PMATH 352 (Winter 2013 - 1135)

Complex Analysis

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These notes are currently a work in progress, and as such may be incomplete or contain errors.

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Abstract

The purpose of these notes is to provide an almost primary reference for the content covered in PMATH 352. The official prerequisite for this course is either MATH 237/247 or PMATH 331, but this author personally believes that MATH 237 would be insufficient. This author also recommends that the student taking this course should have a very solid background in analysis and theoretical calculus.

1 Complex Numbers

Recall the definition that $i = \sqrt{-1}$ and $\mathbb{C} := \{x + iy | x, y \in \mathbb{R}\}$ where $\mathbb{R}$ is the set of real numbers. For $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we define

$$\begin{aligned} z_1 + z_2 &= (x_1 + x_2) + i(y_1 + y_2) \\ z_1 z_2 &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \\ \Re(z_1) &= x_1 \\ \Im(z_1) &= y_1 \\ |z_1| &= \sqrt{x_1^2 + y_1^2} \end{aligned}$$

We also define $\forall z \in \mathbb{C}$, $r \in \mathbb{R}^+$,

$$\begin{aligned} \mathcal{B}_r(a) &= \{z \in \mathbb{C} || z - p| < r\} \\ \overline{\mathcal{B}_r(a)} &= \{z \in \mathbb{C} || z - p| \leq r\} \\ \partial \mathcal{B}_r(a) &= \{z \in \mathbb{C} || z - p| = r\} \end{aligned}$$

Definition 1.1. Let $\Omega \in \mathbb{C}$, a subset. We say that Ω is *open* if for all $p \in \Omega$, there exists $\mathcal{B}_r(p) \subseteq \Omega$ and $r > 0$, $\{a_n\} \subseteq \mathbb{C}$. We say that $\lim_{n\to\infty} a_n = a \in \mathbb{C}$ if $\forall \epsilon > 0, \exists N \in \mathbb{P}$ such that for all $n \geq N$, $|a_n - a| < \epsilon$.

Let $A \subseteq \mathbb{C}$ a subset. We say A is *closed* if $\forall \{a_n\} \subseteq A$, $\lim_{n\to\infty} a_n = a$ then $a \in A$.

Let $B \subseteq \mathbb{C}$ be a subset. Define $B^c = \{p \in \mathbb{C} | p \notin B\}$.

Fact 1.1. *Ω is open if and only if Ω^c is closed.*

Definition 1.2. Let Ω be an open subset of $\mathbb{C}$ and $f : \Omega \subseteq \mathbb{R}^2 \mapsto \mathbb{C}$ be a function.

Given $p \in \Omega$ and $\omega \in \mathbb{C}$. We say $f(z) \to \omega$ as $z \to p$ when $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|f(z) - \omega| < \epsilon$ whenever $|z - p| < \delta$. We say that f is continuous at p if $f(z) \to f(p)$

Fact 1.2. *$f(z)$ is* continuous *at p if and only if $\forall \{a_n\} \in \mathbb{C}$ we have $\lim_{n\to\infty} a_n = p \implies \lim_{n\to\infty} f(a_n) = f(p)$*

Definition 1.3. Let $\Omega \subseteq \mathbb{C}$ a subset. We define the following:

$$\begin{aligned} \bar{\Omega} &= \{p \in \mathbb{C} | \exists \{a_n\} \subseteq \Omega, \lim_{n\to\infty} a_n = p\} \supseteq \Omega \\ \Omega^0 &= \{p \in \Omega | \exists r > 0, \mathcal{B}_r(p) \subseteq \Omega\} \subseteq \Omega \\ \partial \Omega &= \bar{\Omega} \backslash \Omega^0 \end{aligned}$$

Fact 1.3. *$\bar{\Omega}$ is closed and Ω^0 is open and if Ω is closed (open), then $\bar{\Omega}(\Omega^0) = \Omega$.*

Definition 1.4. $\forall X, Y \subseteq \mathbb{C}$ subsets, define $dist(X, Y) := \inf\{|z_1 - z_2| z_1 \in X, z_2 \in Y\}$

Remark 1.1. $dist(X, Y)$ does not imply $X \cap Y \neq \emptyset$.

2 Complex Functions

Proposition 2.1. *Let $f : \Omega \mapsto \mathbb{C}$, a function and Ω open. The following are equivalent (TFAE):*

(i) f is continuous on Ω

(ii) $\forall U \subseteq \mathbb{C}$ open, then $f^{-1}(U) = \{p \in \Omega | f(p) \in U\}$ is open

Definition 2.1. Let $\Omega \subseteq \mathbb{C}$ be an open subset and $f : \Omega \mapsto \mathbb{C}$ a function. We say f is analytic (a.k.a. holomorphic) if it is differentiable at every point in Ω.

Definition 2.2. $f'(z) = \lim_{\underbrace{h \to 0}_{\in \mathbb{C}}} \frac{f(z+h)-f(z)}{h}$ exists $\forall z \in \Omega$ and $f'(z)$ is continuous.

2.1 Holomorphic Functions

Remark 2.1. For a function $h : \mathbb{C} \mapsto \mathbb{C}$ we say $h(z) = o(z)$ if $\lim_{z\to 0} \frac{h(z)}{z} = 0$. We can then rewrite the *holomorphic* condition as:

$$\forall z \in \Omega \subseteq \mathbb{C}, \exists f'(z) : \Omega \mapsto \mathbb{C}, s.t.\ f(z+h) = f(z) + f'(z) \cdot h + o(h)$$

and $f(z)$ is continuous.

Proof. Let $\epsilon(h) = \frac{f(z+h)-f(z)}{h} - f'(z)$. Then the holomorphic condition is equivalent to $\lim_{h\to 0} \epsilon(h) = 0$. Thus, we have

$$\lim_{h\to 0} \frac{\epsilon(h) \cdot h}{h} = 0 \implies \epsilon(h) \cdot h = o(h)$$

and so the result follows trivially. □

Proposition 2.2. *Here are some differentiation properties*

(i) $(f \pm g)' = f' \pm g'$

(ii) $(fg)' = f'g + fg'$

(iii) $\left(\frac{f}{g}\right)' = \frac{f'g-fg'}{g^2}$

(iv) $(g \circ f(z))' = g(f(z))f'(z)$

Proof. (ii) $\forall z \in \Omega, h \in \mathbb{C}$,

$$\begin{aligned} f(z+h) &= f(z) + f'(z) \cdot h + o(h) \\ g(z+h) &= f(z) + g'(z) \cdot h + o(h) \end{aligned}$$

and so

$$f(z+h) \cdot g(z+h) = f(z) \cdot g(z) + [f'(z)g(z) + f(z)g'(z)]h + o(h) \cdot (f(z) + f'(z)h + g(z) + g'(z)h)$$

since $f(z), f'(z), g(z), g'(z)$ are continuous and on a small closed ball centered at z which is compact which implies it is bounded. Thus

$$\lim_{h\to 0} \frac{o(h)\,(f(z) + f'(z)h + g(z) + g'(z)h)}{h} = 0, i.e. \text{ it is } o(h)$$

(iv)

$$\begin{aligned} g \circ f(z+h) &= g(f(z) + f'(z) + o(h)) \\ &= g(f(z)) + g'(f(z))(f'(z)h + o(h)) + o(h) \\ &= g(f(z)) + g'(f(z)f'(z) \cdot h + o(h) \end{aligned}$$

and thus $(g \circ f(z))' = g'(f(z)) \cdot f'(z)$. □

Remark 2.2. If $f : \Omega \to \mathbb{C}$ is holomorphic at $z \in \Omega$ then $f(z+h) = f(z)+f'(z)h+o(h)$ for small h. Furthermore $\exists\lambda = f'(z) \in \mathbb{C}$ such that $f(z+h)$ acts locally like a translation using $f(z)$, rotates with $\arg\lambda$, and dilates with scalar $|\lambda|$ (because $f'(z) \in \mathbb{C}$ which is of the form $re^{-i\theta}$).

In other words, if $f(z) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} \approxeq u(x,y) + iv(x,y)$ where $z = x+iy$ then the *Jacobian* at $z = x+iy = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$ must be a dilation, $r \in \mathbb{R}$ and a rotation by θ (the converse is also true). The form must also be like

$$z = x+iy = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = r \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and note that this also implies that $u_x = v_y$ and $u_y = -v_x$. The previous condition is known as the *Cauchy-Riemann condition*.

Notation 1. Let $\lambda = a+bi = re^{i\theta} = r(\cos\theta + i\sin\theta) = r$ cis θ. We also denote

$$M_\lambda := r \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

where it is obvious that M_λ is dilation by r and a rotation by angle θ.

Thus $\forall h \in \mathbb{C}$ and $\bar{h} \in \mathbb{R}^2$ where $\bar{h}$ is the corresponding vector in $\mathbb{R}^2$, we have

$$\underbrace{\lambda h}_{\in\mathbb{C}} \approxeq \underbrace{M_\lambda \bar{h}}_{\in\mathbb{R}^2}$$

So if we have $f : \Omega \subseteq \mathbb{C} \approxeq \mathbb{R}^2 \mapsto \mathbb{C} \approxeq \mathbb{R}^2$ is a function from $\mathbb{R}^2$ to $\mathbb{R}^2$, then if f is holomorphic on Ω then $M_{f'(z)}$ is the Jacobian of f at z.

Proposition 2.3. *The following are equivalent (TFAE):*

(i) $f : \Omega \mapsto \mathbb{C}$*, open* Ω*, is holomorphic*

(ii) $\exists f'(z) : \Omega \mapsto \mathbb{C}$ *continuous such that* $\forall z \in \Omega, h \in \mathbb{C}$*, we have* $f(z+h) = f(z) + f'(z)h + o(h)$

(iii) $\exists f'(z) : \Omega \mapsto \mathbb{C}$ *continuous such that if we identify* $f : \Omega \mapsto \mathbb{R}^2$ *the Jacobian of* f *at* z *is equal to* $M_{f'(z)}, \forall z \in \Omega$

2.2 Cauchy-Riemann Condition

Remark 2.3. *Given* $f(z) = u(x,y) + iv(x,y)$ *where* $z = x+iy$ *then the Jacobian at* z *is*

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

and if f *is holomorphic then*

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = r \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = M_\lambda, \lambda = re^{i\theta}$$

and we also have the relationships $u_x = v_y$, $u_y = -v_x$ *and*

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ -u_y & u_x \end{pmatrix} = \sqrt{u_x^2 + u_y^2} \begin{pmatrix} \frac{u_x}{\sqrt{u_x^2+u_y^2}} & \frac{u_y}{\sqrt{u_x^2+u_y^2}} \\ \frac{-u_y}{\sqrt{u_x^2+u_y^2}} & \frac{u_y}{\sqrt{u_x^2+u_y^2}} \end{pmatrix} = r \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

where $r = \sqrt{u_x^2 + u_y^2}$ *and* θ *is the angle of* $u_x + iu_y$*.*

Definition 2.3. Let $u, v : \Omega \mapsto \mathbb{R}^2$ be two $\mathcal{C}^1$ functions. We say that u, v satisfy the Cauchy-Riemann equations if $u_x = v_y$ and $u_y = -v_x$.

Proposition 2.4. *$f : \Omega \mapsto \mathbb{C}$ where $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic on Ω iff (if and only if) $u, v \in \mathcal{C}^1(\Omega)$ and they satisfy the Cauchy-Riemann equations (C-R, CR).*

Proof. ($\implies$) Let $\epsilon \in \mathbb{R}$. If $f : \Omega \mapsto \mathbb{C}$ is holomorphic and $f(x+iy) = u(x,y) + i(x,y)$ then

$$\lim_{\substack{\epsilon \to 0 \\ \epsilon \in \mathbb{R}}} \frac{f(z+\epsilon) - f(z)}{\epsilon} = \lim_{\substack{\epsilon \to 0 \\ \epsilon \in \mathbb{R}}} \frac{f((x+\epsilon)+iy) - f(x+iy)}{\epsilon} = u_x(x,y) + iv_x(x,y) = f'(z)$$

and also

$$\lim_{\substack{\epsilon \to 0 \\ \epsilon \in \mathbb{R}}} \frac{f(x+i(y+\epsilon)) - f(x+iy)}{i\epsilon} = \frac{1}{i}(u_y(x,y) + iv(x,y)) = v_y(x,y) - iu_y(x,y) = f'(z)$$

and so $u_x = v_y$ and $u_y = -v_x$. □

Corollary 2.1. *If $f$is holomorphic and real valued only, then f is constant.*

Proof. Let $f(x+iy) = u(x,y) + iv(x,y)$. If f is real valued only, then $v(x,y) = 0 \implies u_x = v_y = 0, u_y = -v_x = 0$ and hence u is constant. □

Remark 2.4. Let $f(\Omega) \subseteq$ Line L be open. We can translate L to the origin and rotate it to the real line so $L' = e^{i\theta_0}(L + z_0) \subseteq \mathbb{R}$ and since L' is holomorphic and constant, we have $L = \frac{1}{e^{i\theta_0}}L' - z_0$ which is a constant. We then have that f is a constant.

