L_\infty(A)$. That is, for $f, g \in L_\infty(A)$ and $c \in \mathbb{R}$ we have*

(i) $\|f\|_\infty \ge 0$ *and* $\|f\|_\infty = 0 \iff f = 0$ *in* $L_\infty(A)$

(ii) $\|cf\|_\infty = |c|\|f\|_\infty$

(iii) $\|f + g\|_\infty \le \|f\|_\infty + \|g\|_\infty$

Proof. (i) and (ii) are straightforward (Left as an exercise)

(iii) First note that $f, g \in L_\infty(A)$ implies that $f+g \in L_\infty(A)$. To prove the $\triangle \leq$ it is enough to show that the constant $\|f\|_\infty + \|g\|_\infty$ is an essential upper bound for the function $f+g$. We first claim that $\{x \in A : |(f+g)(x)| > \|f\|_\infty + \|g\|_\infty\}$ is a null set. We begin by noting that

$$\{x \in A : |f(x) > \|f\|_\infty\} = \underbrace{\bigcup_{n=1}^{\infty} \left\{ x \in A_i, |f(x)| > \underbrace{\frac{1}{n} + \|f\|_\infty}_{C_n} \right\}}_{null\ set}$$

which follows from the definition of the essential supremum (each $\frac{1}{n} + \|f\|_\infty$ is part of the set defined in ess $\sup_{x \in A}$). Hence, N is also a null set. Similarly, $\lambda(\{x \in A : |g(x) > \|g\|_\infty\}) = 0$ and so since

$$\{x \in A : |(f+g)(x)| > \|f\|_\infty + \|g\|_\infty\} \subset \{x \in A : |g(x) > \|g\|_\infty\} \cup \{x \in A : |f(x) > \|f\|_\infty\}$$

then $\lambda(\{x \in A : |(f+g)(x)| > \|f\|_\infty + \|g\|_\infty\}) = 0$ so the claim is verified. Hence by the definition of $\|f+g\|_\infty$, we have

$$\|f+g\|_\infty = \|f\|_\infty + \|g\|_\infty$$

□

Theorem 5.2. $(L_\infty(A), \|\cdot\|_\infty)$ *is complete and hence a Banach space.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}} \subset L_\infty(A)$. We will consider each f_n as an essentially bounded function. Suppose that $\sum_{n=1}^\infty \|f_n\| < \infty$. We need to show that $\sum_{n=1}^\infty f_n$ defines an element of $L_\infty(A)$. Let, for each $k \in \mathbb{N}$,

$$E_k = \{x \in A : |f_k(x)| > \|f_k\|_\infty\}$$

where E_k is a null set. Hence $E = \bigcup_{k=1}^\infty E_k$ is also a null set. So, if $x \in A \backslash E$, by absolute convergence, $|\sum_{k=1}^\infty f_k(x)| \leq \sum_{k=1}^\infty \|f_k\|_\infty < \infty$. Hence $\sum_{k=1}^\infty \|f_k\|_\infty$ is an essential upper bound for $f = \sum_{k=1}^\infty f_k$. So $f \in L_\infty(A)$ and $L_\infty(A)$ is complete. Therefore, we proved that if $1 \leq p \leq \infty$ then $L_p(A)$ is a Banach space where $A \in \mathcal{L}(\mathbb{R}), \lambda(A) > 0$. □

Remark 5.2. If $0 < p < 1$, the $\triangle \leq$ fails. (Exercise)

5.5 Containment Relations

We will consider $A = [a,b]$, $\lambda(a) < \infty$ and then $A = \mathbb{R}$ or $(0,\infty)$ where $\lambda(A) = \infty$. First, suppose that $A = [a,b]$, $a < b$, and let $1 \leq p < r < \infty$.

Theorem 5.3. $L_r([a,b]) \subset L_p([a,b])$. *Moreover, if* $f \in L_r([a,b])$ *then* $\|f\|_p \leq \|f\|_r (b-a)^{\frac{r-p}{rp}}$.

Proof. Let $f \in L_r([a,b])$. Then for $|f|^p \in L_{\frac{r}{p}}([a,b])$ we have

$$\int_{A=[a,b]} ||f|^p|^{\frac{r}{p}} = \int_{[a,b]} |f|^r < \infty$$

which is well defined since $\frac{r}{p} \geq 1$. Let q be the *conjugate* to $\frac{r}{p}$. Then $\frac{1}{q} + \frac{1}{r/p} = 1 \implies \frac{1}{q} = \frac{r-p}{r}$ .By Hölder's inequality for $(1,q)$ and $(|f|^p, \frac{r}{p})$,

$$\int_{[a,b]} |f|^p \cdot 1 \leq \||f|^p\|_{\frac{r}{p}} \|1\|_q$$

that is,

$$\|f\|_p = \left(\int_{[a,b]} |f|^p \cdot 1\right)^{\frac{1}{p}} \leq \left(\||f|^p\|_{\frac{r}{p}} \|1\|_q\right)^{\frac{1}{p}}$$

and evaluating through, we get

$$\|f\|_p \leq \left(\int_{[a,b]} \||f|^p|^{\frac{r}{p}}\right)^{\frac{p}{r}\cdot\frac{1}{p}} \left(\int_{[a,b]} 1\right)^{\frac{1}{q}\cdot\frac{1}{p}} = \left(\int_{[a,b]} |f|^r\right)^{\frac{1}{r}} (b-a)^{\frac{r-p}{pr}} = \|f\|_r (b-a)^{\frac{r-p}{pr}}$$

□

Note 10. 1) $L_\infty([a,b]) \subset L_p([a,b])$ for each $1 \leq p < \infty$. (Exercise)

2) If $\phi \in S([a,b])$ then $\lim_{p\to\infty} \|\phi\|_p = \|\phi\|_\infty$.

3) $\overline{S([a,b])} = L_\infty([a,b])$.

4) $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$ for and $f \in L_\infty([a,b])$.

Remark 5.3. $1 \leq p < r < \infty$ do we have $L_p([a,b]) \subset L_r([a,b])$? The answer is no! Let $A = [0,1]$. Then for any $1 \leq p < \infty$ consider $f(x) = \frac{1}{x^{1/r}}$ for a.e. $x \in [0,1]$. Since $\frac{p}{r} < 1$, $\int_{[0,1]} |f|^p = \underbrace{\int_0^1 x^{-p/r}\, dx}_{A3} = \frac{r}{r-p}$ while $\int_{[0,1]} |f|^r = \int_0^1 \frac{1}{x} = \infty$. So $L_p([0,1]) \not\subseteq L_r([0,1])$.

Exercise 5.1. $L_\infty([a,b]) \subset L_p([a,b])$

Remark 5.4. If $A = \mathbb{R}$ or $[0,\infty)$ we ask what happens when $1 \leq r < p < \infty$.

Is $L_p(A) \subset L_r(A)$?

No! Consider the above given function f and define $g(x) = f(x)$ on $[0,1]$ and 0 elsewhere. Then $\int_A |g|^k = \int_A |f|^k$ if $k = p, r$

Is $L_r(A) \subset L_p(A)$?

No! Consider $h(x) = \min\left\{1, \frac{1}{x^{1/p}}\right\}$ to prove that $L_r([0,\infty)) \not\subseteq L_p([0,\infty))$. Check the details (Hint: you will need Q4 of A3).

Definition 5.5. A Banach space $(X, \|\cdot\|)$ is called *separable* if there is a countable subset $\{d_n\}_{n=1}^\infty$ which is *dense* (w.r.t. $\|\cdot\|$) in X. That is, given $x \in X$, $\epsilon > 0$, there is $n \in \mathbb{N}$ such that $\|x - d_n\| < \epsilon$.

Theorem 5.4. *If $A = [a,b]$ is a bounded interval and $1 \leq p < \infty$ then $L_p([a,b])$ is separable.*

Proof. By Q6(e) of A3, $\mathcal{C}([a,b])$ is dense w.r.t. $\|\cdot\|_p$ in $L_p([a,b])$ and by Q6(d) of A3, for any $h \in \mathcal{C}([a,b])$, we have $\|h\|_p \leq c\|h\|_u$ where $c \in \mathbb{R}^{\geq 0}$ a constant which depends on λ ($[a,b]$ and p), and $\|\cdot\|_u = \|\cdot\|_\infty = \sup_{x\in[a,b]} |\cdot|$.

By the Stone-Weierstrass Theorem, $\mathbb{R}[x]$, the set of polynomials is dense in $\mathcal{C}([a,b])$ w.r.t. $\|\cdot\|_u$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\mathbb{Q}[x]$ is dense in $\mathbb{R}[x]$ w.r.t. $\|\cdot\|_u$. But $\mathbb{Q}[x]$ is a countable union of countable sets and thus $\mathbb{Q}[x]$ is countable. We write $\{d_n\}_{n=1}^\infty$. Let $f \in L_p([a,b])$ and $\epsilon > 0$. Since $\overline{C([a,b])}^{\|\cdot\|_p} = L_p([a,b])$, there is $h \in \mathcal{C}([a,b])$ such that

$$\|f - h\|_p < \frac{\epsilon}{2}$$

Let $n \in \mathbb{N}$ be such that

$$\|h - d_n\|_u < \frac{\epsilon}{2c}$$

Therefore,

$$\|f - d_n\|_p \leq \|f - h\|_p + \|h - d_n\|_p < \frac{\epsilon}{2} + c\|h - d_n\|_u < \frac{\epsilon}{2} + c\left(\frac{\epsilon}{2c}\right) < \epsilon$$

□

Theorem 5.5. *For $1 \leq p < \infty$, $L_p(\mathbb{R})$ is separable.*

Proof. The map $\psi_n : L_p([-n,n]) \mapsto L_p(\mathbb{R})$, $f :\mapsto \psi_n(f)$ is defined by

$$\psi_n(f)(x) = \begin{cases} f(x) & x \in [-n,n] \\ 0 & \text{otherwise} \end{cases}$$

a.e. on $\mathbb{R}$. Then for each n, ψ_n is an isometry. That is, for any $f \in L_p([-n,n])$ we have

$$\underbrace{\|\psi_n(f)\|_p}_{\text{p-norm in } L_p(\mathbb{R})} = \underbrace{\|f\|_p}_{\text{p-norm in } L_p([-n,n])}$$

for all $n \in \mathbb{N}$. By the previous theorem, for each $n \in \mathbb{N}$, $L_p([-n,n])$ has a countable dense subset $\{d_m^{(n)}\}_{m=1}^\infty$. Let $f \in L_p(\mathbb{R})$ and for each n, define $f_n = f \cdot \chi_{[-n,n]}$. So, $f_n \in L_p([-n,n])$ and for each n, we have

$$|f_n - f|^p \leq (|f_n| + |f|)^p \leq (|f| + |f|)^p = 2^p|f|^p$$

Consider $\{|f_n - f|^p\}_{n=1}^\infty$. By the LDCT,

$$\lim_{n\to\infty} \|f_n - f\|_p = \left(\lim_{n\to\infty} \int_{\mathbb{R}} |f_n - f|^p\right)^{\frac{1}{p}} = \left(\int_{\mathbb{R}} \lim_{n\to\infty} \underbrace{|f_n - f|^p}_{\to 0}\right)^{\frac{1}{p}} = \int_{\mathbb{R}} 0 = 0$$

So $\exists N \in \mathbb{N}$ such that

$$\|f - f_N\|_p < \frac{\epsilon}{2}$$

and for $f_N \in L_p([-N,N])$, find $d_m^{(N)} \in \{d_m^{(N)}\}_{m=1}^\infty$ such that

$$\|f_N - d_m^N\|_p < \frac{\epsilon}{2}$$

and hence by the $\triangle \leq$,

$$\|f - d_m^N\|_p < \epsilon$$

Therefore, $\{d_m^{(n)}\}_{n,m=1}^\infty$ is dense in $L_p\{\mathbb{R}\}$ w.r.t. $\|\cdot\|_p$. □

Theorem 5.6. *$L_\infty([0,1])$ is not separable.*

Proof. Recall that $\left|\{0,1\}^{\mathbb{N}}\right| = c$. Hence, there are c many sequences $\eta = \{\eta_n\}_{n=1}^\infty$, $\eta_n \in \{0,1\}$. Let $\eta \in \{0,1\}^{\mathbb{N}}$ and $\phi_\eta = \sum_{n=1}^\infty \eta_n \chi_{\left(\frac{1}{n+1},\frac{1}{n}\right]}$. This implies that $\forall \eta$, $\phi_\eta \in L_\infty([0,1])$. If $\eta \neq \eta'$ in $\{0,1\}^{\mathbb{N}}$ then

$$\|\phi_\eta - \phi_{\eta'}\|_\infty = 1$$

Since $\eta_n \chi_{\left(\frac{1}{n+1},\frac{1}{n}\right]} \neq \eta'_n \chi_{\left(\frac{1}{n+1},\frac{1}{n}\right]}$ since $\left(\frac{1}{n+1},\frac{1}{n}\right]$ is non-zero length. Consider $\left\{\mathcal{B}_{\frac{1}{2}}(\phi_\eta)\right\}_{\eta\in\{0,1\}^{\mathbb{N}}}$ disjoint open balls in $L_\infty([0,1])$. That is, suppose that there was a dense subset $\{d_n\}_{n=1}^\infty$ of $L_\infty([0,1])$ such that for each $\eta \in \{0,1\}^{\mathbb{N}}$, $\exists n(\eta) \in \mathbb{N}$ such that $\|\phi_\eta - d_{n(\eta)}\|_\infty < \frac{1}{2}$. Note that $n(\eta) \neq n(\eta')$ if $\eta \neq \eta'$ because otherwise

$$\|\phi_\eta - \phi_{\eta'}\|_\infty \leq \|\phi_\eta - d_{n(\eta)}\|_\infty + \|\phi_{\eta'} - d_{n(\eta')}\|_\infty < 1$$

since $d_{n(\eta)} = d_{n(\eta')}$. So $\eta \mapsto n(\eta)$ is an injective map and hence $\left|\{0,1\}^{\mathbb{N}}\right| \leq |\mathbb{N}|$ which is impossible. □

5.6 Functional Analytic Properties of L_p-Spaces

Recall that for $1 \leq p \leq \infty$, $L_p(A)$ is a Banach space.

Definition 5.6. Let X, Y be Banach spaces. A linear map $T : X \mapsto Y$ is *bounded* if the operator norm $\|\cdot\|$ of T, defined by

$$|||T||| = \sup\{\|T(x)\| : x \in X, \|x\| < 1\}$$

is finite ($< \infty$). If $Y = \mathbb{R}$ we call $f : X \mapsto \mathbb{R}$ a *linear functional*. Define

$$|||f||| = \|f\|_*$$

Proposition 5.4. *Let X, Y be Banach spaces and $T : X \mapsto Y$ linear. Then TFAE*

i) T is continuous

ii) T is bounded

iii) T is Lipschitz, with Lipschitz constant $|||T|||$

Aside. We say that a function $T : X \mapsto Y$ is Lipschitz if there is some constant $L > 0$ such that $\|T(x) - T(x')\| \leq L\|x - x'\|$ for $x, x' \in X$.

Proof. $i) \implies ii)$ Assume that T is continuous which implies that T is continuous at 0_X. That is $T(0_X) = 0_Y$. Consider the open ball $\mathcal{B}_1(0_Y) \subset Y$. Since T is continuous there is some $\delta > 0$ such that

$$T(\mathcal{B}_\delta(0_X)) \subset \mathcal{B}_1(0_Y)$$

Let $x \in X$ be such that $\|x\| < 1$. Then, $\|\delta x\| = \delta\|x\| < \delta$ and $\delta x \in \mathcal{B}_\delta(0_X)$. Thus,

$$T(\delta x) \in \mathcal{B}_1(0_Y) \implies \|T(\delta x)\| < 1 \implies \delta\|T(x)\| < 1 \implies \|T(x)\| < \frac{1}{\delta}$$

where the far right side is a constant. Taking the sup of all $\|x\|$ we get that

$$\|T\| = \sup\{\|T(x)\| : x \in X, \|x\| < 1\} \leq \frac{1}{\delta} < \infty$$

and hence T is bounded.

$ii) \implies iii)$ If $x \in X$, $\epsilon > 0$ and $\left\|\frac{x}{\|x\|+\epsilon}\right\| < 1$ with $\frac{x}{\|x\|+\epsilon} \in X$ then

$$T\left(\frac{x}{\|x\| + \epsilon}\right) \leq \|T\|$$

by definition. Thus,

$$\|T(x)\| \leq \|T\|(\|x\| + \epsilon) \implies \|T\|\|x\|$$

for all $x \in X$ since ϵ was arbitrary. Therefore,

$$\|T(x) - T(x')\| = \|T(x - x')\| \leq \|T\|\|x - x'\|$$

and so T is Lipschitz with $\|T\|$ as the Lipschitz constant. We also have that if $c \leq \|T\|$ then c is not a Lipschitz constant (Exercise).

$iii) \implies i)$ Suppose that T is Lipschitz. Then by PMATH 351, T is uniformly continuous and continuous. □

Theorem 5.7. *Let $A = [a, b]$ or $A = \mathbb{R}$ and $1 < p < \infty$. Let q be the conjugate of p. If $g \in L_q(A)$ then the map $\tau_g : L_p(A) \mapsto \mathbb{R}$ given by $f \mapsto \int_A fg$ is a bounded linear map (bounded functional) on $L_p(A)$ with norm $\|\tau_g\| = \|g\|_q$.*

Proof. We will need to verify:

1) τ_g is well defined ($\forall f \in L_p(A)$, fg is integrable):

If $f \in L_p(A)$, then by Hölder's inequality, $fg \in L_1(A)$ and hence τ_g is well-defined.

2) τ_g is linear:

This follows from the definition of multiplication and integration.

3) τ_g is bounded:

Again, by Hölder's inequality,

$$|\tau_g(f)| = \left|\int_A fg\right| \leq \int_A |fg| \leq \|f\|_p\|g\|_q$$

and so if $\|f\|_p < 1$ then

$$|\tau_g(f)| \leq \|f\|_p\|g\|_q < \|g\|_q$$

with

$$\||\tau_g|\| = \sup\{|\tau_g(f)| : \|f\|_p < 1\} \leq \|g\|_q$$

so τ_g is bounded.

4) $\|\tau_g\| = \|g\|_q$:

We already proved one side the of the inequality above so we want to now find $f \in L_p(A)$ such that $\|f\|_p < 1$ and $\|\tau_g(f)\| \geq g$. This can be imitated from the equality case of Hölder's inequality by letting $|f|^p = c|g|^q$ if such f and c exist. Let $f = c|g|^{q/p} \cdot \text{sgn}(g)$ where c is some constant. Then f is Borel measurable (check the measurability of $\text{sgn}(\cdot)$).

We claim that $f \in L_p(A)$. To show this, remark that

$$\begin{aligned} \|f\|_p^p &= \int_A |f|^p = \int_A \left|c|g|^{q/p} \cdot \text{sgn}(g)\right|^p \\ &= \int_A f^p|g|^q|\text{sgn}(g)| = c^p \int_A |g|^q \end{aligned}$$

and observe that $\|f\|_p = c\|g\|_q^{q/p}$. Choose

$$c = \frac{1}{\|g\|_q^{q/p} + \epsilon}$$

and note that $\|f\|_p < 1$. Hence, we get that

$$\begin{aligned} \|\tau_g\| &= \sup\{|\tau_g(f)| : f \in L_p(A), \|f\|_p < 1\} \\ &\geq \left|\tau_g\left(\frac{1}{\|g\|_q^{q/p} + \epsilon}|g|^{q/p}\text{sgn}(g)\right)\right| \\ &= \left|\int_A \frac{1}{\|g\|_q^{q/p} + \epsilon}|g|^{q/p} \underbrace{\text{sgn}(g) \cdot g}_{|g|}\right| \\ &= \left|\int_A \frac{1}{\|g\|_q^{q/p} + \epsilon}|g|^{(q/p)+1}\right| \\ &= \frac{1}{\|g\|_q^{q/p} + \epsilon}\|g\|_q^q \\ &\geq \frac{1}{\|g\|_q^{q/p}}\|g\|_q^q = \|g\|_q^{q\left(1-\frac{1}{p}\right)} = \|g\|_q \end{aligned}$$

since $\frac{q}{p} + 1 = q\left(\frac{1}{p} + \frac{1}{q}\right) = q$. Together with the inequality from 3), we get that $\|\tau_g\| = \|g\|_q$ as required. □

Fact 5.2. *Any linear functional $\tau : L_p(A) \mapsto \mathbb{R}$ is of the form $\tau_g = \tau$ for some $g \in L_p(A)$. (PMATH 454)*

[Midterm Content Ends Here]

Theorem 5.8. *Let $A \in \mathcal{L}(\mathbb{R})$ be s.t. $0 < \lambda(A) < \infty$. Let ϕ. Define $\Gamma_\phi : L_1(A) \mapsto \mathbb{R}$ by $\Gamma_\phi(f) = \int_A f \cdot \phi$. Then Γ_ϕ is a bounded linear functional with $\|\Gamma_\phi\| = \|\phi\|_\infty$.*

Proof. Linearity follows easily. To show *boundedness,* remark that $|\phi \cdot f| \leq \|\phi\|_\infty \cdot |f|$ a.e. so $\int |\phi \cdot f| \leq \|\phi\|_\infty \cdot \int |f| = \|\phi\|_\infty \cdot \|f\|_1$. This implies that

$$|\Gamma_\phi(f)| \leq \int |\phi \cdot f| \leq \|\phi\|_\infty \|f\|_1 \quad \Longrightarrow \quad \Gamma_\phi \text{ is bounded}$$

We show that $\|\Gamma_\phi\| \leq \|\phi\|_\infty$ by definition:

$$\begin{aligned} \|\Gamma_\phi\| &= \sup\{|\Gamma_\phi(f)| : \|f\|_1 \leq 1\} \\ &\leq \sup\{\|\phi\|_\infty \cdot \|f\|_1 : \|f\|_1 \leq 1\} \\ &\leq \|\phi\|_\infty \end{aligned}$$

To show the reverse inequality ($\|\Gamma_\phi\| \geq \|\phi\|_\infty$) let $\epsilon > 0$. We'll find f_ϵ such that $|\Gamma_\phi(f_\epsilon)| \geq \|\phi\|_\infty - \epsilon$. Let

$$A_\epsilon = \{x \in A : \|\phi\|_\infty - \epsilon \leq |\phi(x)|\}$$

and by definition of $\|\phi\|_\infty$ we have $0 < \lambda(A_\epsilon) \leq \lambda(A)$ since $\|\phi\|_\infty - \epsilon \leq \|\phi\|_\infty$. Define

$$f_\epsilon = \frac{1}{\lambda(A_\epsilon)} \cdot \chi_{A_\epsilon} \cdot \text{sgn}(\phi)$$

and check that $\|f_\epsilon\| \leq 1$:

$$\|f_\epsilon\|_1 = \int_A \left| \frac{1}{\lambda(A_\epsilon)} \cdot \chi_{A_\epsilon} \cdot \text{sgn}(\phi) \right| = \frac{1}{\lambda(A_\epsilon)} \int_A \chi_{A_\epsilon} = \frac{1}{\lambda(A_\epsilon)} \cdot \lambda(A_\epsilon) = 1$$

Since $\|f_\epsilon\| \leq 1$, we find that

$$\begin{aligned} \|\Gamma_\phi\| \geq |\Gamma_\phi(f_\epsilon)| &= \left| \int_A \phi \cdot \frac{1}{\lambda(A_\epsilon)} \cdot \chi_{A_\epsilon} \cdot \text{sgn}(\phi) \right| \\ &= \left| \int_A |\phi| \cdot \frac{1}{\lambda(A_\epsilon)} \cdot \chi_{A_\epsilon} \right| = \frac{1}{\lambda(A_\epsilon)} \int_A |\phi| \cdot \chi_{A_\epsilon} \\ &\geq \frac{1}{\lambda(A_\epsilon)} \int_A (\|\phi\|_\infty - \epsilon) \cdot \chi_{A_\epsilon} \\ &= \left(\frac{1}{\lambda(A_\epsilon)} \int_A \|\phi\|_\infty \right) - \epsilon = \|\phi\|_\infty - \epsilon \end{aligned}$$

because $|\phi| \cdot \chi_{A_\epsilon} \geq (\|\phi\|_\infty - \epsilon) \cdot \chi_{A_\epsilon}$. So thus $\|\Gamma_\phi\| \geq \|\phi\|_\infty - \epsilon$ and letting $\epsilon \to 0$ we find that $\|\Gamma_\phi\| \geq \|\phi\|_\infty$ and hence

$$\|\Gamma_\phi\| = \|\phi\|_\infty$$

□

Theorem 5.9. *Let $1 \leq p < \infty$ and $A \in \mathcal{L}(\mathbb{R})$ with $\lambda(A) < \infty$. Let $\phi \in L_\infty(A)$. Define $M_\phi : L_p(A) \mapsto L_p(A)$ by $f \mapsto \phi \cdot f$. Then M_ϕ is a linear operator with $\|M_\phi\| = \|\phi\|_\infty$.*

Proof. (Exercise) □

Theorem 5.10. *Let $a < b$ in $\mathbb{R}$. Then,*

(a) If $f \in L_1([a,b])$ then the functional $\Gamma_f : L_\infty([a,b]) \mapsto \mathbb{R}$ given by $\Gamma_f(\phi) = \int_{[a,b]} f \cdot \phi$ is linear and bounded with $\|\Gamma_f\| = \|f\|_1$.

(b) Furthermore we consider $\Gamma_f : \mathcal{C}([a,b]) \mapsto \mathbb{R}$. Then

$$\|\Gamma_f\| = \sup\{|\Gamma_f(h)| : h \in \mathcal{C}([a,b]), \|h\|_\infty \leq 1\} = \|f\|_1$$

Proof. (a) We start with boundedness and one half of the two inequalities and then move on to the second inequality.

$\|\Gamma_f\| \leq \|f\|_1$: By definition,

$$\|\Gamma_f\| = \sup\left\{\left|\int_{[a,b]} f\cdot\phi\right| : \|\phi\|_\infty \leq 1\right\} \leq \sup\left\{|\|\phi\|_\infty\|f\|_1| : \|\phi\|_\infty \leq 1\right\} \leq \|f_1\|$$

$\|\Gamma_f\| \geq \|f\|_1$: Consider $\phi = \text{sgn}(f)$. Then since $\|\phi\|_\infty \leq 1$ we have

$$\|\Gamma_f\| \geq \left|\int_A f\cdot \text{sgn}(f)\right| = \|f\|_1 \implies \|\Gamma_f\| \geq \|f\|_1$$

Aside. From Assignment 3 Question 6, $\exists\{h_n\} \subset \mathcal{C}([a,b])$, such that $\|h_n\| \leq 1$, $\lim_{n\to\infty} h_n = \text{sgn}(f)$ a.e. on $[a,b]$ and $h_n \cdot f \to |f|$ a.e.

(b) Let's show $\int h_n f \to \int |f|$. To do this, note that $|h_n f| \leq |f|$ a.e. and since $f \in L^1([a,b])$ by the LDCT, $\lim_{n\to\infty}\int h_n f \to \int |f|$. Returning to the problem,

$$\|\Gamma_f\| \geq \sup_n \left|\int_{[a,b]} f\cdot h_n\right| \geq \lim_{n\to\infty}\left|\int_{[a,b]} f\cdot h_n\right| = \|f\|_1$$

and $\|\Gamma_f\| \geq \|f\|_1$. The reverse inequality is left as an exercise. □

6 Fourier Analysis

Definition 6.1. A function on $A \in \mathcal{L}(\mathbb{R})$, $f : A \mapsto \mathbb{C}$ is said to be measurable if $\Im(f), \Re(f) : A \mapsto \mathbb{R}$ are both measurable. Furthermore, we say $f : A \mapsto \mathbb{C}$ is integrable if both $\Re(f)$ and $\Im(f)$ are integrable. In this case, we define

$$\int_A f = \int_A \Re(f) + i\int_A \Im(f)$$

Fact 6.1. *1) Let $A \in \mathcal{L}(\mathbb{R})$. Then*

$$\mathcal{M}_{\mathbb{C}}(A) = \{f : A \mapsto \mathbb{C} : f \textit{ measurable}\} \supset \mathcal{M}(A)$$

is an algebra of functions w.r.t. pointwise operations.

2) MCT and Fatou's Lemma require the order structure of $\mathbb{R}$ and hence they are theorems about $\mathbb{R}-$valued functions. Still they may be applied to real and imaginary parts of $\mathbb{C}-$valued functions.

3) LDCT works for $\mathbb{C}-$valued functions but we need a proof without Fatou's Lemma (Exercise) [i.e. $f_n \mapsto f$ a.e. on A and $\underbrace{|f_n|}_{\mathbb{C}-\text{modulus}} \leq g$ a.e. on A, $g \in L(A)$ then $\int_A f_n \to \int_A f$)

Remark 6.1. Furthermore, Hölder's and Minkwoski's Theorems are valid for $\mathbb{C}-$valued functions. To see this, consider $A = [a,b]$ a compact interval in $\mathbb{R}$ ($a < b$). Define

$$\mathcal{C}([a,b]) = \{f : [a,b] \mapsto \mathbb{C} : f \text{ is cts}\}$$

equipped with the uniform/infinity norm. For $1 \leq p < \infty$, define

$$L_p([a,b]) = \{f : [a,b] \mapsto \mathbb{C} : f \text{ is measurable and } |f|^p \text{ is integrable}\}/\sim$$

$$L_\infty([a,b]) = \{f : [a,b] \mapsto \mathbb{C} : f \text{ is measurable and } |f| \text{ is essentially boune}\}/\sim$$

equipped with the $\|\cdot\|_p$ norm for $1 \leq p \leq \infty$.

Definition 6.2. A function $f : \mathbb{R} \mapsto \mathbb{C}$ is called $\theta-$periodic ($\theta \in \mathbb{R}$) if

$$f(t+\theta) = f(t), \text{ a.e. for } t \in \mathbb{R}$$

We make the following remarks with regards to this definition.

- Notice that if we define $e^n : \mathbb{R} \mapsto \mathbb{T}$ by $t \mapsto e^{i(nt)}$ with $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ then for each $n \in \mathbb{N}$, e^n is 2π periodic.
- If $f : \mathbb{R} \mapsto \mathbb{C}$ is 2π periodic, then so are $\Re(f)$ and $\Im(f)$
- Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Then the map $\mathbb{R} \mapsto \mathbb{T}$ defined by $t \mapsto e^{it}$ carries $\mathbb{R}$ onto $\mathbb{T}$. So we let

$$\begin{aligned} \mathcal{C}(\mathbb{T}) &= \{f : \mathbb{R} \mapsto \mathbb{C} : f \text{ is cts and } 2\pi\text{periodic}\} \\ &\approxeq \{f \in \mathcal{C}([-\pi,\pi]) : f(-\pi) = f(\pi)\} \end{aligned}$$

  and for $1 \le p \le \infty$,

$$L_p(\mathbb{T}) = \left\{ f : \mathbb{R} \mapsto \mathbb{C} : f \text{is } 2\pi\text{periodic and } f\Big|_{[-\pi,\pi]} \in L_p([-\pi,\pi]) \right\}$$

- Note that $f \in L_p(\mathbb{T}) \nRightarrow f$ is integrable on $\mathbb{R}$ with $f\Big|_{[-\pi,\pi]} \in L_p([-\pi,\pi])$ meaning $\int_{[-\pi,\pi]} |f|^p < \infty$. In fact, $\int_{\mathbb{R}} |f|^p$ is ∞ if $f \neq 0$ as an element of L_p.
- If $1 \le p < \infty$ we equip $L_p(\mathbb{T})$ with the norm

$$\|f\|_p = \left(\frac{1}{2\pi} \int_{[-\pi,\pi]} |f|^p \right)^{1/p}$$

- If $p = \infty$ we equip $L_\infty(\mathbb{T})$ with $\|f\|_\infty = \text{ess sup}_{t\in[-\pi,\pi]} |f(t)|$. Note that

$$L_1(\mathbb{T}) \supset L_p(\mathbb{T}) \supset L_\infty(\mathbb{T}) \supset \mathcal{C}(\mathbb{T}), 1 < p < \infty$$

Problem 6.1. Given a 2π periodic function $f \in L(\mathbb{T})$ we want to represent this function as a Fourier series. That is, we want to find $\{c_n\}_{n\in\mathbb{Z}}$ such that

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

for a.e. $t \in [-\pi,\pi]$. If we allow interchanging of the sum and the integral (ignoring questions of convergence) we observe that for any $k \in \mathbb{Z}$,

$$\underbrace{\int_{[-\pi,\pi]}}_{\text{Lebesgue Integral}} f(t)e^{-ikt}dt = \sum_{n=-\infty}^{\infty} \int_{[-\pi,\pi]} e^{int}e^{-ikt}dt = \sum_{n=-\infty}^{\infty} \int_{[-\pi,\pi]} \underbrace{e^{i(n-k)t}}_{\text{cts fn}} dt$$

By Assignment 3, Question 3, Riemann integrals imply that

$$\int_{[-\pi,\pi]} e^{i(n-k)t}dt = \int_{[-\pi,\pi]} \cos((n-k)t)dt + i\int_{[-\pi,\pi]} \sin((n-k)t)dt = \begin{cases} 2\pi & n = k \\ 0 & n \neq k \end{cases}$$

Therefore, $\int_{[-\pi,\pi]} f(t)e^{-ikt}dt = 2\pi c_k$ for any $k \in \mathbb{Z}$.