Notation 2. Let $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$.

Proposition 2.5. *If f is holomorphic, then $\frac{\partial f}{\partial z} = f'$ and $\frac{\partial f}{\partial \bar{z}} = 0$.*

Proof. Recall $f'(z) = \frac{\partial f}{\partial x}$ and $if'(x) = \frac{\partial f}{\partial y}$ and hence

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{1}{2}(f' + f') = f'$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}(f' - f') = 0$$

□

Example 2.1. We give some examples of some holomorphic functions.

1) Polynomials $p(z)$ in $\mathbb{C}$ are holomorphic.

2) Rational functions $\frac{p(z)}{q(z)}$ are holomorphic except the set $Z(q(z)) := \{z \in \mathbb{C} | q(z) = 0\}$

3) $\sin(z) = -\frac{z^3}{3!} + \frac{z^5}{5} - \ldots$ and $\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots$ are holomorphic.

2.3 Harmonic Functions

Recall $f(x) = \sin x, f''(x) = -\sin x$ and hence $f + f'' = 0 \implies f = -f'' \implies f^{(n)} = -f^{(n+2)}, \forall n \in \mathbb{N}$ and $f \in \mathcal{C}^\infty$.

Remark 2.5. Let $f(x+iy) = u(x,y) + iv(x,y)$. If f is holomorphic, we have $u_x = v_y$, $u_y = -v_x$ by CR and also $u_{xx} + u_{yy} = v_{yx} - v_{xy}$. The right side is equal to 0 if $u, v \in \mathcal{C}^2$. So from now on, we would like to assume that our holomorphic functions are in $\mathcal{C}^2$. Similarly on the left side, $v_{xx} + v_{yy} = 0$.

Definition 2.4. Let $\triangle$ denote the differential operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ which we call the *Laplacian*. We say that a function f is harmonic if $\triangle f = 0 \iff \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$.

Remark 2.6. From partial differential equation (PDE) theory, if u is harmonic, then u is analytic $\implies u \in \mathcal{C}^\infty$.

Theorem 2.1. *Suppose $f, g \in \mathcal{C}^1(\Omega), \Omega \subseteq \mathbb{R}^2$ where Ω is an open rectangle or an open disk such that $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$. Then $\exists h \in \mathcal{C}^2(\Omega)$ such that $\frac{\partial h}{\partial x} = f$ and $\frac{\partial h}{\partial y} = g$. This also implies $h_{xy} = h_{yx}$.*

Proof. Define $h(x,y) = \int_a^x f(t,b)dt + \int_b^y g(x,s)ds$ where (a,b) is the center of Ω. Now

$$\begin{aligned}
\frac{\partial h}{\partial x}(x_0,y_0) &= \left(\frac{\partial}{\partial x}\int_a^x f(t,b)dt\right)\Big|_{x=x_0} + \left(\frac{\partial}{\partial x}\int_b^y g(x,s)ds\right)\Big|_{x=x_0} \\
&= f(x_0,b) + \left(\int_b^y \frac{\partial}{\partial x} g(x,s)ds\right)\Big|_{x=x_0} \\
&= f(x_0,b) + \int_b^{y_0} \frac{\partial f}{\partial y}(x_0,s)ds \\
&= f(x_0,b) + (f(x_0,y_0) - f(x,b)) \\
&= f(x_0,y_0)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial h}{\partial x}(x_0,y_0)\frac{\partial h}{\partial y}(x_0,y_0) &= \left(\frac{\partial}{\partial y}\int_a^x f(t,b)dt\right)\Big|_{(x_o,y_0)} + \left(\frac{\partial}{\partial y}\int_b^y g(x,s)ds\right)\Big|_{(x_0,y_0)} \\
&= 0 + g(x_0,y_0)
\end{aligned}$$

and so h satisfies the requirements. □

Remark 2.7. Recall that $u \in \mathcal{C}(\Omega), \Omega \subseteq \mathbb{C} \approx \mathbb{R}^2$ open is harmonic if and only if $u_{xx} + u_{yy} = 0$ and if $f = u + iv$ is holomorphic then u and v are *harmonic*.

Definition 2.5. Let $u, v \in \mathcal{C}^2(\Omega), \Omega \subseteq \mathbb{C}$ open. If $f = u + iv$ are holomorphic, we say u is a *harmonic conjugate* of v (and vice versa).

2.4 Holomorphic Function Construction

Problem 2.1. We would like to construct holomorphic functions from its real/imaginary part. In other words, given $u \in \mathcal{C}^2(\Omega)$ which is harmonic, we would like to find its harmonic conjugate.

Theorem 2.2. *Let $u \in \mathcal{C}^2(\Omega)$ where Ω is nice (rectangle or disk) and u be harmonic. Then there exists a unique harmonic conjugate v up to a constant.*

Proof. Let $f = -u_y$ and $g = u_x$. Then

$$\frac{\partial f}{\partial y} = -u_{yy} = u_{xx} = \frac{\partial g}{\partial x}$$

using the harmonic property of u. Thus, by the previous theorem, there exists $v \in \mathcal{C}^2(\Omega)$ such that $\frac{\partial v}{\partial x} = f, \frac{\partial v}{\partial y} = g$ which implies that

$$v_x = -u_y, v_y = u_x$$

and so $f = u + iv$ satisfies the CR equations and f is holomorphic. So v is a harmonic conjugate of u. Assume that v_1 and v_2 are 2 harmonic conjugates of u. Then that means $f_1 = u + iv_1$ and $f_2 = u + iv_2$ are holomorphic. Also, $h = f_1 - f_2 = i(v_1 - v_2)$ is holomorphic as well. Thus, h is only imaginary valued and by the CR equations, h is a constant c and therefore

$$i(v_1 - v_2) = c \implies v_1 = v_2 + (-ic)$$

□

Example 2.2. Let $u(x,y) = \frac{1}{2}\ln(x^2+y^2)$. Note that $u_x = \frac{2x}{2(x^2+y^2)} = \frac{x}{x^2+y^2}$ and similarly $u_y = \frac{y}{x^2+y^2}$. Taking partials again,

$$u_{xx} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \implies u_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

and $u_{xx} + u_{yy} = 0$. So u is harmonic on $\mathbb{C}\backslash\{0\}$. Now define

$$v_x = -u_y = \frac{-y}{x^2+y^2}, v_y = u_x = \frac{x}{x^2+y^2}$$

and using one of our previous theorems, we can define

$$\begin{aligned} v(x,y) &= \int_1^x v_x(t,0)dt + \int_0^y v_y(x,s)ds \\ &= \int_1^x 0dt + \int_0^y \frac{x}{x^2+s^2}ds \\ &= 0 + \int_0^y \frac{1}{1+\left(\frac{s}{x}\right)^2} d\left(\frac{s}{x}\right) \\ &= \arctan\left(\frac{y}{x}\right) \end{aligned}$$

This can also be seen as

$$v(x,y) = v(z) = \arg z \quad \left(-\frac{\pi}{2} \le \arg \le \frac{\pi}{2}\right)$$

and so we define

$$f = u + iv = \ln\sqrt{x^2+y^2} + i\arctan\left(\frac{y}{x}\right) = \ln|z| + i\arg z$$

In fact, $\log(z) = \log|z| + i\arg z$. [1] Note that this function cannot be defined on the whole plane (excluding $\{0\}$) because it would not be continuous on equivalent angles like $-\frac{\pi}{2}$ and $\frac{3\pi}{2}$. So we restrict it as follows. Let

$$\begin{aligned} \Omega &= \text{right half of } \mathbb{C} \\ &= \{z \in \mathbb{C} | \Re(z) > 0\} \end{aligned}$$

and by a previous theorem, we can define $\ln z$ as a holomorphic function on this domain. Thus the condition Ω being "nice" is essential.

3 Sequences and Series

Definition 3.1. Let $\{a_n\}$ be some sequence in $\mathbb{C}$. We say that $\lim_{n\to\infty} a_n = a$ if $\forall\epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n \ge N$,

$$|a_n - a| < \epsilon$$

In this case, we say that $\{a_n\}$ converges to a. If $\{|a_n|\}$ converges, we say that a_n converges absolutely.

Remark 3.1. (1) (Cauchy condition on $\{a_n\}$. We say that $\{a_n\}$ satisfies the Cauchy condition if $\forall\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n, m \ge N$

$$|a_n - a_m| < \epsilon$$

and if $\{a_n\}$ satisfies the Cauchy condition then there is some $a \in \mathbb{C}$ such that $\lim_{n\to\infty} a_n = a$.

(2) We say the series $\sum_{n=1}^\infty a_n$ converges if $\left\{S_N = \sum_{n=1}^N s_n\right\}$ converges. We can define the absolute convergence if $\sum_{n=1}^\infty |a_n|$ converges. If $\sum a_n$ converges but not absolutely, then we say $\sum a_n$ converges conditionally.

Definition 3.2. A sequence of function $f_n : X \mapsto \mathbb{C}$ converges to f if $\forall\epsilon > 0, \exists N \in \mathbb{N}$ such that

$$\sup_{z\in X} |f_n(z) - f(z)| < \epsilon, \forall n \ge N$$

[1] Why? If $z = |z| \cdot e^{i\arg z}$ then $\ln z = \ln|z| + \ln\left(e^{i\arg z}\right) = \ln|z| + i\arg z$.

3.1 Uniform Continuity

Proposition 3.1. *The uniform limit of continuous functions is continuous.*

Proof. Let $f_n \in \mathcal{C}(X)$, the set of continuous function on X, $f_n \to f$ uniformly. $\forall z \in Z, \forall \epsilon > 0$, pick N such that $\forall n \geq N$,

$$\sup_{\omega \in X} |f_n(\omega) - f(\omega)| < \frac{\epsilon}{3}$$

and since f_N is continuous, there exists a $\delta > 0$ such that for any $\omega \in X$, we have

$$|\omega - z| < \delta \implies |f_N(\omega) - f_N(z)| < \frac{\epsilon}{3}$$

and so using the triangle inequality, we get

$$\begin{aligned} |f(\omega) - f(z)| &\leq |f(\omega) - f_N(\omega)| + |f_N(\omega) - f_N(z)| + |f(z) - f_N(z)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

□

Definition 3.3. Let $\Omega \subseteq \mathbb{C}$ open and $f_n \in \mathcal{C}(\Omega)$. We say that $f_n \to f$ uniformly converges on compact sets (U.C.C.) if $\forall k \subseteq \Omega$, $f_n\Big|_k \to f\Big|_k$ uniformly.

Remark 3.2. (1) $f_n \to f$ u.c.c. on Ω if and only if $\forall \mathcal{B}_r(z) \subseteq \Omega$, $f_n\Big|_{\overline{\mathcal{B}_r(z)}} \to f\Big|_{\overline{\mathcal{B}_r(z)}}$ uniformly.

(2) If $f_n \to f$ u.c.c. and $f_n \in \mathcal{C}(\Omega)$ then $f \in \mathcal{C}(\Omega)$

Theorem 3.1. *(Weierstrass M-Test) $\forall X \subseteq \mathbb{C}$,$u_k \in \mathcal{C}(X)$, assume $\|u_k\|_X := \sup_{z \in X} |u_k(z)| \leq M_k < \infty$ and $\sum M_k$ converges absolutely. Then $f_n = \sum_{k=1}^{n} u_k$ converges absolutely and uniformly to a continuous function $f(z)$. We denote $\sum u_k = f$.*

Proof. Since $f_n = \sum_{k=1}^{n} u_k$ are continuous, it enough to show that f_n converges uniformly to a function f. Since $\sum M_k$ converges, the sequence $\tilde{M}_n := \sum_{k=1}^{n} M_k$ converges. That is, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n > m \geq N$

$$\left|\tilde{M}_n - \tilde{M}_m\right| < \epsilon \implies \left|\sum_{k=m+1}^{n} M_k\right| < \epsilon$$

Now we have $\forall z \in X$, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n > m \geq N$

$$\left|\sum_{k=m+1}^{n} M_k\right| < \epsilon$$

and so then

$$|f_m(z) - f_n(z)| \leq \left|\sum_{k=m+1}^{n} u_k(z)\right| \leq \sum_{k=m+1}^{n} |u_k(z)| \leq \sum_{k=m+1}^{n} \|u_k\|_X \leq \sum_{k=m+1}^{n} M_k < \epsilon$$

Thus, $\{f_n(z)\}$ satisfies the Cauchy condition and $f_n(z) \to f(z)$ for some $f(z) \in \mathbb{C}$. Since from the assignment, $f_n(z) \to f(z)$ is independent off z, $f_n \to f$ uniformly. □

Remark 3.3. Let $\Omega \subseteq \mathbb{C}$ open. If $\forall \overline{\mathcal{B}_r(z)} \subseteq \Omega$ the sequence of continuous functions $u_k \subseteq \mathcal{C}(\Omega)$ has the property that

$$\|u_k\|_{\overline{\mathcal{B}_r(z)}} \leq M_{k,r,z}$$

and if $\sum M_{k,r,z}$ converges, then $\sum u_k \to f$ u.c.c. on Ω.