Definition 6.3. If $f \in L(\mathbb{T})$ and $k \in \mathbb{Z}$ the k^{th} *Fourier coefficient* of f is given by

$$c_k(f) = \frac{1}{2\pi} \int_{[-\pi,\pi]} f(t)e^{-ikt}dt = \frac{1}{2\pi} \int_{[-\pi,\pi]} fe^{-k}$$

with the exponential function $e^k(t)$ as $t \mapsto e^{-ikt}$. Note that if $f = g$ a.e. on $[-\pi,\pi]$ then $fe^{-k} = ge^{-k}$. That is, c_k is well-defined on $L_1(\mathbb{T})$.

Goal. Let's restate our goal: Let $f \in L(\mathbb{T})$ or $L_p(\mathbb{T})$ or $C(\mathbb{T})$. Then does the following hold?

$$f = \sum_{n=-\infty}^{\infty} c_n(f)e^n = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n(f)e^n$$

Pointwise? A.e. ? In L_1? In L_p? Uniformly?

6.1 The Fourier Approximation

Definition 6.4. (Fourier Approximation) For $f \in L(\mathbb{T})$ define

$$S_n(f) = \sum_{k=-n}^{n} c_k(f)e^k, S_n(f,t) = S_n(f)(t) = \sum_{k=-n}^{n} c_k(f)e^{ikt}$$

where $S_n(f)$ is a continuous 2π periodic function.

Remark 6.2. We observe that

$$\begin{aligned} S_n(f,t) = \sum_{k=-n}^{n} c_k(f)e^{ikt} &= \sum_{k=-n}^{n}\left(\frac{1}{2\pi}\int_{[-\pi,\pi]} f(s)e^{-iks}ds\right)e^{ikt} \\ &= \frac{1}{2\pi}\int_{[-\pi,\pi]} f(s)\sum_{k=-n}^{n} e^{ik(t-s)}ds \end{aligned}$$

and let $D_n = \sum_{k=-n}^{n} e^k \implies D_n(x) = \sum_{k=-n}^{n} e^{ikx}$ which we call the *Dirichlet kernel of order* n. Then,

$$S_n(f,t) = \frac{1}{2\pi}\int_{[-\pi,\pi]} f(s)\sum_{k=-n}^{n} e^{ik(t-s)}ds = \frac{1}{2\pi}\int_{[-\pi,\pi]} f(s)D_n(t-s)ds$$

and setting $\sigma = s - t$ gives us, by translation invariance,

$$\begin{aligned} S_n(f,t) &= \frac{1}{2\pi}\int_{[-\pi-t,\pi-t]} f(\sigma+t)D_n(-\sigma)d\sigma \\ &= \frac{1}{2\pi}\int_{[-\pi,\pi]} f(\sigma+t)D_n(-\sigma)d\sigma \\ &= \frac{1}{2\pi}\int_{[-\pi,\pi]} f(t-s)D_n(s)ds, s=-\sigma \\ &:= D_n * f(t) \end{aligned}$$

which we will call the *convolution* of D_n with f. That is to study the behaviour of $S_n(f)$ we need to study the behaviour of D_n. Remark that inversion invariance follows from the symmetry of the domain.

We will first study the notion of "convolution" in a more rigourous and theoretical way.

6.2 Convolution

Definition 6.5. A *homogeneous Banach space* over $\mathbb{T}$ is a Banach space $B \subset L_1(\mathbb{T})$ which is equipped with its own norm $\|\cdot\|_B$ (Note that $(B, \|\cdot\|)$ is a Banach space) if the following conditions hold

1. $\text{span}\{e^k\}_{k=-\infty}^{\infty} \subset B$ where we denote $\text{span}\{e^k\}_{k=-\infty}^{\infty} = \text{Trig}(\mathbb{T})$ with elements called the *trigonometric polynomials*.

2. If $s \in \mathbb{R}$, $f \in B$ then $s * f \in B$ where $s * f(t) = f(t-s)$
3. $\|\cdot\|_B$ satisfies:
   (a) $\|s * f\|_B = \|f\|_B$ for all $s \in \mathbb{R}$, $f \in B$
   (b) The mapping $\mathbb{R} \mapsto (B, \|\cdot\|_B)$ given by $s \mapsto s * f$ is continuous for any $f \in B$

Example 6.1. $(\mathcal{C}(\mathbb{T}), \|\cdot\|_\infty)$ is a homogeneous Banach space over $\mathbb{T}$.

Proof. We check the conditions:

[1] Clearly Trig$(\mathbb{T}) \subset \mathcal{C}(\mathbb{T})$ and in fact $\overline{\text{Trig}(\mathbb{T})}^{\|\cdot\|_\infty} = \mathcal{C}(\mathbb{T})$ by the Stone-Weierstrass Theoerem.

[2 + 3(a)] Let $s \in \mathbb{R}, f \in \mathcal{C}(\mathbb{T})$ then $t \mapsto t - s \mapsto f(t-s)$ are also continuous mappings and so is $s * f$. Consider

$$\begin{aligned} \|s * f\|_\infty &= \max_{t\in\mathbb{R}} |s * f(t)| \\ &= \max_{t\in\mathbb{R}} |f(t-s)| \\ &= \max_{t\in\mathbb{R}} |f(t)| = \|f\|_\infty \end{aligned}$$

So 2 and 3(a) are satisfied.

[3(b)] Let $f \in \mathcal{C}(\mathbb{T})$ be fixed and take any $\epsilon > 0$. Note that if f is continuous then it is continuous on any compact interval and in particular, $[-3\pi, 3\pi]$. From the above, there is $\delta > 0$ such that $|s - s'| < \delta \implies |f(s) - f(s')| < \epsilon$. We want $|s - s'|$ small enough such that

$$\|s * f - s' * f\|_\infty < \epsilon \iff \max_{t\in\mathbb{R}} \|f(t-s) - f(t-s')\| < \epsilon$$

To do this, let $t \in \mathbb{R}$ and choose $n \in \mathbb{Z}$ large enough such that

$$t + 2\pi n \in [-\pi, \pi]$$

So if $s, s' \in [-2\pi, 2\pi]$ with $|s - s'| < \delta$ then $t + 2\pi n - s, t + 2\pi n - s' \in [-3\pi, 3\pi]$ and so

$$|(t-s) - (t-s')| = |(t + 2\pi n - s) - (t + 2\pi n - s')| < \delta$$

and by continuity,

$$\begin{aligned} |s * f(t) - s' * f(t)| &= |f(t-s) - f(t-s')| \\ &= |f(t + 2\pi n - s) - f(t + 2\pi n - s')| < \epsilon \end{aligned}$$

Since t was arbitrary,

$$\|s * f - s' * f\|_\infty < \epsilon$$

and $s \mapsto s * f$ is continuous. □

Example 6.2. For $1 \le p < \infty$, $L_p(\mathbb{T})$ is a homogeneous Banach space over T.

Proof. We have that Trig$(\mathbb{T}) \subset \mathcal{C}(\mathbb{T}) \subset L_p(\mathbb{T})$. If $s \in \mathbb{R}$ and $f \in L_p(\mathbb{T})$, then $s * f \in L_p(\mathbb{T})$ by the translation invariance of the Lebesgue integral. Again from translation invariance, $\|s * f\|_p = \|f\|_p$. Finally, if $f \in L_p(\mathbb{T})$ and $\epsilon > 0$ then we an find $h \in \mathcal{C}(\mathbb{T})$ such that

$$\|f - h\|_p < \frac{\epsilon}{3}$$

and we can find $\delta > 0$ such that if $s, s' \in \mathbb{R}$ with $|s - s'| < \delta$ then

$$\|s * h - s' * h\|_\infty < \frac{\epsilon}{3}$$

Hence we get

$$\begin{aligned} \|s * f - s' * f\|_p &= \|s * f - s * h\|_p + \|s * h - s' * h\|_p + \|s' * f - s' * h\|_p \\ &\leq \frac{\epsilon}{3} + \|s * h - s' * h\|_\infty + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

□

Example 6.3. $(L_\infty(\mathbb{T}), \|\cdot\|_\infty)$ is NOT a homogeneous Banach space over $\mathbb{T}$.

Proof. 3(b) fails. Consider $f = \sum_{n\in\mathbb{Z}} \chi_{[\pi 2n, \pi 2(n+1)]}$. Prove that if $0 < |s| < \pi$ then $\|s * f - f\|_\infty = 1$ so $s \mapsto s * f$ can not be continuous at $s = 0$ as an exercise. □

Remark 6.3. Let $B \subset L_1(\mathbb{T})$ be a homogeneous Banach space over $\mathbb{T}$. Let $h \in \mathcal{C}(\mathbb{T})$, $f \in B$. Define the convolution of h and f as

$$h * f = \frac{1}{2\pi} \int_{[-\pi,\pi]} \underbrace{h(s)}_{\in\mathbb{C}} \underbrace{(s * f)}_{t \mapsto f(t-s)} \; ds$$

which is a vector valued Riemann integral. If we put $F(s) = \frac{1}{2\pi} h(s)(s * f)$ which is a function $\mathbb{R} \mapsto L(\mathbb{T})$. In Assignment 4, we will show:

1) $f \in B \implies F(s) \in B$

2) $F(s)$ is a vector-valued continuous function on $[-\pi, \pi]$

Therefore, $h * f$ is well defined and we have for a.e. $t \in \mathbb{R}$,

$$\begin{aligned} h * f(t) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} h(s) f(t - s) \; ds \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} h(s + t) f(-s) \; ds \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} h(t - s) f(s) \; ds \end{aligned}$$

by translation invariance and inversion invariance. For any $h \in \mathcal{C}(\mathbb{T})$ we can define

$$\begin{aligned} C(h) : \quad & B \mapsto B \\ & f \mapsto h * f \end{aligned}$$

that is $C(h)_f = h * f$ for all $f \in B$.

Proposition 6.1. *If $h \in \mathcal{C}(\mathbb{T})$ and $C(h) : B \mapsto B$ denotes the convolution operator, then $C(h)$ is a bounded linear operator with*

$$|||C(h)|||_B \leq \|h\|_1$$

Proof. We have

$$\begin{aligned}
\|C(h)_f\|_B &= \left\| \frac{1}{2\pi} \int_{-\pi}^{\pi} h(s)(s * f)\, ds \right\|_B \\
&\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \| \underbrace{h(s)}_{\in \mathbb{C}} (s * f) \|_B \, ds \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} |h(s)| \underbrace{\|s * f\|_B}_{=\|f\|_B \text{ by defn of B. spc over } \mathbb{T}} ds \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} |h(s)| \|f\|_B \, ds \\
&= \|f\|_B \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} |h(s)|\, ds}_{\in L_1(\mathbb{T})} \\
&= \|f\|_B \|h\|_1 \leq \|h\|_1 \text{ if } \|f\|_B \leq 1
\end{aligned}$$

So by definition, $|||C(h)|||_B \leq \|h\|_1$. □

Note 11. We will see that if $B = L_1(\mathbb{T})$ or $\mathcal{C}(\mathbb{T})$ then $|||C(h)|||_B = \|h\|_1$, but it can be smaller in general.

Theorem 6.1. *Let* $h \in \mathcal{C}(\mathbb{T})$ *then*

(i) $|||C(h)|||_{\mathcal{C}(\mathbb{T})} = \|h\|_1$

(ii) $|||C(h)|||_{L_1(\mathbb{T})} = \|h\|_1$

Proof. We will only check the $\geq$ inequality since the reverse was proven above.

(i) Let $f \in \mathcal{C}(\mathbb{T})$. Then for $t = 0$

$$\begin{aligned}
h * f(0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} h(s) f(0 - s)\, ds \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} h(-s) f(s)\, ds \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \breve{h}(s) f(s)\, ds, \breve{h}(s) = h(-s) \\
&= \Gamma_{\hat{h}}(f)
\end{aligned}$$

by inversion invariance and where Γ is from our function analysis section, where $\breve{f}(x) = f(-x)$. Hence, we have

$$\|C(h)_f\|_\infty = \|h * f\|_\infty \geq |h * f(0)| = |\Gamma_{\breve{h}}(f)|$$

Recall that

$$\begin{aligned}
|||C(h)|||_{\mathcal{C}(\mathbb{T})} &= \sup\{\|C(h)_f\|_\infty : f \in \mathcal{C}(\mathbb{T}), \|f\|_\infty \leq 1\} \\
&\geq \sup\{|\Gamma_{\breve{h}}(f)| : f \in \mathcal{C}(\mathbb{T}), \|f\|_\infty \leq 1\} \\
&= \|\breve{h}\|_1 = \|h\|_1
\end{aligned}$$

and together with the previous proposition, we get $|||C(h)|||_{\mathcal{C}(\mathbb{T})} = \|h\|_1$.