3.2 Power Series

Definition 3.4. A *power series* is a series of the form

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

and remark that this notation is not well defined unless we prove existence.

Example 3.1. $\sum_{k=1}^{\infty} z^k$ is a power series with equivalent value $\frac{1}{1-z}$ for $|z|<1$. If we define $f_m := \sum_{n=1}^{m} z^n = \frac{1-z^{m+1}}{1-z}$ with $|z|<1$, then

$$\lim_{m\to\infty} f_m = \lim_{m\to\infty} \frac{1-z^{m+1}}{1-z} = \frac{1}{1-z}$$

Claim 3.1. Let $\Omega = \mathcal{B}_1(0)$ and $u_n = z^n$. Then $f_m = \sum_{n=1}^{m} u_n$ u.c.c. on Ω and thus $\lim_{m\to\infty} f_m = f = \sum_{n=1}^{\infty} u_m = \sum_{n=1}^{\infty} z^n$ exists and is continuous on Ω.

Proof. Observe that any closed disk $\bar{\mathcal{B}}$ in Ω is contained in a closed disk $\overline{\mathcal{B}_r(0)}$ centered at 0. Thus, we only need to check the condition on closed disks centered at zero. Given an $0<r<1$ and let $M_{0,r,n} = r^n$. We see that $\forall z \in \mathcal{B}_r(0)$

$$|u_n(z)| = |z^n| = |z|^n \le r^n = M_{0,r,n} \implies \|u_n(z)\|_{\overline{\mathcal{B}_r(0)}} \le M_{0,r,n}$$

and since

$$\sum_{n=0}^{\infty} M_{0,r,n} = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} < \infty$$

by the Weierstrass M-Test, $\sum_{n=0}^{\infty} u_n$ exists and is equal to some continuous function f. □

Remark 3.4. On the proof above, $f(z) = \frac{1}{1-z}$ on Ω and note that $f(z)$ is unbounded on Ω by $f(z)$ is bounded on every closed disk $\overline{\mathcal{B}_r(0)}$ for $0<r<1$. Thus, it is possible that in the M-Test, f is unbounded on Ω but the corollary is still true.

Theorem 3.2. *(Hadamard Theorem) Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series. Define $R\in[0,\infty]$ by*

$$R := \frac{1}{\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}}$$

with $R=0$ if $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \infty$ and $R=\infty$ if $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = 0$.

The series

(1) converges absolutely for $z \in \mathcal{B}_R(z_0)$

(2) converges uniformly on $\overline{B_r(z_0)}$ for $0\le r<R$

(3) diverges on $\{z\in\mathbb{C} \mid |z-z_0|>R\}$

(4) cannot be said about on $z\in\partial\mathcal{B}_R(z_0)$

and this R is called the radius of convergence.

Example 3.2. Consider the power series $\sum_{n=0}^{\infty} z^n$ where $a_n = 1$, $z_0 = 0$. We have

$$R := \frac{1}{\limsup_{n\to\infty} |1|^{\frac{1}{n}}} = \frac{1}{1} = 1$$

and by the above theorem, $\sum_{n=0}^{\infty} z^n$ converges on $\mathcal{B}_1(0)$ and uniformly on $\mathcal{B}_r(0)$ for $0\le r<1$. Moreover, for $|z|>1$, $\sum_{n=0}^{\infty} z^n$ diverges (by divergence test; $\lim_{n\to\infty} z^n \ne 0$) and for $|z|=1$ also diverges by the divergence test.

Example 3.3. Consider the power series $\sum_{n=0}^{\infty} \frac{z^n}{n^2}$ where $a_n = \frac{1}{n^2}$, $z_0 = 0$. We have

$$R := \frac{1}{\limsup_{n\to\infty} \left|\frac{1}{n^2}\right|^{\frac{1}{n}}} = \frac{1}{1} = 1$$

where we use the fact that

$$\lim_{n\to\infty} \sqrt[n]{n^k} = 1, k < n$$

using the binomial theorem and $\sum_{n=0}^{\infty} \frac{z^n}{n^2}$ converges on $|z| = 1$ by the p-series test ($p = 2$).

Example 3.4. Consider the power series $\sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n}$ where $a_n = \frac{(-1)^n}{n}$, $z_0 = 0$. This series converges on $z = 1$ (alternating series) and diverges on $z = -1$ (harmonic series).

Remark 3.5. From assignments, if

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$$

exists, then $R = \frac{1}{L}$.

Example 3.5. When $a_n = \frac{1}{n!}$ then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right| = \lim_{n\to\infty} \frac{1}{n+1} = 0 = L$$

and $R = \frac{1}{L} = \infty$. So $\sum a_n$ exists on the whole of $\mathbb{R}$.

Definition 3.5. Define the complex exponential function by

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Theorem 3.3. *(a) The series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ has radius of convergence ∞ and therefore e^z is well defined on the whole $\mathbb{C}$*

(b) $e^w e^z = e^{w+z}, \forall w, z \in \mathbb{C}$

(c) $e^{x+iy} = e^x(\cos y + i \sin y)$

(d) $\overline{e^z} = e^{\bar{z}}$

(e) $|e^z| = e^{\Re(z)}$

Proof. (a) Given $a_n = \frac{1}{n!}$, we have

$$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \limsup_{n\to\infty} \frac{1}{(n!)^{\frac{1}{n}}} = \frac{1}{\infty} = 0$$

using $\forall N \in \mathbb{N}, n! \geq M_N \cdot N^n$ for all n and some $M_n \in \mathbb{R}$. So $\sqrt[n]{n!} \geq \sqrt[n]{M_N \cdot N^n} = \sqrt[n]{M_N} \cdot N$ and taking $N \to \infty$ we get $\sqrt[n]{n!} \to \infty$.

(b) By definition,

$$\begin{aligned} e^w e^z &= \left(\sum_{k=0}^{\infty} \frac{w^k}{k!}\right)\left(\sum_{l=0}^{\infty} \frac{z^l}{l!}\right) \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{w^k z^l}{k!l!} \\ &= \sum_{n=0}^{\infty}\sum_{k+l=n} \frac{w^k z^l}{k!l!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{k=0}^{n} \binom{n}{k} w^k z^{n-k}\right) \\ &= \sum_{n=0}^{n} \frac{1}{n!}(w+k)^n \\ &= e^{w+k} \end{aligned}$$

(c) By definition,

$$\begin{aligned} e^{iy} &= \sum_{n=0}^{\infty} \frac{1}{n!}(iy)^n \\ &= \sum_{k=0}^{\infty} \frac{1}{(2k)!}(iy)^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(iy)^{2k+1} \\ &= \left(\sum_{k=0}^{\infty} \frac{1}{(2k)!}(-1)^k (y)^{2k}\right) + i\left(\sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(-1)^k (y)^{2k+1}\right) \\ &= \cos y + i \sin y \end{aligned}$$

(d) By part (b) and (c),

$$e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$$

(e) By definition,

$$\overline{e^z} = \overline{\left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)} = \sum_{n=0}^{\infty} \frac{\overline{z}^n}{n!} = e^{\overline{z}}$$

(f) By defintion,

$$|e^z| = \left|e^{\Re(z)+\Im(z)}\right| = e^{\Re(z)} \underbrace{\sqrt{\cos^2(\Im(z)) + \sin^2(\Im(z))}}_{=1} = e^{\Re(z)}$$

and in particular, $\left|e^{iy}\right| = 1$. □

Remark 3.6. Define

$$\cos z := \frac{e^{iz} + e^{-iz}}{2}, \sin z := \frac{e^{iz} - e^{-iz}}{2}$$

and we can check that

$$\cos^2 z + \sin^2 z = 1$$

for all $z \in \mathbb{C}$.

Theorem 3.4. *(Differentiate term by term) If $f(z) := \sum_{n=0}^{\infty} a_n (z-z_0)^n$ has radius of convergence $R > 0$, then f is holomorphic on $\mathcal{B}_R(z_0)$ and $f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$ and has the same radius of convergence.*

Proof. We know that

$$R = \left(\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}\right)^{-1}$$

and so

$$\left(\limsup_{n\to\infty} |na_n|^{\frac{1}{n}}\right)^{-1} = \left(\lim_{n\to\infty} n^{\frac{1}{n}}\right)^{-1}\left(\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}\right)^{-1} = (1)(R) = R$$

by a theorem in one of MATH 147, MATH 148 and MATH 247. So $\sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}$ exists, but we haven't shown that $\frac{d}{dz}(f(z)) = \sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}$. To see this, let

$$g(z) = \sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}, F(z,w) = \sum_{n=1}^{\infty} a_n\left(\sum_{k=0}^{n-1} z^k w^{n-1-k}\right)$$

and we claim that F is u.c.c. on $\overline{\mathcal{B}_R(0)} \times \overline{\mathcal{B}_R(0)}$. To prove this, let r be such that

$$|z| \le r \le R, |w| \le r \le R$$

and let

$$u_n(z,w) := a_n \sum_{k=1}^{n-1} z^k w^{n-1-k}$$

and note the bound

$$\|u_n(z,w)\| \le |a_n| \sum_{k=1}^{n-1} |z|^k |w|^{n-1-k} \le n|a_n|r^{n-1}$$

with $\sum_{n=1}^{\infty} n|a_n|r^{n-1} < 0$ since the radius of convergence of $g(z)$ is R. Thus, by the M-test, $\sum u_n$ converges uniformly on $\overline{\mathcal{B}_R(0)} \times \overline{\mathcal{B}_R(0)}$ and therefore $F(z,w)$ is u.c.c. on $\overline{\mathcal{B}_R(0)} \times \overline{\mathcal{B}_R(0)}$ with the claim that $F(w,z)$ is continuous. Now observe that

$$F(z,z) = \sum_{n=1}^{\infty} na_n z^{n-1} = g(z)$$

and if $w \neq z$ then

$$\begin{aligned} F(w,z) &= \sum_{n=0}^{\infty} a_n\left(\frac{w^n - z^n}{w-z}\right) \\ &= \frac{1}{w-z}\left(\sum_{n=0}^{\infty} a_n w^n - \sum_{n=0}^{\infty} a_n z^n\right) \\ &= \frac{1}{w-z}(f(w) - f(z)) \\ &= \frac{f(w)-f(z)}{w-z} \end{aligned}$$

Since $F(z,w)$ is continuous on $\overline{\mathcal{B}_R(0)} \times \overline{\mathcal{B}_R(0)}$,

$$\lim_{w\to z} F(w,z) = F(z,z) \implies \lim_{w\to z} \frac{f(w)-f(z)}{w-z} = g(z)$$

and therefore $f(z)$ is holomorphic and $f'(z) = \sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}$. □

Corollary 3.1. $\frac{d(e^z)}{dz} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = \sum_{n=1}^{\infty} \frac{n\cdot z^n}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$.

Example 3.6. Given

$$f(z) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(z-1)^n}{n}$$

we have that

$$R = \left(\limsup_{n\to\infty}\left(\left|\frac{(-1)^{n-1}}{n}\right|\right)^{\frac{1}{n}}\right)^{-1} = \frac{1}{\lim_{n\to\infty} n^{\frac{1}{n}}} = 1$$

and so

$$f'(z) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}n(z-1)^{n-1}}{n} = \sum_{n=1}^{\infty}(-(z-1))^{n-1} = \frac{1}{1-(-(z-1))} = \frac{1}{z}$$

We hope to have $f(z) = \ln z$ for $z < |1|$. We need to only check that $f(e^z) = z$. Remark that

$$[f(e^z)]' = f'(e^z)(e^z)' = \frac{1}{e^z}\cdot e^z = 1$$

Thus, $(f(e^z) - z)' = 1 - 1 = 0$ which implies that $\exists c \in \mathbb{C}$ such that $f(e^z) = z + C$. Set $z = 0 \implies f(e^0) = f(1) = 0 + C \implies c = 0$. Therefore,

$$f(e^z) = z \implies f(z) = \ln z$$

for $z \in \mathcal{B}_1(1)$.

Corollary 3.2. *Let $f(z) = \sum_{i=1}^{\infty} a_n(z-z_0)^n$ be a power series of radius of convergence $R > 0$. Then it is $\mathcal{C}^\infty(\mathcal{B}_R(z_0))$ and $f^{(n)}(z_0) = n!a_n$.*

Proof. $f'(z) = \sum_{i=1}^{\infty} na_n(z-z_0)^{n-1}$ is a power series of radius of convergence R. Thus, $f'(z)$ is holomorphic. By induction,

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)...(n-k)a_n(z-z_0)^{n-k} \implies f^{(k)}(z_0) = k!a_n$$

□

Remark 3.7. If $f(z)$ is holomorphic and $f(z) = \sum_{i=1}^{\infty} a_n(z-z_0)^n$ on $\mathcal{B}_R(z_0)$ then $a_n = \frac{f^{(n)}(z_0)}{n!}$.