(ii) Similarly it is enough to show that $|||C(h)|||_{L_1(\mathbb{T})} \geq \|h\|_1$. For $n \in \mathbb{N}$, define $f_n = n\pi \chi_{[-\frac{1}{n}, \frac{1}{n}]}$. Then

$$\|f_n\| = \frac{1}{2\pi} \int_{[-\pi,\pi]} n\pi \chi_{[-\frac{1}{n}, \frac{1}{n}]} = 1$$

and for a.e. $t \in \mathbb{R}$ we have

$$\begin{aligned} h * f(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(s) f_n(t-s)\, ds \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(s+t)\underbrace{f(-s)}_{=f_n(s)}\, ds \\ &= \frac{n}{2}\int_{-\frac{1}{n}}^{\frac{1}{n}} h(s+t)\, ds \end{aligned}$$

and recall that h is continuous. So for $\epsilon > 0$ choose a $\delta > 0$ such that

$$|s| < \delta \implies |h(t) - h(s+t)| < \epsilon$$

If $n \geq \frac{1}{\delta}$, Then $\sup_{s \in [-\frac{1}{n}, \frac{1}{n}]} |h(t) - h(s-t)| \leq \epsilon$. Hence, if $n \geq \frac{1}{\delta}$ then

$$\begin{aligned} \|h - h * f_n\|_1 &= \frac{1}{2\pi}\int_{[-\pi,\pi]} \left| h(t) - \frac{n}{2}\int_{-\frac{1}{n}}^{\frac{1}{n}} h(s+t)\, ds \right| dt \\ &= \frac{1}{2\pi}\int_{[-\pi,\pi]} \left| \frac{n}{2}\int_{-\frac{1}{n}}^{\frac{1}{n}} (h(t) - h(s+t))\ ds \right| dt \\ &\leq \frac{1}{2\pi}\int_{[-\pi,\pi]} \frac{n}{2}\int_{-\frac{1}{n}}^{\frac{1}{n}} |h(t) - h(s+t)|\ dsdt \\ &\leq \frac{1}{2\pi}\int_{[-\pi,\pi]} \left(\frac{n}{2}\cdot\epsilon\cdot\frac{2}{n}\right) dt \\ &= \frac{1}{2\pi}\cdot 2\pi\cdot\epsilon = \epsilon \end{aligned}$$

and $\|h - h * f\|_1 \leq \epsilon$ for all n large enough. Since ϵ was arbitrary, we conclude that

$$\lim_{n\to\infty} \|h - h * f\|_1 = 0 \implies \||C(h)\||_{L_1(\mathbb{T})} = \sup\{\|C(h)_f\|_\infty : f \in L_1(\mathbb{T}), \|f\|_\infty \leq 1\} \geq \lim_{n\to\infty} \|h * f\|_1 = \|h\|_1$$

□

6.3 The Dirichlet Kernel

Theorem 6.2. *(Properties of Dirichlet Kernel)*

The Dirichlet kernel (of order n) satisfies the following properties:

(1) D_n is real-valued, $2\pi-$periodic and even

(2) $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n = 1$

(3) For $t \in [-\pi, \pi]$, $D_n = \begin{cases} \frac{\sin\left[\left(n+\frac{1}{2}\right)t\right]}{\sin\left[\frac{1}{2}t\right]} & t \neq 0 \\ 2n+1 & t = 0 \end{cases}$

(4) Let $L_n = \|D_n\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n|$ which we call the Lebesgue constant. *Then* $\lim_{n\to\infty} L_n = \lim_{n\to\infty} \|D_n\|_1 = +\infty$

Proof. (1) $D_n(t) = \sum_{k=-n}^{n} e^{ikt}$ and so 2π periodicity is clear. Evenness and real-valuedness will follow from (3).

(2) We observe that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-n}^{n} e^{ikt}\,dt = \frac{1}{2\pi}\sum_{k=-n}^{n}\int_{-\pi}^{\pi} e^{ikt}\,dt = \frac{1}{2\pi}\cdot 2\pi = 1$$

(3) Let $t \in [-\pi, \pi]$ then

$$\begin{aligned} D_n(t)\sum_{k=-n}^{n} e^{ikt} \implies D_n(t)\left[e^{-i\frac{1}{2}t} - e^{i\frac{1}{2}t}\right] &= \left[e^{-i\left(n+\frac{1}{2}\right)t} + \ldots + e^{i\left(n+\frac{1}{2}\right)t}\right] + \left[e^{-i\left(n-\frac{1}{2}\right)t} + \ldots + e^{i\left(n-\frac{1}{2}\right)t}\right] \\ &= e^{-i\left(n+\frac{1}{2}\right)t} - e^{i\left(n+\frac{1}{2}\right)t} \end{aligned}$$

If $t \neq 0$ then

$$\begin{aligned} D_n &= \frac{e^{-i\left(n+\frac{1}{2}\right)t} - e^{i\left(n+\frac{1}{2}\right)t}}{e^{-i\frac{1}{2}t} - e^{i\frac{1}{2}t}} \\ &= \frac{\cos\left(\left(n+\frac{1}{2}\right)t\right) - i\sin\left(\left(n+\frac{1}{2}\right)t\right) - \left(\left(\cos\left(n+\frac{1}{2}\right)t\right) + \sin\left(\left(n+\frac{1}{2}\right)t\right)\right)}{\cos\left(\frac{1}{2}t\right) - i\sin\left(\frac{1}{2}t\right) - \left(\cos\left(\frac{1}{2}t\right) + \sin\left(\frac{1}{2}t\right)\right)} \\ &= \frac{-2i\sin\left(\left(n+\frac{1}{2}\right)t\right)}{-2i\sin\left(\frac{1}{2}t\right)} = \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{\sin\left(\frac{1}{2}t\right)} \end{aligned}$$

Now if $t = 0$ then $D_n(0) = \sum_{k=-n}^{n} e^{ik0} = 2n+1$.

(4) Note that $|\sin\theta| \leq |\theta|$ for $\theta \in \mathbb{R}$. Then

$$L_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}|D_n| = \frac{1}{\pi}\int_{0}^{\pi}|D_n| = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{\sin\left(\frac{1}{2}t\right)}\right| dt \geq \frac{1}{\pi}\int_{0}^{\pi}\frac{\left|\sin\left(\left(n+\frac{1}{2}\right)t\right)\right|}{\frac{1}{2}t}dt$$

since D_n is even and $\left|\sin\left(\frac{1}{2}t\right)\right| \leq \left|\frac{1}{2}t\right|$. Using

$$s = \left(n+\frac{1}{2}\right)t \implies ds = \left(n+\frac{1}{2}\right)dt \implies t = \frac{2}{2n+1}s$$

we get

$$\begin{aligned} \frac{1}{\pi}\int_{0}^{\pi}\frac{\left|\sin\left(\left(n+\frac{1}{2}\right)t\right)\right|}{\frac{1}{2}t}dt &= \frac{2}{\pi}\int_{0}^{\left(n+\frac{1}{2}\right)\pi}\frac{|\sin s|}{s/\left(n+\frac{1}{2}\right)}\cdot\left(\frac{1}{n+\frac{1}{2}}\right)ds \\ &= \frac{2}{\pi}\int_{0}^{\left(n+\frac{1}{2}\right)\pi}\underbrace{\frac{|\sin s|}{s}}_{\geq 0}ds \\ &\geq \frac{2}{\pi}\int_{0}^{n\pi}\frac{|\sin s|}{s}ds \\ &= \frac{2}{\pi}\sum_{j=1}^{n}\int_{(j-1)\pi}^{j\pi}\frac{|\sin s|}{s}ds \\ &\geq \frac{2}{\pi}\sum_{j=1}^{n}\frac{1}{j\pi}\underbrace{\int_{(j-1)\pi}^{j\pi}|\sin s|ds}_{=1} = \frac{2}{\pi^2}\sum_{j=1}^{n}\frac{1}{j} \end{aligned}$$

and in short, $L_n \geq \frac{2}{\pi^2}\sum_{j=1}^{n}\frac{1}{j}$ for each n. As $n \to \infty$, the right side converges to the harmonic series, which diverges, and so L_n must diverge. That is $L_n \to \infty$ as required. □

Corollary 6.1. *$|||C(D_n)|||_{L_1(\mathbb{T})} = |||D_n|||_1 = L_n \to \infty$ and $|||C(D_n)|||_{\mathcal{C}(\mathbb{T})} = |||D_n|||_1 = L_n \to \infty$ as $n \to \infty$. We want to use $\lim_{n\to\infty} L_n$ to show that if $f \in \mathcal{C}(\mathbb{T})$ then $S_n(f,t) \nrightarrow f$ as $n \to \infty$ in the uniform sense.*

Theorem 6.3. *(Banach -Steinhaus Theorem) Let X, Y be Banach spaces (usually $Y = X$ or $Y = \mathbb{C}$), $\mathcal{F}$ be a family of bounded linear operators from X to Y. Suppose that U is a set of second category in X (So U is not 1^{st} category, i.e. U cannot be written as a countable union of nowhere dense sets. Also note that since X is a Banach space, then any open subset of X is of second category by the Baire category theorem).*

If for each $x \in U$ we have $\sup\{\|Tx\| : T \in \mathcal{F}\} < \infty$ where $T(x) = Tx$ and T is linear, then $\sup\{|||T||| : T \in \mathcal{F}\} < \infty$.

Proof. Let for each $n \in \mathbb{N}$,

$$F_n = \{x \in U : \|Tx\| \leq n, \text{ for each } T \in \mathcal{F}\}$$

Then each F_n is closed and $U = \bigcup_{n=1}^{\infty} F_n$. Since U is not of 1^{st} category there is $n_0 \in \mathbb{N}$ such that $\text{int}(F_{n_0}) \neq \emptyset$. Hence there is $x_0 \in X$ and $r > 0$ such that

$$\mathcal{B}_r(x_0) = \{x \in X : \|x_0 - x\| < r\} \subset F_{n_0}$$

If $y \in \mathcal{B}_r(x_0)$ then $\|Ty\| \leq n_0$ for all $T \in \mathcal{F}$. Let $x \in X$ with $\|x\| \leq 1$. Then

$$x_0 + \frac{r}{2}x, x_0 - \frac{r}{2}x \in \mathcal{B}_r(x_0)$$

and

$$x = \frac{1}{r}\left[\left(x_0 + \frac{r}{2}x\right) - \left(x_0 - \frac{r}{2}x\right)\right]$$

Hence

$$Tx = \frac{1}{r}\left[T\left(x_0 + \frac{r}{2}x\right) - T\left(x_0 - \frac{r}{2}x\right)\right]$$

which by triangle inequality gives us

$$\begin{aligned} \|Tx \quad \| &\leq \frac{1}{r}\left[\left\|T\left(x_0 + \frac{r}{2}x\right)\right\| + \left\|T\left(x_0 - \frac{r}{2}x\right)\right\|\right] \\ &\leq \frac{2n_0}{r} \end{aligned}$$

If $T \in \mathcal{F}$ then

$$|||T||| \leq \frac{2n_0}{r} \implies \sup\{|||T||| : T \in \mathcal{F}\} < \infty$$

□

Corollary 6.2. *If X, Y are Banach spaces, $\{T_n\}_{n\in\mathbb{N}}$ is sequence of bounded linear maps from X to Y s.t. $\sup_{n\in\mathbb{N}} |||T_n||| = \infty$, then there is a non-empty set $U \subseteq X$ whose complement is first category s.t. $\sup_{n\in\mathbb{N}} \|T_n x\| = \infty$ for any $x \in U$.*

Proof. Suppose that $\sup_{n\in\mathbb{N}} |||T_n||| = \infty$. Consider

$$V = \left\{x \in X : \sup_{n\in\mathbb{N}} \|T_n x\| < \infty\right\}$$

Then V is of first category (if not, V is of second category and by Banach-Steinhaus, $\sup_{n\in\mathbb{N}} |||T_n||| < \infty$ which creates a contradiction). Let $U = X \backslash V$ and since X is of second category (from the Baire Category Theorem), $X \neq V \implies X\backslash V \neq \emptyset$ and $U \neq \emptyset$. □

Note 12. If $F_1, F_2, \ldots$ are sets of first category, then $\bigcup_{n=1}^{\infty} F_n$ is also first category. Hence, if $U_1, U_2, \ldots$ are sets whose complements are of first category then $\bigcap_{n=1}^{\infty} U_n$ is also of second category.

Theorem 6.4. *Consider $\{C(D_n)\}_{n\in\mathbb{N}}$. We have the following results.*

1) There is a set $U \subset L_1(\mathbb{T})$ whose complement is of first category such that $\sup_{n\in\mathbb{N}} \|S_n(f)\|_1 = \infty$ for any $f \in U$.

2) There is $U \subset \mathcal{C}(\mathbb{T})$ whose complement is of first category such that $\sup_{n\in\mathbb{N}} \|S_n(f)\|_\infty = \infty$ for $f \in U$.

Proof. 1) We know that $S_n(f) = D_n * f = C(D_n)(f)$ and $\forall n, |||C(D_n)|||_{L_1(\mathbb{T})} = \|D_n\|_1$. Hence $|||C(D_n)|||_{L_1(\mathbb{T})} \to \infty$ as $n \to \infty$. By the above corollary, the set

$$F = \left\{ f \in L_1(\mathbb{T}) : \underbrace{\sup_{n\in\mathbb{N}} \|C(D_n)(f)\|_1}_{=\sup_{n\in\mathbb{N}} \|D_n * f\|_1} < \infty \right\}$$

(when considering $\{C(D_n)\}_{n\in\mathbb{N}}$) is of first category. Since $L_1(\mathbb{T})$ is not of first category, then $U = L_1(\mathbb{T})\backslash F$ is non-empty and of second category.

2. This is similar to the above. □

In light of the above theorem, there are two ways we can proceed:

- (An idea due to Fejer) We can average te Fourier series
- (Dini's Theorem) We can look at specific functions where convergence holds

6.4 Averaging Fourier Series

Definition 6.6. If X is a vector space and $x = \{x_n\}_{n=1}^{\infty} \subseteq X$ we let the n^{th} *Cesaro mean* (average) of X be defined by

$$\sigma_n(x) = \frac{x_1 + ... + x_n}{n}$$

Proposition 6.2. *If X is a normed vector space and $x = x_n{}_{n=1}^{\infty}$ is sequence converging to $x_0 \in X$ then the sequence of Cesaro means $\{\sigma_n(X)\}_{n=1}^{\infty}$ converges to x_0 too.*

Definition 6.7. If $f \in L(\mathbb{T})$ we define

$$\sigma_n(f) = \frac{1}{n+1}\sum_{j=0}^{n} S_j(f) = \frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=-j}^{j} c_k(f)e^k$$

called the n^{th} *Cesaro mean* of f. Note that

$$\begin{aligned}\sigma_n(f) &= \frac{1}{n+1}\left(S_0(f) + ... + S_n(f)\right) \\ &= \frac{1}{n+1}\left(D_0 * f + ... + D_n * f\right) = \left(\frac{1}{n+1}\sum_{j=0}^{n} D_j\right) * f\end{aligned}$$

Thus, if we let $K_n = \frac{D_0 + ... + D_n}{n+1}$ we have $\sigma_n(f) = K_n * f$ for each $n \in \mathbb{N}$. We call each K_n the n^{th} *Ferjer Kernel*.

Theorem 6.5. *(Properties of the Fejer Kernel) The Ferjer Kernel of order n, K_n satisfies the following:*

(i) K_n is real-valued, 2π-periodic and even.

(ii) We have

$$K_n(t) = \begin{cases} \frac{1}{n+1}\left(\frac{\sin\left[\frac{1}{2}(n+1)\right]t}{\sin\left[\frac{1}{2}t\right]}\right)^2 & t \neq 0 \\ n+1 & t = 0 \end{cases}, t \in [-\pi, \pi]$$

(iii) $\|K_n\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |K_n| = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n = 1$

(iv) If $0 < |t| \leq \pi$ then $0 \leq K_n(t) \leq \frac{\pi^2}{(n+1)t^2}$

Proof. (i) Follows from the properties of the Dirichlet Kernel.

(ii) First, we observe that

$$\begin{aligned} K_n(t) &= \frac{1}{n+1}\sum_{j=0}^{n} D_j(t) = \frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=-k}^{j} e^{ikt} \\ &= \frac{1}{n+1}\left[e^{-int} + 2e^{-i(n-1)t} + ... + ne^{-it} + (n+1) + ne^{it} + ... + e^{int}\right] \end{aligned}$$

Thus, if we multiply both sides by $(n+1)(e^{it} - 2 + e^{it})$ we get

$$(n+1)K_n(t)(e^{-it} - 2 + e^{it}) = e^{-i(n+1)t} - 2 + e^{i(n+1)t}$$

and if $t \in [-\pi, \pi]\backslash\{0\}$ then

$$K_n(t) = \frac{1}{n+1}\cdot\frac{e^{-i(n+1)t} - 2 + e^{i(n+1)t}}{e^{-it} - 2 + e^{it}} = \frac{1}{n+1}\left(\frac{\sin\left[\frac{1}{2}(n+1)\right]t}{\sin\left[\frac{1}{2}t\right]}\right)^2$$

while

$$K_n(0) = \frac{1}{n+1}\sum_{j=0}^{n} D_j(0) = \frac{1}{n+1}\sum_{j=0}^{n}(2j+1) = n+1$$

(iii) To see this, note that since $K_n \geq 0$ on $[-\pi, \pi]$ hence

$$\begin{aligned} \|K_n\|_1 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n = \frac{1}{2\pi(n+1)}\sum_{j=0}^{n}\int_{-\pi}^{\pi} D_j \\ &= \frac{1}{2\pi}\frac{1}{n+1}(n+1)2\pi = 1 \end{aligned}$$

(iv) If $0 < \theta \leq \frac{\pi}{2}$ then $\frac{2\theta}{\pi} \leq \sin\theta$. Thus, for $0 < t < \pi$ we have

$$\frac{1}{\sin\frac{1}{2}t} \leq \frac{1}{t/\pi} = \frac{\pi}{t}$$

Therefore, $\theta \leq K_n(t) = \frac{1}{n+1}\left(\frac{\sin\left[\frac{1}{2}(n+1)t\right]}{\sin\frac{1}{2}t}\right)^2 \leq \frac{1}{(n+1)\left[\sin\frac{1}{2}t\right]^2} \leq \frac{1}{(n+1)\left(\frac{t}{\pi}\right)^2} = \frac{1}{n+1}\left(\frac{\pi}{t}\right)^2.$ □

Definition 6.8. A *summability kernel* is a sequence $\{k_n\}_{n=1}^{\infty}$ of 2π periodic bounded and piecewise continuous functions such that

(i) $\frac{1}{2\pi}\int_{-\pi}^{\pi} k_n = 1$

(ii) $\sup_{n\in\mathbb{N}}\|k_n\|_1 < \infty$

(iii) For any $0 < \delta \leq \pi$ we have $\lim_{n\to\infty}\left(\int_{-\pi}^{-\delta}|k_n| + \int_{\delta}^{\pi}|k_n|\right) = 0$ (as $n \to \infty$, the mass k_n concentrates at 0).

Example 6.4. The Fejer Kernel $\{k_n\}_{n=1}^{\infty}$ is a summability kernel.

Proof. (i) and (ii) follow from the previous theorem. We need to prove (iii). For $0 < \delta \leq \pi$ fixed then

$$0 \leq \int_{\delta}^{\pi}|K_n(t)| \leq \int_{\delta}^{\pi}\frac{\pi^2}{(n+1)t^2}dt = \frac{\pi^2}{n+1}\left(\frac{1}{\delta} - \frac{1}{\pi}\right)$$

By symmetry, we also get $\int_{-\pi}^{-\delta}|K_n| \to 0$. □

Example 6.5. The Diriclet Kernel $\{D_n\}_{n=1}^{\infty}$ is a not a summability kernel since (ii) fails. That is, $L_n = \|D_n\|_1 \to \infty$.

Example 6.6. (a) The sequence $\{k_n\}_{n=1}^{\infty} = \left\{n\pi\chi_{[-\frac{1}{n},\frac{1}{n}]}\right\}_{n=1}^{\infty}$ on $[-\pi,\pi]$, extend 2π periodically to $\mathbb{R}$. Then $\{k_n\}$ is a summability kernel.

(b) Similarly, $\{k_n\}_{n=1}^{\infty} = \left\{2n\pi\chi_{[0,\frac{1}{n}]}\right\}$, extend 2π periodically, is a measurability kernel

Proof. Exercise. □

Theorem 6.6. *(Abstract Summability Kernel Theorem (ASKT)) Let B be a homogeneous Banach space over $\mathbb{T}$. If $\{k_n\}_{n=1}^{\infty}$ is a summability kernel, then*

$$\lim_{n\to\infty} \|k_n * f - f\|_B = 0$$

for any $f \in B$.

Proof. Let $f \in B$ be fixed. Suppose that $\|f\|_B > 0$ and consider

$$k_n * f(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(s)\underbrace{f(t-s)}_{s*f(t)}\,ds$$

Let $F : \mathbb{R} \mapsto B$ given by $S \mapsto F(s) = s * f$. Since B is a homogeneous Banach space then F is continuous. Since f is 2π periodic then F is 2π periodic and

$$\|F(s)\|_B = \|s * f\|_B = \|f\|_B$$

for all $s \in \mathbb{R}$. Finally, $F(0) = 0 * f = f$ and so

$$\begin{aligned} k_n * f - f &= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(s)F(s)ds\right) - F(0)\left(\underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(s)ds}_{=1}\right) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(s)\left[F(s) - F(0)\right]ds \end{aligned}$$

which is a vector valued Riemann integral. So we have

$$\begin{aligned} \|k_n * f - f\|_B &= \left\|\frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(s)\left[F(s) - F(0)\right]ds\right\| \\ &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |k_n(s)|\|F(s) - F(0)\|_B ds \end{aligned}$$

from a result from assignment 1 since F is continuous. Let $\epsilon > 0$ be given. Put $\sup_{n\in\mathbb{N}} \|k_n\|_1 = M > 0$ and find $\delta > 0$ (by the continuity of F at $s = 0$) such that if $|s| < \delta$ then $\|F(s) - F(0)\|_B < \frac{\epsilon}{M}$. Next, we choose N large enough so that

$$\frac{1}{2\pi}\int_{[-\pi,-\delta]\cup[\delta,\pi]} |k_n| < \frac{\epsilon}{4\|f\|_B}, \text{ for any } n \ge N$$

by the summability kernel definition in (iii). Then for any $n \ge N$ we get that

$$\begin{aligned} \|k_n * f - f\|_B &\le \frac{1}{2\pi}\int_{[-\pi,-\delta]\cup[\delta,\pi]} |k_n(s)|\|F(s) - F(0)\|_B ds + \frac{1}{2\pi}\int_{[-\delta,\delta]} |k_n(s)|\|F(s) - F(0)\|_B ds \\ &\le 2\|f\|_B \frac{1}{2\pi}\int_{[-\pi,-\delta]\cup[\delta,\pi]} |k_n(s)|ds + \frac{\epsilon}{2M}\underbrace{\frac{1}{2\pi}\int_{[-\delta,\delta]} |k_n(s)|ds}_{\le M} \\ &\le 2\|f\|_B \frac{\epsilon}{4\|f\|_B} + \frac{\epsilon}{2} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

since

$$\|F(s) - F(0)\|_B \leq \|F(s)\|_B + \|F(0)\|_B = \|s * f\|_B + \|f\|_B = 2\|f\|_B$$

In short, if $n \geq N$ and $\|k_n * f - f\| < \epsilon$. □

Corollary 6.3. *(1) For $f \in \mathcal{C}(\mathbb{T})$ we have*

$$\lim_{n\to\infty} \|\sigma_n(f) - f\|_\infty = 0$$

That is $\sigma_n(f) \to f$ uniformly as $n \to \infty$.

(2) If $1 \leq p < \infty$, for $f \in L_p(\mathbb{T})$ we have

$$\lim_{n\to\infty} \|\sigma_n(f) - f\|_p = 0$$

Fact 6.2. *Note that $f = g$ a.e. on $[-\pi, \pi] \implies c_n(f) = c_n(g)$ for all $n \in \mathbb{Z}$ in $L(\mathbb{T})$.*

Corollary 6.4. *Suppose that $f, g \in L(\mathbb{T})$ and $c_k(f) = c_k(g)$ for each $k \in \mathbb{Z}$. then $f = g$ a.e. on $[-\pi, \pi]$.*

Proof. We have

$$\sigma_n(f,t) = \frac{1}{n+1}\sum_{j=0}^{n} S_j(f,t) = \frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=-j}^{j} c_k(f)e^{ikt} = \sigma_n(g,t)$$

for all $n \in \mathbb{N} \cup \{0\}$. We then have

$$\|f - g\|_1 = \|f - \sigma_n(f) + \sigma_n(g) - g\| \leq \|f - \sigma_n(f)\| + \|\sigma_n(g) - g\| \to 0$$

as $n \to \infty$ by our previous theorem. Hence $\|f - g\|_1 = 0 \implies f - g = 0$ a.e. on $[-\pi, \pi] \implies f = g$ a.e. on $[-\pi, \pi]$. □

Problem 6.2. If $f \in L(\mathbb{T})$ and $t \in \mathbb{R}$ (or $t \in [-\pi, \pi]$) then do we have $\sigma_n(f,t) \to f(t)$ pointwise as $n \to \infty$?

Definition 6.9. Consider $f \in L(\mathbb{T})$ (or $f \in L_1(\mathbb{T}) = L(\mathbb{T})/\infty$) and $s \in \mathbb{R}$ (usually $s \in [-\pi, \pi]$). We let

$$w_f(s) = \frac{1}{2}\lim_{h\to 0^+} [f(s+h) + f(s-h)]$$

This limit may fail to exist (note that the limit can be $+\infty$ or $-\infty$). If $w_f(s)$ exists, thorugh, we call it the mean value of f at s.

Note 13. If $s \in \mathbb{R}$ is a point of continuity for $f \in L(\mathbb{T})$ then clearly $w_f(s)$ exists and $w_f(s) = f(s)$.

Theorem 6.7. *(Fejer's Theorem) There are two parts:*

(1) If $f \in L(\mathbb{T})$ and $x \in [-\pi, \pi]$ such that $w_f(x)$ exists, then $\lim_{n\to\infty} \sigma_n(f,x) = w_f(x)$. In particular, $\lim_{n\to\infty} \sigma_n(f,x) = f(x)$ if f is continuous at x.

(2) If I is an open interval on which f is continuous then for any closed and bounded subinterval $J$of I we have

$$\lim_{n\to\infty}\sup_{t\in J} |\sigma_n(f,t) - f(t)| = 0$$

that is $\lim_{n\to\infty} \sigma_n(f,t) = f(t)$ uniformly on J.

Proof. Note that $\sigma_n(f,x) = K_n * f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{K_n(s)}_{\text{Fejer kernel}} f(x-s)ds$. Recall that

i) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n = 1$

ii) Each K_n is even and non-negative

iii) If $0 < |t| \leq \pi$, $K_n(t) \leq \frac{\pi^2}{(n+1)t^2}$ and $\delta < 0$ then $\sup_{t\in[\delta,\pi]} K_n(t) \leq \frac{\pi^2}{\delta(n+1)}$

Now suppose that $w_f(x)$ is finite (the cases $\pm\infty$ are exercises). Let $\epsilon > 0$ be given. Then $\exists\delta > 0$ such that for any $0 < |s| \leq \delta$ we have

$$\left| w_f(x) - \frac{1}{2}\left(f(x-s) + f(x+s)\right) \right| < \epsilon$$

and so

$$\begin{aligned}
|\sigma_n(f,x) - w_f(x)| &= \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(s) f(x-s) ds - w_f(x) \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n}_{=1} \right| \\
&= \frac{1}{2\pi}\left| \int_{-\pi}^{\pi} K_n(s)\left[f(x-s) - w_f(x)\right] ds \right| \\
&\leq \frac{1}{2\pi}\left| \int_{-\delta}^{\delta} K_n(s)\left[f(x-s) - w_f(x)\right] ds \right| + \frac{1}{2\pi}\left| \left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} \right) K_n(s)\left[f(x-s) - w_f(x)\right] ds \right|
\end{aligned}$$

and for each n we have

$$\int_{-\delta}^{\delta} K_n(s)\left[f(x-s) - w_f(x)\right] ds = \int_{-\delta}^{\delta} \underbrace{K_n(-s)}_{=K_n(s)} \left[f(x+s) - w_f(x)\right] ds = \int_{-\delta}^{\delta} K_n(s)\left[f(x+s) - w_f(x)\right] ds$$

by translation invariance. Consider

$$\begin{aligned}
A &= \frac{1}{2\pi}\int_{-\delta}^{\delta} K_n(s)\left[f(x-s) - w_f(x)\right] ds = \frac{A}{2} + \frac{A}{2} \\
&= \frac{1}{4\pi}\int_{-\delta}^{\delta} K_n(s)\left[f(x-s) - w_f(x)\right] ds + \frac{1}{4\pi}\int_{-\delta}^{\delta} K_n(s)\left[f(x+s) - w_f(x)\right] ds \\
&= \frac{1}{2\pi}\int_{-\delta}^{\delta} K_n(s)\left[\frac{1}{2}\left(f(x-s) + f(x+s) - w_f(x)\right)\right] ds
\end{aligned}$$

by our choice of $\delta > 0$ then

$$\begin{aligned}
\frac{1}{2\pi}\left| \int_{-\delta}^{\delta} K_n(s)\left[f(x-s) - w_f(x)\right] ds \right| &= \frac{1}{2\pi}\left| \int_{-\delta}^{\delta} K_n(s)\left[\frac{1}{2}\left(f(x-s) + f(x+s) - w_f(x)\right)\right] ds \right| \\
&\leq \frac{1}{2\pi}\int_{-\delta}^{\delta} K_n(s) \underbrace{\left| \frac{1}{2}\left(f(x-s) + f(x+s) - w_f(x)\right) \right|}_{\leq\epsilon} ds \\
&\leq \frac{\epsilon}{2\pi}\int_{-\delta}^{\delta} K_n(s) ds \leq \frac{\epsilon}{2\pi}\int_{-\pi}^{\pi} K_n(s) ds = \epsilon
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{1}{2\pi}\left|\left(\int_{-\pi}^{-\delta}+\int_{\delta}^{\pi}\right)K_n(s)\left[f(x-s)-w_f(x)\right]ds\right| &\leq \frac{1}{2\pi}\left(\int_{-\pi}^{-\delta}+\int_{\delta}^{\pi}\right)\underbrace{K_n(s)}_{\leq\frac{\pi^2}{\delta^2(n+1)}}\ |f(x-s)-w_f(x)|\,ds \\
&\leq \frac{1}{2\pi}\cdot\frac{\pi^2}{\delta^2(n+1)}\left(\int_{-\pi}^{-\delta}+\int_{\delta}^{\pi}\right)\left|\underbrace{f(x-s)}_{=\check{f}(s-x)=x*\check{f}(s)}\ -w_f(x)\right|ds \\
&= \frac{1}{2\pi}\cdot\frac{\pi^2}{\delta^2(n+1)}\left(\int_{-\pi}^{-\delta}+\int_{\delta}^{\pi}\right)\underbrace{\left|x*\check{f}(s)-w_f(x)\right|}_{\geq 0}ds \\
&\leq \frac{\pi^2}{\delta^2(n+1)}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\underbrace{\left|x*\check{f}(s)-w_f(x)\right|}_{\geq 0}ds\right) \\
&= \frac{\pi^2}{\delta^2(n+1)}\|x*\check{f}-w_f(x)\|\to 0
\end{aligned}$$

as $n\to\infty$ (exercise). Hence it follows that $\lim_{n\to\infty}|\sigma_n(f,x)-w_f(x)|\leq\epsilon$ and since $\epsilon>0$ was arbitrary, the conclusion follows.

(2) Since f is uniformly continuous on J the $\delta>0$ can be chosen to work for all $x\in J$. Hence the limit will be uniform. □

Corollary 6.5. *Suppose $f\in L(\mathbb{T})$, $x\in[-\pi,\pi]$ and $w_f(x)$ exists. Then if $\lim_{n\to\infty}S_n(f,x)$ exists, we have*

$$\lim_{n\to\infty}S_n(f,x)=w_f(x)$$

Proof. $\lim_{n\to\infty}\sigma_n(f,x)=\lim_{n\to\infty}S_n(f,x)$ and since $w_f(x)=\lim_{n\to\infty}\sigma_n(f,x)$ by Fejer's Theorem. □

Definition 6.10. If $f\in L([a,b])$ a point $x\in(a,b)$ is called a *Lebesgue point* of f if

$$\lim_{h\to 0}\frac{1}{h}\int_0^h\left|\frac{f(x+s)+f(x-s)}{2}-f(x)\right|ds=0$$

Fact 6.3. *For any $f\in L([a,b])$, it is the case that almost every $x\in(a,b)$ is a Lebesgue point.*

Proof. (Lebesgue Differentiation Theorem (PMATH 451)) □

Theorem 6.8. *If $x\in[-\pi,\pi]$ is a Lebesgue point for some $f\in L(\mathbb{T})$ then $w_f(x)=\lim_{n\to\infty}\sigma_n(f,t)$. In particular, for a.e. $x\in[-\pi,\pi]$, $\sigma_n(f,x)\to w_f(x)$ in $\mathbb{C}$.*

In short, given $f\in L(\mathbb{T})$ ($L_1(\mathbb{T})$) f has Fourier series defined as

$$\sum_{-\infty}^{\infty}c_k(f)e^k$$

We know that it is 'rarely' the case that f is equal to its Fourier series. However, we have

$$\begin{aligned}
f &= \left(L_1-\lim_{n\to\infty}\right)\sigma_n(f)=\left(L_1-\lim_{n\to\infty}\right)\frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=-j}^{j}c_k(f)e^k \\
&= \left(L_1-\lim_{n\to\infty}\right)\sum_{k=-n}^{n}\frac{n+1-|k|}{n+1}c_k(f)e^k
\end{aligned}$$

where $(L_1-\lim_{n\to\infty})$ is with respect to $\|\cdot\|_1$.