4 The Extended Complex Plane

Definition 4.1. Define $C_\infty = \mathbb{C}_\infty = \mathbb{C}\cup\{\infty\}$ is called the *extended complex plane*. We say $\{a_n\} \subseteq \mathbb{C}$ converges to ∞ if and only if for any $r > 0$, $\exists N \in \mathbb{N}$ such that $|a_n| > r$, $\forall n \geq N$ and also a function $f : C_\infty \to C_\infty$ is continuous at ∞ if

$$\lim_{n\to\infty} a_n \to \infty \implies f(a_n) \text{ converges to } f(\infty)$$

4.1 S^3 Riemann Sphere

Definition 4.2. Define the *Riemann sphere* as the set $S^3 = S^{(3)} := \{(a,b,c) \in \mathbb{R}^3 | a^2 + b^2 + c^2 = 1\}$.

Problem 4.1. We want to identify C_∞ with S^3.

Definition 4.3. In C_∞, a ball around ∞ is $\mathcal{B}_r(\infty) = \{z \in \mathbb{C} | |z| > r\} \cup \{\infty\}$. We can project from S^3 to C_∞ as follows, using the North pole, $N = (0,0,1)$, as a pivot.

$$\Pi((0,0,1)) = \Pi(N) = \infty, \Pi((a,b,c)) = \frac{a+bi}{1-c}$$

this follows from the fact that the line through (a,b,c) is

$$t(a,b,c) + (1-t)(0,0,1) = (ta, tb, tc + (1-t))$$

and setting $tc+(1-t)=0 \implies (1-c)t=1 \implies t=\frac{1}{1-c}$. So $x=ta=\frac{a}{1-c} \implies y=tb=\frac{b}{1-c}$ and $\Pi((a,b,c))=x+iy=\frac{a+bi}{1-c}$. From the picture in class, it is obvious that Π is 1-1 and onto. However, we write Π^{-1} explicitly. For $z=x+iy$ and $\Pi^{-1}(z)=(a,b,c)$. Recall that

$$\begin{aligned}\overrightarrow{NZ} &= \{t(x,y,0)+(1-t)(0,0,1)|t\in\mathbb{R}\}\\ &= \{(tx,ty,1-t)|t\in\mathbb{R}\}\end{aligned}$$

and

$$\begin{aligned}(tx)^2+(ty)^2+(1-t)^2=1 &\implies t(tx^2+ty^2-2+t)=0\\ &\implies t=0,\frac{2}{1+x^2+y^2}\\ &\implies t=\frac{z}{1-|z|^2}\end{aligned}$$

since $t=0$ is the north pole. Thus,

$$\Pi^{-1}(z)=\left(\frac{zx}{1+|z|^2},\frac{zy}{1+|z|^2},\frac{|z|^2-1}{1+|z|^2}\right)$$

Summary 1. To sum up, $\Pi:S^3\mapsto C_\infty$ is defined by

$$\Pi((a,b,c))=\begin{cases}\Pi(N)=\infty & c=1\\ \frac{a+bi}{1-c} & c\neq 1\end{cases}$$

and $\Pi^{-1}:C_\infty\mapsto S^3$ by

$$\Pi^{-1}(\infty)=N,\Pi^{-1}(z)=\left(\frac{2\Re(z)}{1+|z|^2},\frac{2\Im(z)}{1+|z|^2},\frac{|z|^2-1}{1+|z|^2}\right)$$

where it is easy to check that Π and Π^{-1} are continuous.

Theorem 4.1. *Π preserves circles and angles. That is, it is a conformal mapping. The circles on C_∞ includes the lines which are circles through ∞.*

Proof. We know that Π is bijective and

$$z=u+iv=\Pi\left(\frac{2u}{1+|z|^2},\frac{2v}{1+|z|^2},\frac{|z|^2-1}{1+|z|^2}\right)$$

where a circle on S^3 is intersection of a plane

$$\alpha a+\beta b+\gamma c=\delta$$

and S^3 where $\alpha,\beta,\gamma,\delta\in\mathbb{R}$. Thus, the projection of the circle satisfies

$$\begin{aligned}\alpha\left(\frac{2u}{1+|z|^2}\right)+\beta\left(\frac{2v}{1+|z|^2}\right)+\gamma\left(\frac{|z|^2-1}{1+|z|^2}\right) &= \delta\\ 2\alpha u+2\beta v+(u^2+v^2-1) &= \delta(u^2+v^2+1)\\ (r-\delta)(u^2+v^2)+2\alpha u+2\beta v &= r+\delta\end{aligned}$$

Case 1. If $r=\delta$, then

$$2\alpha u+2\beta v=r+\delta \implies \alpha u+\beta v=\delta \implies \gamma=0$$

which is a line (circle through ∞). This is the circle through through the north pole.

Case 2. If $r\neq\delta$,

$$u^2+v^2+\left(\frac{2\alpha}{r-\delta}\right)u+\left(\frac{2\beta}{r-\delta}\right)v=\frac{r+\delta}{r-\delta}$$

which is a circle on $\mathbb{C}$. □

Definition 4.4. A mapping $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ is said to be conformal (preserving angles) if for any two curves, c_1 and c_2, through a point p with angle θ, then $f(c_1)$ and $f(c_2)$ has the angle θ at $f(p)$.

Example 4.1. Let $f : \Omega \mapsto \mathbb{C}$ be a holomorphic function. Then, f is conformal.

Proof. Recall that f is holomorphic if and only if locally, f behaves like a translation, dilation, and a rotation and therefore it is conformal. $\square$

Remark 4.1. Let $c_1, c_2 \subseteq S^3$ through $(a, b,)$ with angle θ. Recall that the angle only depends on the tangent line of c_1 and c_2 at (a, b, c). Thus, we can replace c_1, c_2 by two circles with the tangent lines. Moreover, we can choose the circles through N. The angles between c_1 and c_2 is the same as the angle through N. By the definition of projection, Π preserves angles at N and since Π maps circles to circles, it preserves the angle at (a, b, c).

4.2 Möbius Mappings

Definition 4.5. A *fractional linear transformation* or a *Möbius map* is a function $T : \mathbb{C}_\infty \mapsto \mathbb{C}_\infty$ by

$$T(z) := \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$ and

$$T(\infty) = \begin{cases} \frac{a}{c} & c \neq 0 \\ \infty & c = 0 \end{cases}, T\left(-\frac{d}{c}\right) = \infty \text{ if } c \neq \infty$$

Remark 4.2. T is continuous on $\mathbb{C}_\infty$.

Proposition 4.1. *If* $T(z) := \frac{az+b}{cz+d}$ *and* $S(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$ *then*

$$(T \circ S)(z) = \frac{Az+B}{Cz+D} \text{ where } \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

Proof. By direct evaluation,

$$\begin{aligned} (T \circ S)(z) &= \frac{a\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)+b}{c\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)+d} \\ &= \frac{a(\alpha z+\beta)+b(\gamma z+\delta)}{c(\alpha z+\beta)+d(\gamma z+\delta)} \\ &= \frac{(a\alpha+br)z+(a\beta+b\delta)}{(c\alpha+dr)z+(c\beta+d\delta)} \\ &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{aligned}$$

where $A = a\alpha + br$, $B = a\beta + b\delta$, $C = c\alpha + dr$, and $D = c\beta + d\delta$ as above. $\square$

Remark 4.3. To make the Möbius maps non-trivial, we add an extra condition that

$$\det \begin{pmatrix} a & c \\ c & d \end{pmatrix} = ad - bc \neq 0$$

Corollary 4.1. *Any Möbius map is 1-1 and onto.*

Proof. For ∞ points, it is left as an exercise. Let $T = \frac{az+b}{cz+b}$. Then $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ and $S = \frac{\alpha z+\beta}{\gamma z+\delta}$ is the inverse of T. Thus, T is 1-1 and onto on $\mathbb{C}_\infty$. $\square$

Fact 4.1. *The group*

$$GL(2,\mathbb{C}) := \textit{the set of } 2{\times}2 \textit{ invertible matrices with } \mathbb{C}\textit{coefficients}$$

is generated by three kinds of matrices, namely

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}_{dilation}, \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}_{translation}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{rotation}$$

That is, any invertible matrix A can be written as a product of those three kinds of matrices.

Example 4.2. Recall that in Linear Algebra any invertible matrix is row (column) [RREF form and CREF] equivalent to $id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. That is, there is a set of row and column operations in $GL(2,\mathbb{C})$ such that they can reduce any invertible matrix into the identity matrix.

Definition 4.6. The cross ratio of z_1, z_2, z_3 and z_4 is defined by

$$(z_1, z_2, z_3, z_4) := \left(\frac{z_1 - z_3}{z_2 - z_3}\right)\left(\frac{z_2 - z_4}{z_1 - z_4}\right)$$

Remark 4.4. Let

$$S(z) = (z, z_2, z_3, z_4) = \left(\frac{z - z_3}{z_2 - z_3}\right)\left(\frac{z_2 - z_4}{z - z_4}\right)$$

Then S is a Möbius map provided that $z_i \neq z_j$, $2 \leq i < j \leq 4$. Also note that $S(z_2) = 1$, $S(z_3) = 0$ and $S(z_4) = \infty$.

Lemma 4.1. *If a Möbius map has three fixed points, i.e. $\exists z_1, z_2, z_3$ all different points such that $T(z_1) = z_1$, $T(z_2) = z_2$ and $T(z_3) = z_3$ then $T = id \implies T(z) = z$.*

Proof. Let $T = \frac{az+b}{cz+b}$. Then $z = T(z) = \frac{az+b}{cz+b} \iff cz^2 + dz = az + b \iff cz^2 + (d-a)z - b = 0$ provided that $z \neq \infty$. For $z = \infty$, then

$$\infty = T(\infty) = \begin{cases} \frac{a}{c} & c \neq 0 \\ \infty & c = 0 \end{cases}$$

which implies that ∞ is a fixed point.

Case 1: $T(\infty) = \infty$

If $z \neq \infty$ and z is a fixed point, $c = 0$ and $(d-a)z - b = 0 \implies (d-a)z = b$. Since T has three different fixed points, $(d-a)z = b$ has at least two solutions. From Linear Algebra, $d = a, b = 0$ and $T(z) = \frac{az+b}{cz+d} = \frac{az+0}{0+a} = z$.

Case 2: $T(\infty) \neq \infty$

In this case, $c \neq \infty$. However, the quadratic equation $cz^2 + (d-a)z - b = 0$ has at most two different solutions which is a contradiction. Hence only case 1 is valid and $T(z) = z$. □

Corollary 4.2. *If T and S are two Möbius mappings and they take the same values at three different points, then $T = S$. In particular, $S(z) = (z, z_2, z_3, z_4)$ is the only Möbius map that sends $S(z_2) = 1$, $S(z_3) = 0$, $S(z_4) = \infty$.*

Proof. Suppose that $\exists z_2, z_3, z_4 \in \mathbb{C}$ all different such that $T(z_2) = S(z_2)$, $T(z_3) = S(z_3)$, and $T(z_4) = S(z_4)$. Then $T^{-1}S(z) = z$ for $z \in \{z_2, z_3, z_4\}$. By the above lemma, $T^{-1}S$ must be the identity and therefore $T = S$. □

Corollary 4.3. *For any Möbius map S we have*

$$S(z) = (z, S^{-1}(1), S^{-1}(0), S^{-1}(\infty))$$

Proposition 4.2. *The Möbius maps preserves the cross ratio. That is, for any $z_1, z_2, z_3, z_4 \in \mathbb{C}$ all different, then*

$$(z_1, z_2, z_3, z_4) = (T(z_1), T(z_2), T(z_3), T(z_4))$$

for any Möbius map T.

Proof. Let $S(z) = (z, z_2, z_3, z_4)$ and $A = S \circ T^{-1}$. Remark that $A(T(z_2)) = S(z_2) = 1$, $A(T(z_3)) = S(z_3) = 0$, $A(T(z_4)) = S(z_4) = \infty$. and so

$$A(z) = (z, T(z_2), T(z_3), T(z_4))$$

and

$$A(T(z_1)) = (T(z_1), T(z_2), T(z_3), T(z_4)) = S(z_1) = (z_1, z_2, z_3, z_4)$$

and thus they are equal. □

Proposition 4.3. *Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$ be four distinct points. Then z_1, z_2, z_3, z_4 lie on a circle if and only if $(z_1, z_2, z_3, z_4) \in \mathbb{R}_\infty$.*

Proof. Let $S(z) = (z, z_2, z_3, z_4)$ be a Möbius mapping which has the property that $S(z_2) = 1$, $S(z_3) = 0$, $S(z_4) = \infty$, and $\mathbb{R}_\infty$ be the circle $\mathbb{R} \cup \{\infty\}$ (real line). Assume that $S^{-1}(\mathbb{R}_\infty)$ is a a circle. Then if z_1, z_2, z_3, z_4 lie on a circle, then that circle must be $S^{-1}(\mathbb{R}_\infty)$ and therefore, $S(z_1) \in \mathbb{R}_\infty$.

On the other hand, if $(z_1, z_2, z_3, z_4) \in \mathbb{R}_\infty$ (i.e. $S(z_1) \in \mathbb{R}_\infty$) then $z_1 \in S^{-1}(\mathbb{R}_\infty)$ which is the circle through z_2, z_3, z_4. Thus, z_1, z_2, z_3, z_4 lie on a circle. Thus, it is enough to show that $S^{-1}(\mathbb{R}_\infty)$ is a circle. To show this, let $z \in S^{-1}(\mathbb{R}_\infty)$. Then it is equivalent to $S(z) \in \mathbb{R}_\infty$ and $S(z) = \overline{S(z)}$. Thus, the necessary and sufficient condition for $z \in S^{-1}(\mathbb{R}_\infty)$ is

$$S(z) = \frac{az+b}{cz+d} = \overline{\left(\frac{az+b}{cz+d}\right)} = \overline{S(z)}$$

and expanding we get

$$\frac{az+b}{cz+d} = \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}} \iff (a\bar{c} - \bar{a}c)|z|^2 + (a\bar{d} - \bar{b}c)z + (b\bar{c} - \bar{a}d)\bar{z} + (b\bar{d} - \bar{b}d) = 0$$

Let $\alpha = a\bar{d} - \bar{b}c, \beta = b\bar{d}$.

Case 1: $a\bar{c}$ is real. That is $a\bar{c} - \bar{a}c = 0$ and we get the equation $(\alpha z + \beta) - \overline{(\alpha z + \beta)} = 0 \implies 2\Im(\alpha z + \beta) = 0$ which is a line.

Case 2: $a\bar{c} - \bar{a}c \neq 0$. For some $\gamma, \delta \in \mathbb{C}$, the equation becomes $|z|^2 + \bar{\gamma}z + \gamma\bar{z} - \delta = 0$. Note that $|z+\gamma|^2 = (z+\gamma)\overline{(z+\gamma)} = |z|^2 + \bar{\gamma}z + \gamma\bar{z} + |\gamma|^2$ and thus the equation can be re-written as $|z+\gamma|^2 = |\gamma|^2 + \delta = R^2$ where

$$R^2 = \left|\frac{a\bar{d} - \bar{b}c}{a\bar{c} - \bar{a}c}\right|^2 + \frac{\bar{b}d - b\bar{d}}{a\bar{c} - \bar{a}c} = \left|\frac{ad - bc}{a\bar{c} - \bar{a}c}\right|^2 > 0$$

Thus, $S^{-1}(\mathbb{R}_\infty)$ is a circle whose center $-\gamma$ and radius $R > 0$. □

Corollary 4.4. *(i) Möbius mappings preserve circles.*

(ii) Any circle can be taken to any other circle by a Möbius map.

Proof. (i) Let the circle C be the circle through z_2, z_3, z_4 where $z \in C \iff (z, z_2, z_3, z_4) \in \mathbb{R}_\infty$. Let T be any Möbius map and $\tilde{C}$ be the circle through $T(z_2), T(z_3), T(z_4)$. We need to show $T(z) \in \tilde{C}$. However, $T(z) \in \tilde{C} \iff (T(z), T(z_2), T(z_3), T(z_4)) \in \mathbb{R}_\infty \iff (z, z_2, z_3, z_4) \in \mathbb{R}_\infty \iff z \in C$.

(ii) Let C_1, C_2 be two circles on $\mathbb{C}_\infty$. Let $z_2, z_3, z_4 \in C_1$ be distinct. Then $S(z) = (z, z_2, z_3, z_4)$ sends C_1 to $\mathbb{R}_\infty$. On the other hand, if $z_2', z_3', z_4' \in C_2$ are distinct, the $S'(z) = (z, z_2', z_3', z_4')$ sends C_2 to $\mathbb{R}_\infty$. Thus, $(S')^{-1}S$ sends C_1 to C_2. □

Theorem 4.2. *(Done in Assignments) Let T be a Möbius mapping and $\mathbb{D} := \overline{\mathcal{B}_1(0)}$. Then $T(\mathbb{D}) = \mathbb{D}$ if and only if $T(z) = e^{i\theta}\left(\frac{z-w}{1-\bar{w}z}\right)$ where $\theta \in \mathbb{R}$, $|w| < 1$ for some θ and w.*

Proposition 4.4. *Let T be a Möbius map and $\mathbb{D} := \overline{\mathcal{B}_1(0)} = \{z \in \mathbb{C} : |z| \leq 1\}$. Then $T(\mathbb{D}) = \mathbb{D}$ if and only if $\exists w \in \mathbb{C}$, $|w| < 1$, $\theta \in \mathbb{R}$ such that*

$$T(z) = e^{i\theta}\left(\frac{z-w}{1-\bar{w}z}\right)$$

Proof. ($\Longleftarrow$) If $|z| = 1$ then $z\bar{z} = 1 \iff \bar{z} = \frac{1}{z}$ and so if $|z| = 1$ then

$$|T(z)| = \left|e^{i\theta}\right| \left|\frac{z-w}{1-\bar{w}z}\right| = \left|\frac{z-w}{1-\frac{\bar{w}}{\bar{z}}}\right| = |\bar{z}| \left|\frac{z-w}{\bar{z}-\bar{w}}\right| = |\bar{z}| \left|\frac{z-w}{\overline{z-w}}\right| = 1 \cdot 1 = 1$$

so T sends the unit circle to the unit circle. One the other hand, T^{-1} sends the unit circle to the unit circle and $\forall |z| < 1, \left|T^{-1}(z)\right| < 1$ and thus $T(\mathbb{D}) = \mathbb{D}$.

The other direction is an assignment question. □

Remark 4.5. In fact, if T, a Möbius map, sends a circle to a circle, then T will send the interior disk of the circle to either the interior of the image circle or the exterior part of the circle. It is followed by this assignment. Since we can tranfer the unit circle to any circle by a Möbius map.

The stronger version of the assignments is that if T sends the unit circle to the unit circle, then $T = e^{i\theta}\left(\frac{1-\bar{w}z}{z-w}\right)$, $|w| < 1$ and $T(\mathbb{D}) = \overline{\mathbb{D}^c}$.

5 Line Integrals

Theorem 5.1. *(Recall: Local Primitive Theorem) Let $\Omega \subseteq \mathbb{C}$ be "nice" and $f : \Omega \mapsto \mathbb{C}$ holomorphic. Then $\exists h : \Omega \mapsto \mathbb{C}$ holomorphic such that $h' = f$.*

Definition 5.1. If $\gamma : [a, b] \mapsto \mathbb{C}$ is a piecewise $\mathcal{C}^1$ curve (i.e. $r(t) = u(t) + iv(t)$ where u, v are continuous and differentiable, continuous derivative, except on finitely many points). Define f continuous and

$$\oint_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt$$

Let $\Im(\gamma) := \{\gamma(t) \in \mathbb{C} | a \le t \le b\} = \gamma^*$ where $\gamma^* \subseteq \Omega$ is an open subset of $\mathbb{C}$ and $f : \Omega \mapsto \mathbb{C}$ is continuous.

Proposition 5.1. *(Re-parametrization of γ^* does not change the integral)*

(a) If $\phi : [c, d] \mapsto [a, b]$ is an increasing, piecewise-$\mathcal{C}^1$ function and $\phi(x) = a$, $\phi(d) = b$. Let $\tilde{\gamma} = \gamma \circ \phi : [c, d] \mapsto \gamma^*$. Then $\oint_{\tilde{\gamma}} f(z)\, dz = \oint_\gamma f(z)\, dz$.

(b) If $\gamma : [0, 1] \mapsto \mathbb{C}$ piecewise-$\mathcal{C}^1$ and $\tilde{\gamma}(t) = \gamma(1-t)$ then $\oint_{\tilde{\gamma}} f(z)\, dz = -\oint_\gamma f(z)\, dz$

Proof. (a) We have

$$\oint_{\tilde{\gamma}} f(z)\, dz = \int_c^d f(\tilde{\gamma}(t))\tilde{\gamma}'(t)\, dt = \int_c^d f(\gamma \circ \phi(t))(\gamma \circ \phi(t))'\, dt = \int_c^d f(\gamma \circ \phi(t))\gamma' \circ \phi(t)\phi'(t)\, dt$$

and set $s = \phi(t)$ and $ds = \phi'(t)\, dt$. Thus,

$$\oint_{\tilde{\gamma}} f(z)\, dz = \int_a^b f(\gamma(s))\gamma'(s)\, ds = \oint_\gamma f(z)\, dz$$

□

(b) We have

$$\oint_{\tilde{\gamma}} f(z)\, dz = \int_0^1 f(\tilde{\gamma}(t))\tilde{\gamma}'(t)\, dt = \int_0^1 f(\phi(1-t))(\phi(1-t))'\, dt = \int_0^1 f(\phi(1-t))\phi(1-t)(-1)\, dt$$

and so with $s = 1 - t$ we have

$$\oint_{\tilde{\gamma}} f(z)\ dz = -\int_0^1 f(\gamma(s))\gamma'(s)\ ds = -\oint_{\gamma} f(z)\ dz$$

Remark 5.1. (1) If we use the arc-length parametrization of γ^* then the line integral will be be unique. Thus, we denote

$$\oint_{\gamma^*} f(z)\ dz$$

be the line integral by arclength parametrization.

(2) The line integral only depends on γ^* and its direction (multiplicity).

Definition 5.2. Let $|\gamma^*| =$ the length of $\gamma = \int_a^b |\gamma'(t)|\ dt$.

Proposition 5.2. *If* $\left|\oint_\gamma f(z)\ dz\right| \leq \|f\|_\infty |\gamma^*|$ *where* $\|f\|_\infty := \sup_{z\in\gamma^*} |f(z)|$

Proof. Trivial. □

Example 5.1. Let $\mathbb{D}_r = \overline{\mathcal{B}_r(0)}$, $\partial\mathbb{D}_r = \{re^{i\theta} | 0 \leq \theta \leq 2\pi\}$, and $\gamma : [0, 2\pi] \mapsto \partial\mathbb{D}_r \implies \gamma : \theta \mapsto re^{i\theta}$. Let $f(z)$ be some holomorphic function on $\partial\mathbb{D}_r$. Then

$$\oint_\gamma f(z)\ dz = \int_0^{2\pi} f(re^{i\theta})(ir^{i\theta})\ d\theta$$

Problem 5.1. Let $f(z) = z^n$ for $n \in \mathbb{N}$. What is $\oint_\gamma z^n\ dz$? We will show that

$$\oint_\gamma z^n\ dz = \begin{cases} -0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}$$

Proof. We have

$$\oint_\gamma z^n\ dz = \int_0^{2\pi} (re^{i\theta})^n rie^{i\theta}\ d\theta = \int_0^{2\pi} (ir^{n+1})e^{i(n+1)\theta}\ d\theta$$

Case 1: $n \neq 1$

By direct computation,

$$\oint_\gamma z^n\ dz = (ir^{n+1})\frac{1}{i(n+1)}e^{i(n+1)\theta}\Big|_0^{2\pi} = \frac{r^{n+1}}{n+1}\left(e^{2\pi i(n+1)} - e^0\right) = \frac{r^{n+1}}{n+1}(1-1) = 0$$

Case 2:

Also by direct computation,

$$\oint_\gamma z^{-1}\ dz = i\int_0^{2\pi} 1\ d\theta = 2\pi i$$

□

Definition 5.3. A piecewise $\mathcal{C}^1$ curve $\gamma : [a, b] \mapsto \mathbb{C}$ is a *closed curve* if $\gamma(a) = \gamma(b)$.

Theorem 5.2. *Let $\Omega \subseteq \mathbb{C}$ be "nice" and $f : \Omega \mapsto \mathbb{C}$ holomorphic. Let $\gamma : [a, b] \mapsto \mathbb{C}$ be a piecewise $\mathcal{C}^1$ curve. Then there exists a holomorphic function $h : \Omega \mapsto \mathbb{C}$ independent of r such that*

$$\oint_\gamma f(z)\ dz = h(\gamma(b)) - h(\gamma(a))$$

Proof. Since Ω is nice, by the local primitive theorem, there exists $h : \Omega \mapsto \mathbb{C}$ holomorphic such that $h' = f$. So we ha e

$$\oint_\gamma f(z)\, dz = \int_a^b h'(\gamma(t))\gamma'(t)\, dt = \int_a^b (h(\gamma(t))' = h(\gamma(t))\Big|_a^b = h(\gamma(b)) - h(\gamma(a))$$

□

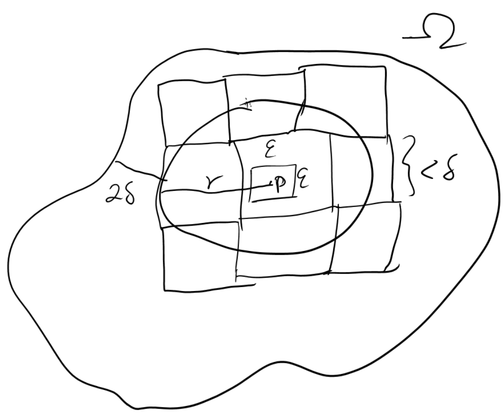
5.1 Cauchy Integral Formula

Corollary 5.1. *(Cauchy Integral Formula v0)*

As above, if γ is a closed curve, then $\oint_\gamma f(z)\, dz = 0$.