Note 14. (Abel means and Abel summation) The idea is to consider a series of complex numbers $\sum_{k=0}^{\infty} c_k$ where $c_k \in \mathbb{C}$. We say that such a series is *Abel summable* to $s \in \mathbb{C}$ if for every $0 \leq r < 1$ the series

$$A(r) = \sum_{k=0}^{\infty} c_k r^k$$

which we call an *Abel mean* for some r, converges and $\lim_{r \to 1} A(r) = s$. Note that if $\sum_{k=0}^{\infty} c_k$ converges to some s then $A(r) \to s$ as $r \to 1$.

Definition 6.11. We define

$$A_r(f)(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} c_n(f) e^{in\theta}, f \in L(\mathbb{T})$$

We easily see that

$$A_r(f) = \left(\sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} \right) * f = P_r(\theta)$$

which we call the *Poisson Kernel.*

Fact 6.4. *A given series converges $\Longrightarrow$ Cesero summable $\Longrightarrow$ Abel summable. However, NONE of the converse statements hold. (cf. Stein & Shakarchi, "Fourier Analysis", Section 2.5.)*

6.5 Fourier Coefficients

Suppose that we are given $f \in L(\mathbb{T})$, $\{c_k(f)\}_{k=-\infty}^{\infty}$ a sequence of $\mathbb{C}$-numbers. We will study the behaviour between the two.

Problem 6.3. Now suppose that we are given a sequence $\{a_n\}_{n=-\infty}^{\infty}$. Is there a function $f \in L(\mathbb{T})$ such that $f \sim \lim_{n\to\infty} \sum_{k=-n}^{n} a_k e^k$? Or $c_k(f) = a_k$ for each $k \in \mathbb{Z}$? (The answer is: No!)

Lemma 6.1. *If $f \in L_1(\mathbb{T})$ then for all $k \in \mathbb{Z}$, $|c_k(f)| \leq \|f\|_1$.*

Proof. Observe that

$$\begin{aligned} |c_k(f)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt \right| &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)| \underbrace{|e^{-ikt}|}_{=1} dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)| dt = \|f\|_1 \end{aligned}$$

□

Notation 6. Let $c_0(\mathbb{Z})$ denote the Banach space of all sequences (indexed by $\mathbb{Z}$), $\{a_n\}_{n\in\mathbb{Z}}$ such that

$$\lim_{|n|\to\infty} |a_n| = 0$$

(with pointwise operations and norm $\|\{a_k\}_{k\in\mathbb{Z}}\| = \sup_{k\in\mathbb{Z}} |a_k|$)

Theorem 6.9. *(Riemann-Lebesgue Lemma) If $f \in L_1(\mathbb{T})$ then $\lim_{|n|\to\infty} |c_n(f)| = 0$. From our above notation, this theorem says that $\{c_k(f)\}_{k\in\mathbb{Z}} \in c_0(\mathbb{Z})$ for $f \in L_1(\mathbb{T})$.*

Proof. Let $\epsilon > 0$ be given. It follows by the Abstract Summability Kernel Theorem that

$$\left(L_1 - \lim_{n\to\infty} \right) \sigma_n(f) = f$$

That is, there is $n_0 \in \mathbb{N}$ such that $\|\sigma_n(f) - f\|_1 < \epsilon$ if $|n| > n_0$. Note that

$$\sigma_n(f) = \frac{1}{n+1} \sum_{j=0}^{n} \sum_{k=-j}^{j} c_k(f) e^k = \sum_{k=-n}^{n} \frac{n+1-|k|}{n+1} c_k(f) e^k$$

Let $b_j = \frac{n_0+1-|j|}{n_0+1} c_j(f)$ for any j which implies that $\sigma_{n_0}(f) = \sum_{j=-n}^{n_0} b_j e^j$. Then for any $|k| > n_0$ we have

$$\begin{aligned} c_k(\sigma_{n_0}(f) - f) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\sigma_{n_0}(f,t) - f(t)) e^{-ikt} dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sigma_{n_0}(f,t) dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt \\ &= c_k(\sigma_{n_0}(f)) - c_k(f) \\ &= \frac{1}{2\pi} \left[\int_{-\pi}^{\pi} \sum_{j=-n_0}^{n_0} b_j e^{j-k} dk \right] - c_k(f) \\ &= -c_k(f) \end{aligned}$$

since for each j, $\int_{-\pi}^{\pi} b_j e^{j-k} = 0$ since $j \neq k$. From the above lemma, $|c_k(f)| = |c_k(\sigma_{n_0}(f) - f)| \leq \|\sigma_{n_0}(f) - f\|_1 < \epsilon$ when $|k| > n_0$. □

Corollary 6.6. *Let $f \in L(\mathbb{T})$. Then,*

1) $\lim_{n\to\infty} \int_{-\pi}^{\pi} f(t)\cos(nt)dt = 0$

2) $\lim_{n\to\infty} \int_{-\pi}^{\pi} f(t)\sin(nt)dt = 0$

Proof. 1) We have

$$\cos(nt) = \frac{1}{2}\left(e^{int} + e^{-int}\right) = \frac{1}{2}(e^n + e^{-n})t$$

and hence

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{1}{2}(e^n + e^{-n})(t)dt \\ &= \frac{1}{2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{1}{2}e^{int}dt\right) + \frac{1}{2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{1}{2}e^{-int}dt\right) \\ &= \frac{1}{2}\left(\underbrace{c_{-n}(f)}_{\to 0} + \underbrace{c_n(f)}_{\to 0}\right) \to \frac{0}{2} = 0 \end{aligned}$$

2) Similarly, $i\sin(nt) = \frac{1}{2}\left(e^{int} - e^{-int}\right)$. Let $A(\mathbb{Z}) = \{\{c_n(f)\}_{n\in\mathbb{Z}} : f \in L(\mathbb{T})\}$ called the Fourier algebra. Then $A(\mathbb{Z}) \subseteq c_0(\mathbb{Z})$. Is $A(\mathbb{Z}) = c_0(\mathbb{Z})$? (Answer: No) □

Theorem 6.10. *(Open Mapping Theorem) Suppose that X, Y are Banach spaces and $T : X \mapsto Y$ is a bounded linear map. If T is surjective, then T is "open" (i.e. if $U \subset X$ open, then $T(U)$ is open in Y).*

Proof. This will take about a week in a standard functional analysis class so we will skip this here. □

Corollary 6.7. *(Inverse Mapping Theorem) Let X, Y be Banach spaces and $T : X \mapsto Y$ be linear and bounded. If T is bijective then $T^{-1} : Y \mapsto X$ is bounded.*

Proof. See PMATH 753. □

Corollary 6.8. $A(\mathbb{Z}) \subsetneq c_0(\mathbb{Z})$

Proof. Recall that $L_1(\mathbb{T})$ and $c_0(\mathbb{Z})$ are Banach spaces. Define $T : L_1(\mathbb{T}) \mapsto c_0(\mathbb{Z})$ as the mapping $f \mapsto \{c_k(f)\}_{k\in\mathbb{Z}}$. T is well defined by the Riemann-Lebesgue Lemma. Clearly, T is linear. If $f \in L_1(\mathbb{T})$ then

$$\|T(f)\|_\infty = \|\{c_k(f)\}_{k\in\mathbb{Z}}\|_\infty = \max_{k\in\mathbb{Z}} |c_k(f)| \leq \|f\|_1$$

Thus,

$$|||T||| = \sup\{\|T(f)\|_\infty : f \in L_1(\mathbb{T}), \|f\| \le 1\} \le 1$$

That is T is bounded. From a corollary of the Abstract Summability Kernel Theorem, $c_k(f) = c_k(g) \implies f = g$ a.e. $\implies f = g$ in $L_1(\mathbb{T}) \implies T$ is one-to-one. We assume for contradiction that T is surjective. That is $A(\mathbb{Z}) = c_0(\mathbb{Z})$. By the Inverse Mapping Theorem, we get

$$T^{-1} : c_0(\mathbb{Z}) \mapsto L_1(\mathbb{Z})$$

is bounded (**). However, consider

$$d_n = \{..., 0, \underbrace{1}_{idx=-n}, 1, ..., 1, \underbrace{1}_{idx=n}, 0, ...\}$$

Clearly, $\{d_n\}_{n\in\mathbb{Z}} \in c_0(\mathbb{Z})$ and $\|d_n\|_\infty = 1$. Consider the Dirichlet Kernel $\{D_n\}_{n\in\mathbb{Z}} \subseteq L_1(\mathbb{T})$. Observe that $T^{-1}(\{d_n\}) = D_n$ (i.e. $T(D_n) = d_n$). We have

$$c_k(D_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n e^{-k} = \frac{1}{2\pi}\sum_{j=-n}^{n} e^{j-k} = \begin{cases} 1 & -n \le k \le n \\ 0 & \text{otherwise} \end{cases}$$

but

$$|||T^{-1}||| \ge \sup_{n\in\mathbb{N}} \|T^{-1}(d_n)\|_1 = \sup_{n\in\mathbb{N}} \|D_n\|_1 = \sup_{n\in\mathbb{N}} L_n = \infty$$

which contradicts the Inverse Mapping Theorem (**). Hence T is not onto. □

6.6 Localization and Dini's Theorem

Recall that in $(L_1(\mathbb{T}), \|\cdot\|_1)$ we have on U (whose complement is of first category) that $\|S_n(f) - f\|_1 \nrightarrow 0$. Before we used averaging to study this. Now, we will consider another method. In particular, we will find elements in $L(\mathbb{T})$ where $S_n(f) \mapsto f$.

If $f \in L(\mathbb{T})$ and $t \in [-\pi, \pi]$ we have

$$\begin{aligned}
\sum_{j=-n}^{n} c_j(f)e^{int} &= S_n(f,t) = D_n * f(t) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(s)f(t-s)ds \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{\frac{\sin\left(n+\frac{1}{2}\right)s}{\sin\frac{1}{2}s}}_{\text{even}} f(t-s)ds
\end{aligned}$$

and we apply inversion invariance to get

$$\sum_{j=-n}^{n} c_j(f)e^{int} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)s}{\sin\frac{1}{2}s} f(t+s)ds$$

which we will call (*).

Lemma 6.2. *If $f \in L(\mathbb{T})$ with $\int_{-\pi}^{\pi}\left|\frac{f(t)}{t}\right| dt < \infty$ then $\lim_{n\to\infty} S_n(f,0) = 0$.*

Proof. Recall that $\sin(x+y) = \sin x\cos y + \sin y\cos x$ and hence

$$D_n(s) = \frac{\sin\left(n+\frac{1}{2}\right)s}{\sin\frac{1}{2}s} = \frac{\sin(ns)\cos\frac{1}{2}s}{\sin\frac{1}{2}s} + \cos(ns)$$

and then by (*)

$$\begin{aligned} S_n(f,0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(s)f(0+s)ds \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sin(ns)\cos\frac{1}{2}s\right]\frac{f(s)}{\sin\frac{1}{2}s}ds + \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(ns)f(s)ds \end{aligned}$$

Note that if $0 \le t \le \frac{\pi}{2}$ we have $\frac{2}{\pi}|t| \le |\sin t|$. Hence if $-\pi < \theta < \pi$ then $\frac{1}{\pi}|\theta| \le \left|\sin\frac{1}{2}\theta\right|$. So

$$\int_{-\pi}^{\pi}\left|\cos\left(\frac{1}{2}s\right)\frac{f(s)}{\sin\frac{1}{2}s}\right|ds \le \pi\int_{-\pi}^{\pi}\left|\frac{f(s)}{s}\right|ds < \infty$$

by assumption. Hence the function $s \mapsto \cos\frac{1}{2}s\frac{f(s)}{\sin\frac{1}{2}s}$ a.e. $s \in [-\pi,\pi]$ (extended 2π periodically to $\mathbb{R}$) defines an element of $L_1(\mathbb{T})$. Thus, by the Riemann Lebesgue Lemma,

$$S_n(f,0) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin(ns)\underbrace{\frac{\cos\left(\frac{1}{2}s\right)f(s)}{\sin\frac{1}{2}s}ds}_{\to 0} + \frac{1}{2\pi}\underbrace{\int_{-\pi}^{\pi}\cos(ns)f(s)ds}_{\to 0} \to 0$$

and thus $S_n(f,0) \to 0$. □

Theorem 6.11. *(Localization Principle) If $f \in L(\mathbb{T})$ and I is an open interval in $[-\pi,\pi]$ on which $f(t) = 0$ a.e. $t \in I$, then for any $t \in I$ we have*

$$\lim_{n\to\infty} S_n(f,t) = 0$$

Corollary 6.9. *If $f, g \in L(\mathbb{T})$ and I is an open subinterval in $[-\pi,\pi)$ on which $f(t) = g(t)$ a.e. $t \in I$. Then for any $t \in I$*

$$\lim_{n\to\infty} S_n(f,t) \text{ exists iff } \lim_{n\to\infty} S_n(g,t) \text{ exists}$$

and the two limits coincide when they exist.

Proof. (of Corollary) Let $h = f - g$. Then observe that

$$S_n(f-g,t) = \lim_{n\to\infty}\left(S_n(f,t) - S_n(g,t)\right)$$

The result now follows from the Localization Principle. □

Proof. (of Local. Principle) Let $t \in I$ be fixed. Let g be defined by

$$g(s) = f(t-s) = \check{f}(s-t) = t * f \implies g \in L(\mathbb{T})$$

Then by our assumption of f, $g(s) = 0$ for a.e. s in some neighbourhood of 0, say for a.e. $s \in (-\delta,\delta)$, $g(s) = 0$. Hence

$$\int_{-\pi}^{\pi}\left|\frac{g(s)}{s}\right|ds = \left(\int_{-\pi}^{\delta} + \int_{\delta}^{\pi}\right)\left|\frac{g(s)}{s}\right|ds + \int_{-\delta}^{\delta}\left|\frac{\overbrace{g(s)}^{=0}}{s}\right|ds = \left(\int_{-\pi}^{\delta} + \int_{\delta}^{\pi}\right)\left|\frac{g(s)}{s}\right|ds$$

Now on $[-\pi,-\delta] \cup [\delta,\pi]$,

$$\left|\frac{1}{s}\right| \le \frac{1}{\delta} \implies \left|\frac{g(s)}{s}\right| \le \frac{|g(s)|}{\delta}$$

so

$$\begin{aligned}\int_{-\pi}^{\pi}\left|\frac{g(s)}{s}\right|ds &\leq \frac{1}{\delta}\left(\int_{-\pi}^{\delta}+\int_{\delta}^{\pi}\right)|g(s)|ds\\ &\leq \frac{1}{\delta}\underbrace{\int_{-\pi}^{\pi}|g(s)|ds}_{<\infty}\\ &= \frac{2\pi}{\delta}\|g\|_1=\frac{2\pi}{\delta}\|t*\breve{f}\|_1=\frac{2\pi}{\delta}\|\breve{f}\|_1=\frac{2\pi}{\delta}\|f\|_1\end{aligned}$$

Thus, by the Lemma, $\lim_{n\to\infty} S_n(f,0)=0$. Now,

$$\begin{aligned}S_n(g,0) &= S_n(t*\breve{f},0)=\frac{1}{2\pi}\int_{-\pi}^{\pi}D_n(s)(t*\breve{f})(s-0)ds\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}D_n(s)f(t-s)ds=S_n(f,t)\end{aligned}$$

That is, $\lim_{n\to\infty} S_n(f,t)=\lim_{n\to\infty} S_n(g,0)=0$. □

Theorem 6.12. *(Dini's Theorem for differentiable functions) If $f\in L(\mathbb{T})$ and f is differentiable at $t\in[-\pi,\pi]$ then $\lim_{n\to\infty} S_n(f,t)=f(t)$.*

Proof. Let $\epsilon>0$ be given. Then there is $\delta>0$ such that $|s|<\delta$ gives

$$\left|\frac{f(t-s)-f(t)}{s}-\underbrace{f'(t)}_{\in\mathbb{C}}\right|<\epsilon$$

Therefore on $(-\delta,\delta)$, the function $s\mapsto\frac{f(t-s)-f(t)}{s}$ bounded (by $|f'(t)|+\epsilon$). Define $g=t*\breve{f}-f(t)$. That is $g(s)=f(t-s)-f(t)$. Then we have

$$\begin{aligned}\int_{-\pi}^{\pi}\left|\frac{g(s)}{s}\right|ds &= \left(\int_{-\pi}^{\delta}+\int_{\delta}^{\pi}\right)\left|\frac{g(s)}{s}\right|ds+\underbrace{\int_{-\delta}^{\delta}\left|\frac{g(s)}{s}\right|ds}_{\leq|f'(t)+\epsilon|}\\ &\leq \frac{1}{\delta}\int_{-\pi}^{\pi}|g|ds+\int_{-\delta}^{\delta}(|f'(t)|+\epsilon)ds\\ &= \frac{1}{\delta}\|t*\breve{f}-f(t)\|_1+2\delta\left(|f'(t)|+\epsilon\right)\\ &< \epsilon\end{aligned}$$

Thus, by the Lemma, $\lim_{n\to\infty} S_n(g,0)=0$ and we observe that

$$S_n(g,0)=S_n(t*\breve{f}-f(t),0)=S_n(t*\breve{f},0)-S_n(f(t),0)=S_n(f,t)-f(t)$$

where the last equaility can be checked as an exercise. Therefore,

$$\lim_{n\to\infty} S_n(f,t)=\lim_{n\to\infty} S_n(g,0)+f(t)=f(t)$$

□

Theorem 6.13. *(Dini's Theorem for Lipschitz functions) Suppose $f\in L(\mathbb{T})$ and f is Lipschitz on an open interval. That is there is some $M>0$ such that*

$$|f(s)-f(t)|\leq M|s-t|$$

for all $t,s\in I$. Then for $t\in I$ we have $\lim_{n\to\infty} S_n(f,t)=f(t)$.

Proof. Fix $t \in I$. Then $(t-\delta, t+\delta) \subset I$ for some $\delta > 0$. For each $s \in (-\delta, \delta)$,

$$g(s) = t * \check{f}(s) - f(t) = f(t-s) - f(t)$$

for $s \in (-\delta, \delta)$ with $s \neq 0$. Then

$$\left|\frac{g(s)}{s}\right| \leq \left|\frac{f(t-s) - f(t)}{(t-s) - t}\right| \leq M$$

and the proof is the same as above. □

7 Hilbert Spaces

Definition 7.1. Let X be a complex vector space. An *inner product* $\langle , \rangle : X \times X \mapsto \mathbb{C}$ is a map such that for $f, g, h \in X$ and $\alpha \in \mathbb{C}$ then

(1) $\langle f, f \rangle \geq 0$

(2) $\langle f, f \rangle = 0 \implies f = 0$

(3) $\langle f, g \rangle = \overline{\langle g, f \rangle}$

(4) $\langle \alpha f, g \rangle = \alpha \langle f, g \rangle$

(5) $\langle f + g, g \rangle = \langle f, h \rangle + \langle g, h \rangle$

We call $(X, \langle , \rangle)$ an inner product space. That that (3) and (5) gives

$$\langle f, g + h \rangle = \langle f, g \rangle + \langle f, h \rangle$$

while (3) and (4) give

$$\langle f, \alpha h \rangle = \bar{\alpha} \langle f, h \rangle$$

Furthermore, we define the induced norm for $f \in X$ by $\|f = \sqrt{\langle f, f \rangle}$ (we can check that is a norm).

Proposition 7.1. *(Cauchy-Schwarz Inequality) If $f, g \in (X, \langle , \rangle)$ we have $|\langle f, g \rangle| \leq \|f\|\|g\|$. Moreover, $|\langle f, g \rangle| = \|f\|\|g\|$ iff $g = tf$ for some $t \geq 0$.*

Proof. Omitted. See course notes. □

Example 7.1. (Kolmogorov's Function) Continuity $\not\Rightarrow$ Pointwise convergence of $S_n f(f, x)$. Consider

$$f(x) = \prod_{k=1}^{\infty} \left(1 + i \frac{\cos 10^k x}{k}\right)$$

Here, f is continuous everywhere but for all $x \in [-\pi, \pi]$, $\{S_n(f, x)\}_{n \in \mathbb{N}}$ is unbounded.

Proposition 7.2. *If $(X, \langle , \rangle)$ is an i.p. sp. (inner product space) the $\|f\| = \sqrt{\langle f, f \rangle}$ defines a norm on X.*

Proof. Let $f, g \in X$ and $\alpha \in \mathbb{C}$. Then,

(1) $\langle f, f \rangle = 0 \iff f = 0$

(2) $\|f\| \geq 0$ (trivially)

(3) $\|\alpha f\| = \sqrt{\langle \alpha f, \alpha f \rangle} = \sqrt{|\alpha|^2 \langle f, f \rangle} = |\alpha| \|f\|$

(4) We have

$$\begin{aligned}\|f+g\|^2 &= \langle f+g, f+g\rangle \\ &= \|f\|^2 + 2\underbrace{\Re\langle f,g\rangle}_{\leq|\langle f,g\rangle|} + \|g\|^2 \\ &\leq \|f\|^2 + 2\,|\langle f,g\rangle| + \|g\|^2 \\ &\leq \|f\|^2 + 2\|f\|\|g\| + \|g\|^2 \\ &= (\|f\|+\|g\|)^2\end{aligned}$$

Taking square roots gives us the result. □

Definition 7.2. A *Hilbert space* $\mathcal{H}$ is an inner product space which is complete w.r.t. $\|\cdot\|$.

Example 7.2. (1) $\mathbb{C}^n$, $\langle x,y\rangle = \sum_{i=1}^n x_i\bar{y}_i \implies \|x\|_2 = \sqrt{\sum_{i=1}^\infty |x_i|^2}$

(2) Let $A \in \mathcal{L}(\mathbb{R})$, $\lambda(A) > 0$. Then $L_2(A)$ has inner product

$$\langle f,g\rangle = \int_A f\bar{g}\,(=\Gamma_f(\bar{g}) = \Gamma_{\bar{g}}(f))$$

If $f,g \in L_2(A) \implies \bar{f} \in L_2(A)$ ($|\bar{g}|^2 = |g|^2$) which implies that $f\bar{g} \in L_1(A)$ (by Hölder's Inequality for $p=q=2$). Hence $\langle,\rangle$ is well defined. The norm on $L_2(A)$ determined by $\langle,\rangle$ then gives

$$\|f\| = \left(\int_A f\bar{f}\right)^{\frac{1}{2}} = \left(\int_A f^2\right)^{\frac{1}{2}} = \|f\|_2$$

and since $(L_2(A), \|\cdot\|_2)$ is complete then $(L_2(A), \langle,\rangle)$ is a Hilbert space. Similarly,

$$L_2(\mathbb{T}) = \left\{f : \mathbb{R} \mapsto \mathbb{C} : f \in \mathcal{M}_{\mathbb{C}}(\mathbb{R}), 2\pi - \text{periodic}, \int_{-\pi}^{\pi} |f|^2 < \infty\right\} \approxeq L_2([-\pi,\pi])$$

together with the inner product

$$\langle f,g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\bar{g}$$

is a Hilbert space.

(3) $\mathcal{C}([a,b])$ can be equipped with

$$\langle f,g\rangle = \int_A f\bar{g}$$

but it is NOT a Hilbert space. This is due to $\mathcal{C}([a,b]) \subsetneq L_2([a,b])$ which is dense in $L_2([a,b])$. This implies that it cannot be complete.

(4) Define the set

$$l_2 = l_2(\mathbb{N}) = \left\{x = \{x_n\}_{n=1}^\infty : \sum_{n=1}^\infty |x_n|^2 < \infty\right\}$$

The inner product on l_2 is defined by

$$\langle x,y\rangle = \sum_{n=1}^\infty x_n\bar{y}_n \implies \|x\|_2 \left(\sum_{n=1}^\infty |x_n|^2\right)^{1/2}$$

Note that

$$\begin{aligned}\sum_{n=1}^{\infty}|x_n\bar{y}_n| &= \lim_{N\to\infty}\sum_{n=1}^{N}|x_n||y_n| \\ &\leq \lim_{N\to\infty}\left(\sum_{n=1}^{N}|x_n|^2\right)^{1/2}\left(\sum_{n=1}^{N}|y_n|^2\right)^{1/2} \\ &= \|x\|_2\|y\|_2 < \infty\end{aligned}$$

So $\sum_{n=1}^{\infty}|x_n\bar{y}_n|$ is convergent. Furthermore, $l_2(\mathbb{N})$ is a vector space. Observe that

$$\begin{aligned}\sum_{n=1}^{\infty}|x_n+y_n|^2 &\leq \sum_{n=1}^{\infty}(|x_n|+|y_n|)^2 \\ &= \sum_{n=1}^{\infty}\left(|x_n|^2+2|x_n||y_n|+|y_n|^2\right) \\ &= \|x\|_2^2+2\sum_{n=1}^{\infty}|x_n||y_n|+\|y_2\|^2 \\ &\leq \|x\|_2^2+2\|x_n\|\|y_n\|+\|y_2\|^2 \\ &= (\|x\|_2+\|y\|_2)^2 < \infty\end{aligned}$$

(5) Define

$$l_2 = l_2(\mathbb{Z}) = \left\{x=\{x_n\}_{n\in\mathbb{Z}} : \sum_{n=-\infty}^{\infty}|x_n|^2<\infty\right\}$$

We will show that $l_2(\mathbb{Z})$ s a Hilbert space isomorphic of $L_2(\mathbb{T})$. (*Plancherel's Theorem*)

Definition 7.3. Let $(X,\langle,\rangle)$ be an i.p. sp. A family of vectors $\{e_i\}_{i\in I}\subseteq X$ is called orthogonal if $\langle e_i,e_j\rangle=0$ for all $i,j\in I$ and $i\neq j$. Moreover, $\{e_i\}_{i\in I}$ is called orthonormal if

$$\langle e_i,e_j\rangle = \begin{cases}0 & i\neq j\\ 1 & i=j\end{cases}$$

Proposition 7.3. *(Pythagorean Principle) If $\{f_1,...,f_n\}$ is an orthogonal set in an i.p. sp. X, then*

$$\|f_1+...+f_2\| = \|f_1\|^2+...+\|f_n\|^2$$

Proof. Exercise. □

Remark 7.1. Recall that in a normed vector space X,

$$\text{dist}(f,E) = \inf\left\{\left\|f-\sum_{i=1}^{n}\alpha_i e_i\right\| : \alpha\in\mathbb{C}\right\}$$

where $f\in X$ and $E=\text{span}\{e_1,...,e_n\}$.

Lemma 7.1. *(Linear Approximation Lemma (LAL)) Suppose that $\{e_1,...,e_n\}$ is an orthonormal set in an i.p. sp. X. Let $E=span\{e_1,...,e_n\}$. Then for $f\in X$,*

$$dist(f,E)^2 = \left\|f-\sum_{i=1}^{n}\langle f,e_i\rangle e_i\right\|^2 = \|f\|^2-\sum_{i=1}^{n}|\langle f,e_i\rangle|^2$$

Moreover, $\sum_{i=1}^{n}\langle f,e_i\rangle e_i$ is the unique vector $e\in E$ s.t. $dist(f,E)=\|f-e\|$.

Proof. Let $g = \sum_{i=1}^n \alpha_i e_i$ be an arbitrary element of E. Remark that

$$\begin{aligned}
\|f-g\|^2 &= \langle f-g, f-g\rangle \\
&= \|f\|^2 - 2\underbrace{\Re\langle f,g\rangle}_{\le|\langle f,g\rangle|} + \|g\|^2 \\
(1) \quad &\ge \|f\|^2 - 2\,|\langle f,g\rangle| + \sum_{i=1}^n |\alpha_i|^2 \\
&= \|f\|^2 - 2\sum_{i=1}^n |\alpha_i|\,|\langle f,e_i\rangle| + \sum_{i=1}^n |\alpha_i|^2 \\
\text{(c-s) } (2) \quad &\ge \|f\|^2 - 2\underbrace{\left(\sum_{i=1}^n |\alpha_i|^2\right)^{1/2}}_{A}\underbrace{\left(\sum_{i=1}^n |\langle f,e_i\rangle|^2\right)^{1/2}}_{B} + \underbrace{\sum_{i=1}^n |\alpha_i|^2}_{A^2} \\
&= \|f\|^2 - \sum_{i=1}^n |\langle f,e_i\rangle|^2 + \sum_{i=1}^n |\langle f,e_i\rangle|^2 - 2\underbrace{\left(\sum_{i=1}^n |\alpha_i|^2\right)^{1/2}}_{A}\underbrace{\left(\sum_{i=1}^n |\langle f,e_i\rangle|^2\right)^{1/2}}_{B} + \underbrace{\sum_{i=1}^n |\alpha_i|^2}_{A^2} \\
&= \|f\|^2 - \sum_{i=1}^n |\langle f,e_i\rangle|^2 + B^2 - 2AB + A^2 \\
&= \|f\|^2 - \sum_{i=1}^n |\langle f,e_i\rangle|^2 + \underbrace{\left[\left(\sum_{i=1}^n |\langle f,e_i\rangle|^2\right)^{1/2} - \left(\sum_{i=1}^n |\alpha_i|^2\right)^{1/2}\right]^2}_{\ge 0} \\
(3) \quad &\ge \|f\|^2 - \sum_{i=1}^n |\langle f,e_i\rangle|^2
\end{aligned}$$

Therefore,

$$\operatorname{dist}(f,E)^2 = \inf\left\{\|f-g\| : g = \sum_{i=1}^n \alpha_i e_i, \alpha_i \in \mathbb{C}\right\} \ge \|f\|^2 - \sum_{i=1}^n |\langle f,e_i\rangle|^2$$

The inequality becomes equality when:

In (1) $\sum_{i=1}^n \overline{\alpha_i}\langle f,e_i\rangle \in \mathbb{R}$,

In (2) $\alpha_i = k\langle f,e_i\rangle, k \in \mathbb{R}$ (equality case of c-s $\le$)

In (3) $\sum_{i=1}^n |\alpha_i|^2 = \sum_{i=1}^n |\langle f,e_i\rangle^2$ (follows from above)

Therefore, we need $\alpha_i = \langle f,e_i\rangle$ for all $1 \le i \le n$. In this case,

$$\operatorname{dist}(f,E)^2 = \left\|f - \sum_{i=1}^n \langle f,e_i\rangle e_i\right\|^2 = \|f\|^2 - \sum_{i=1}^n |\langle f,e_i\rangle|^2$$

□

Proposition 7.4. *Let X be an i.p. sp. and $g \in X$. Then*

$$\Gamma_g : X \mapsto \mathbb{C}$$

given by $\Gamma_g(f) = \langle f,g\rangle$ is linear and bounded with $\||\Gamma|\| = \|g\|$.

Proof. Linearity follows from properties of $\langle,\rangle$. By the Cauchy Schwarz Inequality,

$$|\Gamma_g(f)| = |\langle f,g\rangle| \le \|f\|\|g\|$$

for any $f \in X$. Then $|||\Gamma||| \leq \|g\|$ which implies that it is bounded and hence continuous. If $g = 0$ then $\Gamma_g = 0$ and we are done. If $g \neq 0$ the $\|g\| \neq 0$ adn

$$\left|\Gamma_g\left(\frac{1}{\|g\|}g\right)\right| = \left|\left\langle \frac{1}{\|g\|}g, g\right\rangle\right| = \frac{1}{\|g\|}\left|\langle g, g\rangle\right| = \frac{1}{\|g\|}\|g\|^2 = \|g\|$$

Therefore, $|||\Gamma_g||| \geq \|g\| \implies |||\Gamma_g||| = \|g\|$. □

Remark 7.2. (Riesz Representation Theorem) If $\mathcal{H}$ is a Hilbert space, then every bounded linear functional $\Gamma : \mathcal{H} \mapsto \mathbb{C}$ is of the form $\Gamma = \Gamma_g$ where $g \in \mathcal{H}$.