Theorem 5.3. *Let $\Omega \subseteq \mathbb{C}$ be open and $f : \Omega \mapsto \mathbb{C}$ holomorphic, bounded on $\Omega \backslash \{p\}$, $p \in \Omega$. Let $\gamma(t) = p + re^{it}, 0 \le t \le 2\pi$ and $\gamma^* =$ Image$(\gamma) \subseteq \Omega$. Then $\oint_\gamma f(z)\, dz = 0$.*

Proof. Let $\delta = \frac{1}{2}\text{dist}(\mathcal{B}_r(p), \mathbb{C}\backslash\Omega)$ and $M = \sup|f(z)|_{z\in\Omega\backslash\{p\}} < \infty$. We would like to break γ into several pieces in the following way. We cover $\mathcal{B}_r(p)$ by squares with side less than δ.



So $\gamma = (\sum_i r_i) + r_\epsilon$ where $r_i = \partial(\text{square}_i \cap \mathcal{B}_r(p))$. Thus, we have

$$\left|\oint_\gamma f(z)\, dz\right| = \sum_i \left|\oint_{r_i} f(z)\, dz\right| + \left|\oint_{r_\epsilon} f(z)\, dz\right| \le 0 + 4\epsilon M$$

Taking $\epsilon \to 0$ we get $\oint_\gamma f(z)\, dz = 0$. □

Lemma 5.1. *Let $\gamma : [0, 2\pi] \mapsto \mathbb{C}$, $\gamma(t) = p + re^{it}$, $0 \le t \le 2\pi$. For all $a \in \mathbb{C}\backslash\gamma^*$ where $\gamma^* =$ Image(γ), we have*

$$\oint_\gamma \frac{dw}{w-a} = \begin{cases} 0 & |p-a| > r \\ 2\pi i & |p-a| < r \end{cases}$$

Proof. Case 1: $|p - a| > r$

$\frac{1}{w-a}$ is holomorphic on $\overline{\mathcal{B}_r(p)}$ and by the theorem $\oint_\gamma \frac{1}{w-a}$.

Case 2: $|p - a| < r$

Define $f(z) = \oint_\gamma \frac{dw}{w-z}$ with

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}\left(\oint_\gamma \frac{dw}{w-z}\right) = \oint_\gamma \left[\frac{\partial}{\partial \bar{z}} \frac{1}{w-z}\right] = \oint_\gamma 0\, dw = 0$$

Thus, f is holomorphic on $\mathcal{B}_\gamma(p)$. Now,

$$f' = \frac{df}{dz} = \oint_\gamma \frac{d}{dz}\left(\frac{1}{w-z}\right)\, dw = \oint_\gamma \frac{1}{(w-z)^2}\, dw$$

and when we break γ such that $\gamma = \sum \gamma_i$ with there existing a ball $\mathcal{B}_{a_r}(\gamma_i)$ such that $\gamma^* \subseteq \mathcal{B}_{a_r}(\gamma_i)$ and $\frac{1}{(w-z)^2}$ is holomorphic on each $\mathcal{B}_{a_r}(\gamma_i)$. We then have

$$\begin{aligned} f'(z) &= \oint_\gamma \frac{1}{(w-z)^2} dw \\ &= \sum \oint_{\gamma_i} \frac{1}{(w-z)^2} dw \\ &= \sum [g(\gamma_i(1)) - g(\gamma_i(0))] = 0 \end{aligned}$$

by telescoping where $g(z) = -\frac{1}{w-z}$. Thus, $f(z) = c$ for some constant $c \in \mathbb{C}$. Pick $z = p$, $f(p) = 2\pi i$ and $f(z) = 2\pi i$. To sum up, $\oint_\gamma \frac{dw}{w-z} = 2\pi i$ for $|p-a| < r$. □

Theorem 5.4. *(Cauchy Integral Formula v1)*

Let $f : \Omega \mapsto \mathbb{C}$ *holomorphic and* $p \in \overline{\mathcal{B}_r(p)} \subseteq \Omega$, $\gamma(t) = p + re^{it}$, $0 \le t \le 2\pi$. *Then,* $\forall \omega \in \mathcal{B}_r(p)$,

$$f(w) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-w}\, dz$$

Proof. Let

$$g(z) = \begin{cases} \frac{f(z)-f(w)}{z-w} & z \neq w \\ f'(w) & z = w \end{cases}$$

Since f is holomorphic on Ω, g is continuous on Ω and holomorphic on $\Omega \backslash \{w\}$. Note that $g(z)$ is bounded on $\overline{\mathcal{B}_r(p)}$ and since g is continuous and $\overline{\mathcal{B}_r(p)}$ then $\overline{\mathcal{B}_r(p)}$ is compact. Now,

$$\begin{aligned} 0 = \oint_\gamma g(z)\, dz &= \oint_\gamma \frac{f(z)-f(w)}{z-w}\, dz \\ &= \oint_\gamma \frac{f(z)}{z-w}\, dz - f(w) \oint_\gamma \frac{1}{z-w}\, dz \\ &= \oint_\gamma \frac{f(z)}{z-w}\, dz - 2\pi i \cdot f(w) \implies f(w) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-w}\, dz \end{aligned}$$

□

Remark 5.2. The Cauchy integral formula tells us that for a holomorphic function $f(z)$ on $\overline{\mathcal{B}_r(p)}$, its values are uniquely determined by its values on $\partial\mathcal{B}_r(p)$. In fact, for a harmonic function, the same thing happens.

Corollary 5.2. *Let* f *be a holomorphic function on* $\Omega \subseteq \mathbb{C}$ *open and* $z \in \Omega$, $z \in \overline{\mathcal{B}_r(z)} \subseteq \Omega$. *Then,*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z + re^{i\theta})\, d\theta$$

We call this the Mean Value Property of Holomorphic Functions.

Proof. Take $z = p$ and $\gamma(\theta) = p + re^{i\theta}, 0 \le \theta \le 2\pi$

$$f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(z+re^{i\theta})}{re^{i\theta}}\, d(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(z+re^{i\theta})}{re^{i\theta}} \left(re^{i\theta}\right)\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})\, d\theta$$

□

Corollary 5.3. *(Max Principle) In the same setting as Cauchy's integral formual, with an extra condition that Ω is connected, then the maximum will not occur in Ω unless $f(z)$ is a constant function.*

Proof. It is enough to show that if $\exists z \in \Omega$ such that $|f(z_0)| \geq |f(z)|$ $\forall w \in \Omega$ then $f(z)$ is a constant. Let $z \in \Omega$. Since Ω is connected, there exists $\gamma : [a,b] \mapsto \Omega$ such that $\gamma(a) = z_0$ and $\gamma(b) = z$. Let $\zeta := \frac{1}{2}\text{dist}(\gamma^*, \mathbb{C}\backslash\Omega)$. Then we can cover γ^* by finitely many disks of the form $\mathcal{B}_\zeta(w)$, $w \in \gamma^*$. Thus, it is sufficient to show $f(z) = f(z_0)$ in each $\mathcal{B}_\zeta(w)$. But, we can assume that if we prove that f is constant on $\mathcal{B}_\zeta(z_0)$ then the overlaps on the balls will transfer over and will be constant on all of γ^*.

Claim 1. For any circle $\partial\mathcal{B}_r(z_0)$, $0 < r \leq \zeta$, $f(w) = f(w')$ for $w, w' \in \partial\mathcal{B}_r(z_0)$.

By the mean value property (MVP),

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta$$

if there exists θ_0 such that

$$\left|f(z_0 + re^{i\theta_0})\right| < |f(z_0)|$$

By continuity of $f(z)$, there is $\theta_1, \theta_2 \in \mathbb{R}$ such that $\theta_1 \leq \theta \leq \theta_2$ and $|(f(z_0) + re^{i\theta})| < |f(z_0)|$. Now by the MVP

$$\begin{aligned} |f(z_0)| &= \left|\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta\right| \\ &\leq \left|\frac{1}{2\pi}\int_{[0,2\pi]\backslash[\theta_1,\theta_2]} f(z_0 + re^{i\theta})\right| + \left|\frac{1}{2\pi}\int_{\theta_1}^{\theta_2} f(z_0 + re^{i\theta})\right| \\ &< \frac{1}{2\pi}(2\pi - (\theta_2 - \theta_1))\,|f(z_0)| + \frac{1}{2\pi}(\theta_1 - \theta_2)\,|f(z_0)| = |f(z_0)| \end{aligned}$$

and so we have a contradiction. Thus, $|f(z_0)| = |f(w)|$ for all $w \in \partial\mathcal{B}_r(z_0)$. In particular, $\forall w \in \mathcal{B}_\zeta(z_0)$ $|f(w)| = |f(z_0)|$.

Claim 2. If $|f(w)|$ is a constant on $\mathcal{B}_\zeta(z_0)$ then $f(w)$ is a constant. (Midterm)

Thus, the max principle is true. □

Theorem 5.5. *(Power series expansion for holomorphic functions)*

If f is holomorphic on $\mathcal{B}_R(z_0)$ then there exists a power series

$$g(z) := \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

with radius of convergence greater or equal to R such that $g(z) = f(z)$ on $\mathcal{B}_R(z_0)$. Hence $f(z)$ is analytic and $\mathcal{C}^\infty$. Note that $a_n = \frac{1}{n!}f^{(n)}(z_0)$.

Proof. Let $0 < r < R$ and $\gamma(t) = z_0 + re^{it}$. Then by the Cauchy Integral Formula,

$$f(w) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z - w} dz,\ |w - z_0| < r$$

and

$$\frac{1}{z - w} = \frac{1}{(z - z_0) - (w - z_0)} = \frac{1}{z - z_0} \cdot \frac{1}{1 - \left(\frac{w - z_0}{z - z_0}\right)} = \sum_{n=0}^{\infty} \frac{(w - z_0)^n}{(z - z_0)^{n+1}}$$

which converges absolutely for $(w - z_0) < r$ and uniformly for $(w - z_0) < r_1 < r$. Thus,

$$\begin{aligned} f(w) &= \frac{1}{2\pi i}\oint_\gamma f(z)\left(\sum_{n=0}^{\infty} \frac{(w - z_0)^n}{(z - z_0)^{n+1}}\right) dz,\ |w - z_0| < r \\ &= \sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\right](w - z_0)^n\, dz \end{aligned}$$

and taking $a_n = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz$, if $|f(z)| \le M < \infty$ - which is possible since f is continuous on a compact domain γ^* - $|z-z_0| = r$ then $|a_n| \le \frac{M}{(2\pi)\cdot r^{n+1}} \cdot (2\pi r) \le \frac{M}{r^n}$. Thus,

$$\left(\limsup \sqrt[n]{(a_n)}\right)^{-1} \ge \left(\limsup \sqrt[n]{\frac{M}{r^n}}\right)^{-1} = r\cdot 1 = r$$

and if we let $r \to R$ we get

$$f(w) = g(w) = \sum_{n=0}^{\infty} a_n (w-z_0)^n, a_n = \frac{1}{n!} f^{(n)}(z_0)$$

where the radius at convergence of $g(w)$ is at least R. □

Corollary 5.4. *If f is holomorphic on an open set $\Omega \subseteq \mathbb{C}$ then f' is also holomorphic. Thus, $f \in \mathcal{C}^\infty(\Omega)$.*

Proof. Since locally, f has a power series expansion, f' also has a power series expansion and therefore f' is holomorphic. □

Example 5.2. For $f(z) = \ln z = \ln|z| + \arg z$ on a cut plane $\mathbb{C}\backslash - \mathbb{R}_+ = \{z \in \mathbb{C}, z \ne r, r \in \mathbb{R}, r \le 0\}$ where if the centering is at $z = 1$ then

$$\ln z = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(z-1)^n}{n}, |z-1| < 1$$

By the theorem, we can also get a power series expansion of $\ln z$ at a far away point with large radius of convergence.

Corollary 5.5. *(Cauchy's Estimate) If f is holomorphic on $\mathcal{B}_R(z_0)$ and $0 < r < R$, $M(r) := \sup_t \left|f(z_0 + re^{it})\right|$ then*

$$f^{(n)}(z_0) \le \frac{n! M(n)}{r^n}, \forall n \ge 0$$

Definition 5.4. A function $f(z)$ is called entire if $f(z)$ is holomorphic on the whole complex plane $\mathbb{C}$.

Theorem 5.6. *(Liouville's Theeorem) Any bounded entire function is a constant.*

Proof. Assume that $|f(z)| \le M$ for any $z \in \mathbb{C}$. By Cauchy's estimate $(n = 1)$,

$$f'(0) \le \frac{M}{r}, \forall r \in \mathbb{R}^+$$

Taking $r \to \infty$, $f'(0) = 0$. (Similary, $f^{(n)}(0) = 0, \forall n$). Similarly $f'(z) = 0, \forall z \in \mathbb{C} \implies f(z)$ is constant. □

Corollary 5.6. *(Fundamental Theorem of Algebra) Any non-constant polynomial has a roof in $\mathbb{C}$.*

Proof. Let $p(z)$ be a polynomial which is non-contant. Assume that $p(z)$ has no root in $\mathbb{C}$. Consider $f(z) = \frac{1}{p(z)}$. Then $f(z)$ is entire $(f'(z) = -\frac{p'(z)}{p(z)}$ exists). We claim that $f(z)$ is bounded. To see this consider

$$\left|\frac{p(z)}{z^n}\right| = \left|a_n + \sum_{i=0}^{n-1} \frac{a_i}{z^{n-i}}\right|$$

As $z \to \infty$, $\left|\frac{p(z)}{z^n}\right| \to a_n$. Thus, $\exists R > 0$ such that

$$\left|\frac{p(z)}{z^n}\right| > \frac{1}{2}|a_n| \text{ for } |z| > R$$

and

$$|f(z)| \le \frac{2}{|a_n||z|^n} \le \frac{2}{|a_n|R^n} \text{ for } |z| > R$$

On the other hand, since $\overline{\mathcal{B}_r(0)}$ is compact and $f(z)$ is continuous on $\mathcal{B}_r(0)$ there is some $\tilde{M} > 0$ such that $|f(z)| \leq \tilde{M}$ and $z \in \overline{\mathcal{B}_r(0)} \iff |z| \leq R$. To sum up,

$$|f(z)| \leq \max\left(\tilde{M}, \frac{2}{|a_n| \cdot R^n}\right)$$

and so it is bounded. By Liouville's theorem, $f(z)$ must be a constant and so is $p(z)$ which is impossible. Hence $p(z)$ has a root. □

5.2 Zero Sets

Lemma 5.2. *(Isolated zeroes) If f is holomorphic on an open set U and $\mathcal{B}_r(a) \subseteq U$, $f(a) = 0$ and $f \neq 0$.Then there exists $m \in \mathbb{N}$ - called the order of zero at a - such that $f(z) = (z-a)^m g(z)$, $g(a) \neq 0$ and $g(z)$ is holomorphic on U.*

In particular, $\exists 0 < \delta < r$ such that $g(z) \neq 0$ on $\mathcal{B}_\delta(a)$ and therefore $f(z) \neq 0$ on $\mathcal{B}_\delta(a) \backslash \{a\}$.

Proof. Since $f(z)$ is holomorphic on $\mathcal{B}_r(a)$ and by assumption, let

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

for $z \in \mathcal{B}_r(a)$. Assume that $a_0, a_1, ..., a_{m-1}$ are zeroes and $a_m \neq 0$ since $f \neq 0$. Thus,

$$\begin{aligned} f(z) = \sum_{n=m}^{\infty} a_n(z-a)^n &= a_m(z-a)^m + a_{m+1}(z-a)^{m+1} \\ &= (z-a)^m \left(a_m + a_{m+1}(z-a) + a_m(z-a)^2 + ...\right) \end{aligned}$$

and if we let $\tilde{g}(z) = a_m + a_{m+1}(z-a) + a_m(z-a)^2 + ... = \sum_{n=0}^{\infty} a_{m+n}(z-a)^n$ for $|z-a| < r$. Then $\tilde{g}(a) \neq 0$ and it is holomorphic on $\mathcal{B}_r(a)$. Now define

$$g(z) = \begin{cases} \frac{f(z)}{(z-a)^m} & z \neq a \\ \tilde{g}(a) & z = a \end{cases}$$

and we have that $g(z)$ is holomorphic on $\mathcal{B}_r(a)$ since $g(z) = \tilde{g}(z)$ on $\mathcal{B}_r(a)$. For $z \in U \backslash \{a\}$,

$$g'(z) = \frac{f'(z) \cdot (z-a)^m - m(z-a)^{m-1} f(z)}{(z-a)^{2m}}$$

exists and hence $g(z)$ is holomorphic on U. □

Definition 5.5. Let S be a subset of $\mathbb{R}^n$. We say p is a cluster point of S if $\exists \{s_n\}_{n=1}^{\infty} \subseteq S$ such that $\lim_{n\to\infty} s_n = p$ where $p \notin \{s_n\}_{n=1}^{\infty}$. That is, p is NOT isolated in S.

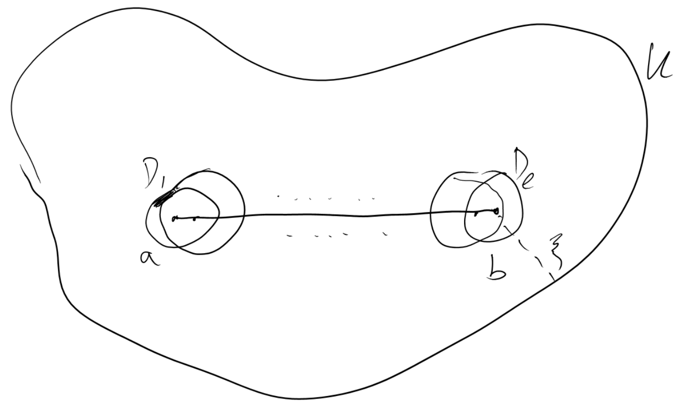
Theorem 5.7. *(Characterization of zero sets) Let $U \subseteq \mathbb{C}$ be a "connected" open subet and $f : U \mapsto \mathbb{C}$ holomorphic. TFAE:*

1. *$f = 0$ the zero function*
2. *$Z(f) = \{z \in U, f(z) = 0\}$ has a cluster point in U*
3. *$\exists a \in U$ such that $f^{(n)}(a) = 0$ or all $n \geq 0$.*

Proof. $1 \implies 2$, $3 \implies 2$, and $1 \implies 3$ are obvious.

$2 \implies 1$: Let a be a cluster point at $Z(f)$ and $a \in U$. By continuity, $f(a) = 0$ and since a is a cluster point of $Z(f)$, $\forall \delta > 0$, $\mathcal{B}_\delta(a) \cap Z(f) \neq 0$. By the theorem of isolation of zeroes, we know that $f(z)$ must be zero on $\mathcal{B}_\delta(a)$ for some $\delta > 0$. Let $b \in U$. Since U is connected, there exists a piecewise $\mathcal{C}^1$ path from a to b, say γ from a to b, for some $b \in U$.

Let $\xi = \frac{1}{4}\text{dist}(\gamma^*, \mathbb{C}\backslash U)$ and we cover γ^* by disks of radius ξ and center on γ. Since γ^* is compact, we can cover γ^* with finitely many such disks. Note that by the choice of ξ, all such disks are contained in U. We label these disks by $D_1, D_2, ..., D_l$ with the property that the center of D_i lies in D_{i-1}, the center of D_1 is a and the center of D_l is b.



We know that $f(z) = 0$ on D_1. That is, from the proof of the theorem, we know $f(z)$ will be zero on $\mathcal{B}_r(a)$ as long as $\mathcal{B}_r(a) \subseteq U$. If is enough to show that if $f(z) = 0$ on D_i then this implies $f(z) = 0$ on D_{i+1}. Consider the Taylor series expansion of $f(z)$ at the center c_{i+1} of D_{i+1}. Since $c_{i+1} \in D_i$ then for all $n \geq 0$, $f^{(n)}(c_{i+1}) = 0$.

Thus, $f(z) = 0$ on any $\mathcal{B}_r(c_{i+1})$ as long as $\mathcal{B}_r(c_{i+1}) \subseteq U$. In particular, $D_{i+1} \subseteq U$ and therefore $f(z) = 0$ on D_{i+1}. By induction, f is 0 on all the balls covering γ^*. □

Corollary 5.7. *If $U \subseteq C$, open, connected, and $\lim_{n\to\infty} z_n = z_0$ and $z_i \neq z_0$ $\forall i$, then for any two holomorphic function f, g on U such that $f(z_i) = g(z_i)$ $\forall i$ we have $f(z) = g(z)$, for all $z \in U$.*

Proof. The holomorphic function $f(z) - g(z)$ must be zero since its zero set has a cluster point z_0 in U. Thus, $f(z) = g(z)$ for any $z \in U$. □

Example 5.3. Let $f(z) = e^{-\frac{1}{z}} - 1$ on $U = \mathbb{C}\backslash\{0\}$. We see that $z_n = \frac{1}{2\pi i n} \to 0$ and $f(z_n) = 0$. Since $0 \notin U$ the theorem's condition does not hold.

Theorem 5.8. *(Schwarz' Lemma) Let f be holomorphic on $\mathbb{D} = \{z : |z| < 1\}$ and $f(0) = 0$, $f(\mathbb{D}) \subseteq \mathbb{D}$. Then $|f(z)| \leq |z|$ for $z \in \mathbb{D}$ and $|f'(0)| \leq 1$. Moreover if $\exists 0 \neq z_0 \in \mathbb{D}$ such that $|f(z_0)| = |z_0|$ then $f(z| = e^{i\theta} z$ for some $\theta \in \mathbb{R}$.*

Proof. Since f is holomorphic on $\mathbb{D}$, we know that $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $|z| < 1$. Moreover, $f(0) = 0 \implies a_0 = 0$. In particular,

$$g(z) = \frac{f(z)}{z} = a_1 + a_2 z + a_3 z^2 ...$$

is holomorphic on $\mathbb{D}$. Since $f(\mathbb{D}) \subseteq \mathbb{D} \implies |f(z)| < 1, \forall z \in \mathbb{D}$,

$$\sup_{|z|\leq r} |g(z)| = \sup_{|z|\leq r} \frac{|f(z)|}{|z|} \leq \frac{1}{r}$$

and taking $r \to 1$ we get

$$\sup_{|z|<1} |g(z)| = 1 \implies \left|\frac{f(z)}{z}\right| \leq 1 \implies |f(z)| \leq |z|$$

for any $0 < r < 1$. So

$$f'(0) = \lim_{z\to 0} \frac{f(z) - 0}{z - 0} = \lim_{z\to 0} \frac{f(z)}{z} \implies |f'(0)| \leq \lim_{z\to 0} \left|\frac{f(z)}{z}\right| \leq 1$$

If $|f(z_0)| = |z_0|$ for some $z_0 \in \mathbb{D}$, then $g(z) = c$ for some constant c. So $f(z) = cz$. In particular, $c = \frac{f(z_0)}{z_0} = e^{i\theta}$ form some $\theta \in \mathbb{R}$. □

Theorem 5.9. *(Homework) Let $f : \mathbb{D} \mapsto \mathbb{D}$ holomorphic one-to-one and onto. Then f is a Möbius map. In particular, $\exists a \in \mathbb{D}$, $\theta \in \mathbb{R}$ such that*

$$f(z) = e^{i\theta} \frac{z-a}{1-\bar{a}z}$$

Proof. Hint: Let $f(a) = b \in \mathbb{D}$. We can find a Möbius map T such that $|T(z)| = 1$ for all $|z| = 1$ and $T(b) = 0$. Then $\tilde{f} = T \circ f$ and $\tilde{f}$ is one-to-one and onto on $\mathbb{D}$ with $\tilde{f}(0) = 0$. Apply Schwartz's lemma. □

Theorem 5.10. *(Morera's Theorem) If f is defined and continuous on an open set $U \subseteq \mathbb{C}$ and $\oint_{\partial R} f(z)\, dz = 0$ around every rectangle R in U, then f is holomorphic.*

Proof. Since holomorphic is a local property, we can assume that $U = \mathcal{B}_r(p)$. Recall the proof of the local primitive theorem. Define

$$g(z) := \int_{x_0}^{x} f(t + iy_0)dt + \int_{y_0}^{y} f(z + is)ds$$

where $z = x+iy$ and $p = x_0+iy_0$. We would like to show that $g'(z) = f(z)$. If so, $g(z)$ is holomorphic and all of its derivatives are holomorphic. In particular, $f(z) = g'(z)$ is holomorphic. Choose $\epsilon > 0$ and let $z_1 = x_1 + iy_1 \in \mathcal{B}_r(p)$ and $h = h_1 + ih_2$ with $\|h\| = \sqrt{h_1^2 + h_2^2}, |f(z_1 + h) - f(z_1)| \leq \epsilon$. Then since

$$z_1 + h = x + h_1 + i\,(y_1 + ih_2)$$

then

$$\begin{aligned} \left|\frac{g(z_1+h) - g(z_1)}{h}\right| &= \left|\frac{1}{h}\left(\int_{x_1}^{x_1+h_1} f(t+iy_1) - f(z_1)dt + i\int_{y_1}^{y_1+h_2} f(x+h_1+is) - f(z_1)ds\right)\right| \\ &\leq \frac{1}{\|h\|}\left(\left|\int_{x_1}^{x_1+h_1} \epsilon\, dt\right| + \left|\int_{y_1}^{y_1+h_1} \epsilon\, ds\right|\right) \\ &\leq \epsilon \cdot \frac{|h_1| + |h_2|}{\sqrt{h_1^2 + h_2^2}} \leq 2\epsilon \end{aligned}$$

Thus, $g'(z) = f(z)$ and we are done. □

Theorem 5.11. *(Goursad's Theorem) If f is differentiable at every point in an open set U then f is holomorphic.*