Theorem 7.1. *(Orthonormal Basis Theorem (OBT)) Let X be an inner product space and $\{e_i\}_{i=1}^{\infty}$ be an orthonormal sequence. Then the following are equivalent.*

(1) span$\{e_i\}_{i=1}^{\infty} = \{\sum_{i=1}^{n} \alpha_i e_i : n \in \mathbb{N}, \alpha_i \in \mathbb{C}\}$ is dense in X.

(2) (Bessel's equality) For every $f \in X$, we have $\|f\|^2 = \sum_{i=1}^{\infty} |\langle f, e_i\rangle|^2$ in $\mathbb{C}$.

(3) For every $f \in X$ we have $f = \lim_{n\to\infty} \sum_{i=1}^{n} \langle f, e_i\rangle e_i = \sum_{n=1}^{\infty} \langle f, e_i\rangle e_i$, w.r.t. $\|\cdot\|$.

(4) (Parseval's identity) For every $f, g \in X$, $\langle f, g\rangle = \sum_{n=1}^{\infty} \langle f, e_i\rangle \langle e_i, g\rangle$ in $\mathbb{C}$.

Remark 7.3. By (3) we are justified to call $\{e_i\}_{i=1}^{\infty}$ an orthonormal basis.

Proof. (of ONBT) We plan to do the proof in the following order: $(1) \iff (3), (2) \iff (3), (3) \implies (4), (4) \implies (2)$.

$(1) \iff (3)$ Let $E_n = \text{span}\{e_1, .., e_n\}$. Then $E_n \subset E_{n+1}$ for each n. So for $f \in X$, $\text{dist}(f, E_n) \geq \text{dist}(f, E_{n+1})$. Therefore,

$$\underbrace{\text{span}\{e_i\}_{i=1}^{\infty} = \bigcup_{n=1}^{\infty} E_n}_{(1)} \iff \underbrace{\left\|f - \sum_{i=1}^{n} \langle f, e_i\rangle e_i\right\| = \text{dist}(f, E_n) \to 0}_{(3)}$$

by the LAL and because span$\{e_i\}_{i=1}^{\infty}$ is dense in X.

$(2) \iff (3)$ By the LAL,

$$\left\|f - \sum_{i=1}^{n} \langle f, e_i\rangle e_i\right\|^2 = \|f\|^2 - \sum_{i=1}^{n} |\langle f, e_i\rangle|$$

for each $n \in \mathbb{N}$. So,

$$\underbrace{\|f\|^2 = \lim_{n\to\infty} \sum_{i=1}^{n} |\langle f, e_i\rangle|}_{(2)} \iff \underbrace{\lim_{n\to\infty} \sum_{i=1}^{n} \left\|f - \sum_{i=1}^{n} \langle f, e_i\rangle e_i\right\|^2 = 0}_{(3)}$$

$(3) \implies (4)$ Let $g \in X$, $\Gamma_g X \mapsto \mathbb{C}$, $f \mapsto \langle f, g\rangle$ is bounded which implies continuity. Then,

$$\langle f, g\rangle = \Gamma_g(f) = \Gamma_g\left(\lim_{n\to\infty} \sum_{i=1}^{n} \langle f, e_i\rangle e_i\right) = \lim_{n\to\infty} \Gamma_g\left(\sum_{i=1}^{n} \langle f, e_i\rangle e_i\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \langle f, e_i\rangle \langle e_i, g\rangle$$

$(4) \implies (2)$ Take $f = g$ and note $\langle f, e_i\rangle \langle e_i, f\rangle = \langle f, e_i\rangle \overline{\langle f, e_i\rangle} = |\langle f, e_i\rangle|^2$. □

Definition 7.4. Any sequence satisfying conditions of the OBT is called an orthonormal basis for X.

Remark 7.4. (Bessel's Inequality) Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal (o.n.) sequence in an i.p. sp. X. Then for $f \in X$, we have

$$\langle f, f\rangle = \|f\|^2 \geq \sum_{i=1}^{\infty} |\langle f, e_i\rangle|^2$$

Proof. If $E_n = \text{span}\{e_1, ..., e_n\}$ then

$$0 \leq \text{dist}(f, E_n)^2 \overset{\text{LAL}}{=} \|f\|^2 - \sum_{k=1}^{n} |\langle f, e_k \rangle|^2$$

Hence $\sum_{k=1}^{n} |\langle f, e_k \rangle|^2 \leq \|f\|^2$ for all $n \in \mathbb{N}$ which implies that

$$\sum_{k=1}^{\infty} |\langle f, e_k \rangle|^2 = \lim_{n\to\infty} \sum_{k=1}^{n} |\langle f, e_k \rangle|^2 = \sup_{n\in\mathbb{N}} \sum_{k=1}^{n} |\langle f, e_k \rangle|^2 \leq \|f\|^2$$

□

Note 15. Equality above holds if $f \in \overline{\text{span}\{e_1, e_2, ...\}}$ closed w.r.t. $\|\cdot\|$.

Theorem 7.2. *Let X be an i.p. sp. and $\{e_i\}_{i=1}^{\infty} \subset X$ be an orthonormal basis in X. Then the operator $U : X \mapsto l_2(\mathbb{N})$ given by $U_f = \{\langle f, e_i \rangle\}_{i=1}^{\infty}$ is an isometry preserving the inner product. That is, $\underbrace{\|U_f\|}_{\text{in } l_2} = \underbrace{\|f\|}_{\text{in } X}$ and $\underbrace{\langle U_f, U_g \rangle}_{\text{in } l_2} = \underbrace{\langle f, g \rangle}_{\text{in } X}$ for $f, g \in X$.*

Proof. By Bessel's equality, for any $f \in X$,

$$\|U_f\|^2 = \sum_{i=1}^{\infty} |\langle f, g \rangle|^2 = \|f\|^2$$

and hence U is a bounded linear operator and isometry on X. We next observe that

$$\begin{aligned} \langle U_f, U_g \rangle &= \langle \{\langle f, e_i \rangle\}_{i=1}^{\infty}, \{\langle g, e_i \rangle\}_{i=1}^{\infty} \rangle \\ &= \sum_{i=1}^{\infty} \langle f, e_i \rangle \overline{\langle g, e_i \rangle} \\ &= \sum_{i=1}^{\infty} \langle f, e_i \rangle \langle e_i, g \rangle \\ &= \langle f, g \rangle \end{aligned}$$

by Parseval's identity. □

Example 7.3. Here are some examples of orthonormal bases.

1. Let $X = l_2(\mathbb{Z})$ with the i.p. $\langle x, y \rangle = \sum_{n=-\infty}^{\infty} x_n \overline{y_n}$. Consider for each $n \in \mathbb{Z}$, the element

$$e_n = (..., 0, \underbrace{1}_{n^{th} \text{ entry}}, 0, ...)$$

   Then we have:

   (a) $\langle e_n, e_m \rangle = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$

   (b) If $x \in l_2(\mathbb{Z})$ then $\langle x, e_n \rangle = e_n$ (n^{th} entry in X)

   (c) If $x \in l_2(\mathbb{Z})$ then $\left\| x - \sum_{k=-n}^{n} \langle x, e_k \rangle e_k \right\|^2 \to 0$ as $n \to \infty$.
   So $\text{span}\{e_k\}_{k\in\mathbb{Z}}$ is dense in l_2 and $\{e_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis (o.n.b.) for $l_2(\mathbb{Z})$.

2. Consider $X = L_2(\mathbb{T})$ with $\langle f, g \rangle = \int_{\mathbb{T}} f\bar{g}$ for $f, g \in L_2(\mathbb{T})$. Consider $\{e^k\}_{k\in\mathbb{Z}} \subset L_2(\mathbb{T})$ where $e^k(t) = e^{ikt}$. Then we have:

   (a) $\{e^k\}_{k\in\mathbb{Z}}$ is orthonormal in $L_2(\mathbb{T})$

   (b) The Abstract Summability Theorem implies that $\{e^k\}_{k\in\mathbb{Z}}$ is an o.n.b for $L_2(\mathbb{T})$

Corollary 7.1. *(L_2 Convergence of Fourier Series) Let $f \in L_2(\mathbb{T})$. Then $\lim_{n\to\infty} \|f - S_n(f)\|_2 = 0$.*

Proof. We have

$$S_n(f) = \sum_{k=-n}^{n} c_k(f)e^k = \sum_{k=-n}^{n} \left\langle f, e^k \right\rangle e^k$$

Since $\{e^k\}_{k\in\mathbb{Z}}$ is an o.n.b. by the OBT, $\lim_{n\to\infty} \left\| f - \sum_{k=-n}^{n} \left\langle f, e^k \right\rangle e^k \right\|_2^2 = 0$. □

Remark 7.5. Let's examine the convergence of Fourier series in various Banach spaces.

(1) Suppose that $f \in L(\mathbb{T})$. In $L_1(\mathbb{T})$, $S_n(f) \to f$ rarely w.r.t. $\|\cdot\|_1$. That is, from the properties of the $D_n's$ (Dirichlet Kernel), $\lim_{n\to\infty} \|S_n(f) - f\|_1 \neq 0$ on $U_1 \subseteq L_1(\mathbb{T})$ where U_1^c is of 1st category.

Suppose that $f \in \mathcal{C}(\mathbb{T})$. Then $\lim_{n\to\infty} \|S_n(f) - f\|_\infty \neq 0$ on $U_\infty \subseteq \mathcal{C}(\mathbb{T})$ where U_∞^c is of 1st category.

(2) Consider $\sigma_n(f,t) = \frac{1}{n+1}\left(\sum_{k=0}^{n} D_k\right) * f(t) = K_n * f(t)$. By the Abstract Summability Kernel Theorem, if $f \in L_p(\mathbb{T})$ for $1 \le p < \infty$ then $\lim_{n\to\infty} \|\sigma_n(f) - f\|_p = 0$.

(3) For $p = 2$, $L_2(\mathbb{T})$ is a Hilbert space. By L_2 convergence of Fourier series, if $f \in L_2(\mathbb{T})$ then $\lim_{n\to\infty} \|S_n(f) - f\|_2 = 0$. To see this, recall that $\||C(D_n)|\|_{L_1(\mathbb{T})} = \|D_n\|_1 \to \infty$ as $n \to \infty$. In L_2, by Bessel's Inequality, $\||C(D_n)|\|_{L_2(\mathbb{T})} \le 1$ for all n (this is in fact, an equality, which is left to be shown as an exercise) on $[-\pi, \pi]$, which implies that $L_2(\mathbb{T}) \subseteq L_1(\mathbb{T})$.

Theorem 7.3. *(Riesz-Fischer Theorem) Let $f \in L_1(\mathbb{T})$. Then $f \in L_2(\mathbb{T})$ if and only if $\sum_{n=-\infty}^{\infty} |c_k(f)|^2 < \infty$*

Proof. ($\implies$) Since $c_k(f) = \left\langle f, e^k \right\rangle$ for $k \in \mathbb{Z}$ then $\|f\|_2^2 \ge \sum_{k=-n}^{n} |c_k(f)|^2$ for each $n \in \mathbb{N}$. Taking sup over $n \in \mathbb{N}$ we get

$$\sum_{k=-\infty}^{\infty} |c_k(f)|^2 = \sup_{n\in\mathbb{N}} \sum_{k=-n}^{n} |c_k(f)|^2 \le \|f\|_2^2 < \infty$$

since $f \in L_2(\mathbb{T})$.

($\impliedby$) Consider $S_n(f) = \sum_{k=-n}^{n} c_k(f)e^k$. Let $n > m$. We have

$$\|S_n(f) - S_m(f)\|_2^2 = \sum_{k=-n}^{-(m+1)} |c_k(f)|^2 + \sum_{k=m+1}^{n} |c_k(f)|^2 \to 0$$

as $n, m \to \infty$, by Pythagoras' Law. Hence $\{S_n(f)\}_{n\in\mathbb{N}}$ is Cauchy in $L_2(\mathbb{T})$. By completeness of $L_2(\mathbb{T})$, there is $\tilde{f} \in L_2(\mathbb{T})$ such that $S_n(f) \to \tilde{f}$ with respect to $\|\cdot\|_2$. That is, $\left\| \tilde{f} - \sum_{k=-n}^{n} c_k(f)e^k \right\|_2 \to 0$. Note that $c_k(\tilde{f}) = c_k(f) \implies \tilde{f} = f$ a.e. on $[-\pi, \pi] \implies \tilde{f} = f$ in $L_2(\mathbb{T})$ and $f \in L_2(\mathbb{T})$. □

Theorem 7.4. *(Abstract Plancherel's Theorem) The map $U : L_2(\mathbb{T}) \mapsto l_2(\mathbb{Z})$ given by $f \mapsto U(f) = \{c_n(f)\}_{n\in\mathbb{Z}}$ is a surjective isometry with $\langle Uf, Ug \rangle_{l_2(\mathbb{Z})} = \langle f, g \rangle_{L_2(\mathbb{T})}$.*

Proof. This is almost a restatement of the Riesz-Fischer Theorem. We will just need to verify surjectivity. Let $\{c_n\}_{n\in\mathbb{Z}} \in l_2(\mathbb{Z})$. Define $f_n = \sum_{k=-n}^{n} c_k e^k$. Then $\{f_n\}_{n=1}^{\infty}$ is Cauchy in $L_2(\mathbb{T})$ (left as an exercise). Hence it converges to $f \in L_2(\mathbb{T})$ and $c_n(f) = c_n$ for any $n \in \mathbb{Z}$. Now recall that U is an isometry as a corollary of Bessel's Inequality and preserves the inner product as a corollary of Parseval's identity.

Alternatively, here is a more rigourous treatment. By Bessel's inequality, for any $f \in X$, $\|U_f\|^2 = \sum_{i=1}^{\infty} |\langle f, e_i \rangle|^2 \le \|f\|^2 < \infty$. So U is indeed a linear map into l_2. Next, we observe that

$$\langle Uf, Ug \rangle = \left(\{\langle f, e_i \rangle\}_{i=1}^{\infty}, \{\langle g, e_i \rangle\}_{i=1}^{\infty}\right) = \sum_{i=1}^{\infty} \langle f, e_i \rangle \langle g, e_i \rangle = \langle f, g \rangle$$

Finally, let $f = g$ to get that $\|Uf\|^2 = \langle Uf, Uf \rangle = \langle f, f \rangle = \|f\|^2$. □

Corollary 7.2. *$l_2(\mathbb{Z})$ is complete $\implies$ It is a Hilbert space.*

Summary 2. Let's summarize the spaces of (almost everywhere equivalent classes of) functions by:

$$\begin{array}{ccccccc} A(\mathbb{T}) & \subset & \mathcal{C}(\mathbb{T}) & \subset & L_2(\mathbb{T}) & \subset & L_1(\mathbb{T}) \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ l_1(\mathbb{Z}) & \subset & C^*(\mathbb{Z}) & \subset & l_2(\mathbb{Z}) & \subset & A(\mathbb{Z}) \subsetneq c_0(\mathbb{Z}) \end{array}$$

Appendix A

This is course is fairly comprehensive in terms of explaining the high level details of measure theory, so instead of using this Appendix to fill in the nitty gritty details I'll leave a few remarks about analysis in general. Others are also welcome to contribute by sending me an e-mail with your contribution.

- Working with ∞ and infintessimals is like playing a game where one side always wins no matter what valid game is being played. (Examples: Continuity, Limit points, Lebesgue measure, $\mathcal{C}^{\infty}$, Sequences, Banach/Hilbert spaces, the real numbers as an equivalence class of Cauchy rational sequences, cardinalities and Cantor's continuum)
- Always leave yourself with $\epsilon > 0$ of room. Don't be afraid to leave too much.
- Kernels are not analogous to the kernels seen in linear algebra; they should be thought of as defining new classes of integrals

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