Proof. By Movera's theorem, it suffices to show that $\oint_{\partial R} f(z)dz = 0$ for any rectangle $R \subseteq U$. Decompose our rectangle into 4 subrectangles:

$$\partial R = \partial R_1 + \partial R_2 + \partial R_3 + \partial R_4$$

where we use arc-length parametrization and counter-clockwise orientation. Thus, $\oint_{\partial R} f(z)dz = \sum_{i=1}^{4} \oint_{\partial R_i} f(z)dz$. We choose one of $\oint_{\partial R_i} f(z)dz$ which has the biggest absolute value, say

$$R = \left|\oint_{\partial R_1} f(z)dz\right| \geq \left|\oint_{\partial R_i} f(z)dz\right| \implies \left|\oint_{\partial R_1} f(z)dz\right| \geq \frac{1}{4}\left|\oint_{\partial R} f(z)dz\right|$$

We can repeat this process. Find $Q_2, Q_3, ...$ such that $Q_2 \supsetneq Q_3 \supsetneq Q_4 \supsetneq ...$ such that

$$\text{diam}(Q_n) = 2^{-n}\text{diam}(R)$$

where $\text{diam}(R) = \sup(z_1 - z_2)$ for $z_1, z_2 \in R$ and

$$\left|\oint_{\partial Q_n} f(z)dz\right| \geq \frac{1}{4^n}\left|\oint_{\partial R} f(z)dz\right|$$

Let

$$z_0 = \bigcap_{n\in\mathbb{N}} Q_n = \lim_{n\to\infty} Q_n \in U$$

Since f is differentiable, we have

$$f(z_0 + h) = f(z_0) + f'(z_0)h + \phi(h)$$

where $\phi(h)$ satisfies the property that $\forall \epsilon > 0$, $\exists \delta > 0$ such that for $|h| < \delta$, $|\phi(h)| \leq \epsilon |h|$. If $\text{diam}(Q_n) < \delta$ we have

$$\begin{aligned}
\left| \oint_{\partial R_n} f(z)dz \right| &= \left| \oint_{\partial R_n} f(z_0)(z - z_0) + \phi(z - z_0)dz \right| \\
&= \left| \oint_{\partial R_n} \phi(z - z_0)dz \right| \\
&\leq |\partial R_n| \sup_{z \in R_n} |\phi(z - z_0)| \\
&\leq 2^{-n}|\partial R| \left(\underbrace{2^{-n}\text{diam} R}_{h} \right) \epsilon \\
&= 4^{-n}|\partial R| \cdot (\text{diam} R)\epsilon
\end{aligned}$$

and so taking $\epsilon \to 0$ we get

$$\oint_{\partial R} f(z)dz = 0$$

□

Corollary 5.8. *(Morera's second theorem) If f_n is holomorphic on Ω open and $f_n \to f$ u.c.c. then f is holomorphic.*

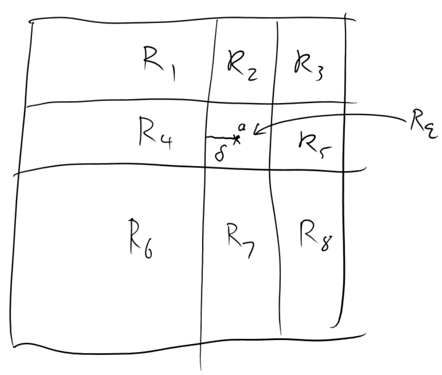
Proof. Let R be a rectangle in Ω. then

$$\oint_{\partial R} f(z)dz = \lim_{n\to\infty} \oint_{\partial R} f_n(z)dz = \lim_{n\to\infty} 0 = 0$$

and so f is holomorphic. □

Corollary 5.9. *If f is holomorphic on $\Omega \backslash \{a\}$ and continuous on Ω, open, then f is holomorphic on Ω.*

Proof. Let R be a rectangle in Ω. If $a \notin R$ then $\oint_{\partial R} f(z)dz = 0$ by Morera's theorem. If $a \in R$ we can divide R into several rectangles as follows:



and so $\partial R = \sum_{i=1}^{\infty} \partial R_i + (\partial R_\epsilon)$ where $\text{diam}(R_\epsilon) = \delta$ where δ is the number such that $|f(z) - f(a)| \leq \epsilon$ for any $|z - a| < \delta$.

We thus have

$$\begin{aligned}\left|\oint_{\partial R} f(z)dz\right| &= \left|\sum_{i=1}^{\infty}\oint_{\partial R_i} f(z)dz + \oint_{\partial R_\epsilon} f(z)dz\right| \\ &= \left|\oint_{\partial R_\epsilon} f(z)dz\right| \\ &= \left|\oint_{R_\epsilon} (f(z)-f(a)) + \oint_{\partial R_\epsilon} f(a)dz\right| \\ &\leq |\epsilon|\cdot 4|\delta| \to 0\end{aligned}$$

as $\epsilon \to 0$ and by Morera's Theorem, f is holomorphic. □

Theorem 5.12. *If f is holomorphic on Ω and $\overline{\mathcal{B}_r(p)} \subseteq \Omega$ then $\forall n \geq 0$, $a \in \mathcal{B}_r(p)$ we have*

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{(z-a)^{n+1}}dz, \gamma(t) = p + re^{it}$$

Corollary 5.10. *If f is holomorphic and $f_n \to f$ u.c.c. on Ω then $f_n^{(k)} \to f^{(k)}$ u.c.c. on Ω for any $k \geq 0$.*

Proof. (of Corollary) We have

$$f^{(k)}(a) = \frac{k!}{2\pi i}\oint_\gamma \frac{f(z)}{(z-a)^{k+1}}dz = \lim_{n\to\infty}\frac{k!}{2\pi i}\oint_\gamma \frac{f_n(z)}{(z-a)^{k+1}}dz = \lim_{n\to\infty} f_n^{(k)}(a)$$

since γ is compact. □

Example 5.4. If $f_n = \frac{1}{n}\sin(n^2 x)$ then $f_n \to f = 0$. However $f_n(x) = n\cos(n^2 x)$.

6 Complex Topology

Fact 6.1. *Let Ω be an open subset of $\mathbb{C}$. If Ω is connected (i.e. $\Omega = U_1 \cup U_2$, U_1, U_2 open then $U_1 = \emptyset$ or $U_2 = \emptyset$), Ω is path connected (i.e. $\forall a, b \in U$, $\exists \gamma :]0,1] \mapsto \Omega$ piecewise $\mathcal{C}^1$,$\gamma(0) = a, \gamma(1) = b$).*

Definition 6.1. A connected open set $\Omega \subseteq \mathbb{C}$ is *simply connected* if any closed curve in U is homotopic to a point. We say that two curves γ_0, γ_1 are *homotopic* on Ω if $\exists \Gamma : [0,1] \times [0,1] \mapsto \Omega$ such that $\Gamma(0,t) = \gamma_0(t)$, $\Gamma(1,t) = \gamma_1(t)$ and Γ is continuous and $\gamma_s(t) = \Gamma(s,t)$ is a continuous family of curves on Ω, $0 \leq s \leq 1$. If $\gamma_1(t) = a$ for some $a \in \Omega$. We say γ_0 is homotopic to a point a.

Exercise 6.1. (Homework) A star shaped region is simply connected. A torus is not simply connected.

6.1 Winding Numbers

Theorem 6.1. *(Winding Number) Let γ be a piecewise $\mathcal{C}^1$ closed functions in $\mathbb{C}$. Set $\Omega = \mathbb{C}\backslash\gamma^*$. Define*

$$Ind_\gamma(w) = \frac{1}{2\pi i}\oint_\gamma \frac{dz}{z-w}$$

Then Ind$_\gamma(w)$ is a continuous, integral-valued function. Hence it is a constant on each connected component, (i.e. max connected subsets in Ω are called connected components) and furthermore it is zero on the unbounded component.

Remark 6.1. If γ_0, γ_1 are closed curves, we make an extra condition on Γ which is $\gamma_s(t) = \Gamma(s,t)$ are closed curves for all $0 \leq s \leq 1$.

Proof. (Winding number) Let $\gamma : [a,b] \mapsto \mathbb{C}$ with $\gamma(a) = \gamma(b)$. Fix $w \in \Omega$ and define

$$g(t) := \int_a^t \frac{\gamma'(s)}{\gamma(s) - w}$$

with $g(a) = 0$, $g(b) = (2\pi i)\operatorname{Ind}_\gamma(w)$. Let $f(t) = e^{-g(t)}(\gamma(t) - w)$ and remark that

$$f'(t) = e^{-g(t)}(-g'(t)(\gamma(t) - w) + \gamma'(t)) = e^{-g(t)}\left(-\frac{\gamma'(t)}{\gamma(t) - w}(\gamma(t) - w) + \gamma'(t)\right) = 0$$

and $f(t)$ is a constant. Now

$$f(a) = f(b) \implies e^{-g(a)}(\gamma(a) - w) = e^{-g(b)}(\gamma(b) - w) \implies 1 = e^{-g(b)} \implies -g(b) = (2\pi i) \cdot m$$

for some $m \in \mathbb{Z}$. Thus $\operatorname{Ind}_\gamma(w) = -m \in \mathbb{Z}$. To show continuity for $w \in \Omega$ let $\delta = \operatorname{dist}(w, \gamma^*) > 0$, $\epsilon > 0$, $\epsilon < \frac{\delta}{2}$. For $|\tilde{w} - w| < \epsilon < \frac{\delta}{2}$ we have

$$\left|\frac{1}{z - w} - \frac{1}{z - \tilde{w}}\right| = \left|\frac{w - \tilde{w}}{(z - w)(z - \tilde{w})}\right| \le \frac{\epsilon}{\delta \cdot \frac{\delta}{2}} = \frac{2\epsilon}{\delta}$$

and hence

$$|\operatorname{Ind}_\gamma(w) - \operatorname{Ind}_\gamma(\tilde{w})| \le \left|\frac{1}{2\pi i}\oint_\gamma \left(\frac{1}{z - w} - \frac{1}{z - \tilde{w}}\right) dz\right| \le \frac{1}{2\pi}|\gamma^*| \cdot \frac{2\epsilon}{\delta^2} \to 0$$

Thus, $\operatorname{Ind}_\gamma(z)$ is continuous and $\operatorname{Ind}_\gamma(z)$ is constant on each connected component. If $w \in$ the unbounded component, let $|w| = R \to \infty$ then

$$|\operatorname{Ind}_\gamma(w)| \le \left|\frac{1}{2\pi i}\oint_\gamma \frac{1}{|z - w|} dz\right| \le \frac{1}{2\pi}|\gamma^*| \cdot \frac{1}{R - L} \to 0$$

as $R \to \infty$ when $L = \sup_t |\gamma(t)|$ and so $\operatorname{Ind}_\gamma(w) = 0$. □

Definition 6.2. Two closed curves γ_1 and γ_2 in an open set Ω are *homologous* in Ω, denoted by by $\gamma_1 \approx \gamma_2$ if $\operatorname{Ind}_{\gamma_1}(w) = \operatorname{Ind}_{\gamma_2}(w)$ for all $w \notin \Omega$.

Definition 6.3. A *cycle* $\gamma = \gamma_1 + \dots + \gamma_n$ is a union of finitely many piecewise $\mathcal{C}^1$ closed curves $\gamma_1, \dots, \gamma_n$. Note that for any f,

$$\oint_\gamma f(z)dz := \sum_{i=1}^n \oint_{\gamma_i} f(z)dz$$

We say that a cycle γ is *homologous to zero,* denoted by $\gamma \approx 0$ in Ω if $\operatorname{Ind}_\gamma(w) = \sum_{i=1}^n \operatorname{Ind}_{\gamma_i}(w) = 0$ for all $w \notin \Omega$.

6.2 General Cauchy Integral Formula

Theorem 6.2. *(General Cauchy Integral Formula) If γ is a cylce in Ω, $\gamma \approx 0$ in Ω, and f is holomorphic on Ω then*

$$\oint_\gamma f(z)dz$$

Remark 6.2. (1) In the old setting, $\Omega = \mathcal{B}_R(p), \gamma(t) = p + re^{it}, r < R$ and we recover the previous Cauchy Integral Formula.

(2) In the old setting, the Cauchy integral formula also works for a rectangle (square). That is

$$\frac{1}{2\pi i}\oint_{\partial R} \frac{f(z)}{z - w} dw = \begin{cases} f(w) & w \in R \\ 0 & w \notin R \end{cases}$$

(3) (Corollary) If f is holomorphic on Ω and a cycle $\gamma \approx 0$ in Ω then $\forall w \notin \gamma^*$,

$$\operatorname{Ind}_\gamma(w) \cdot f(w) = \oint_\gamma \frac{f(z)}{z - w} dw$$

Proof. (of above Corollary) Let

$$g(z) = \begin{cases} \frac{f(z)-f(w)}{z-w} & z \neq w \\ f'(w) & z = w \end{cases}$$

which is holomorphic on Ω. So

$$0 = \frac{1}{2\pi i}\oint_\gamma f(z)dz = \frac{1}{2\pi i}\left[\left(\oint_\gamma \frac{f(z)}{z-w}dz\right) - \left(\oint_\gamma \frac{f(w)}{z-w}dz\right)\right]$$

and we have

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-w}dz = f(w)\left(\frac{1}{2\pi i}\oint_\gamma \frac{1}{z-w}dz\right) = \text{Ind}_\gamma(w)\cdot f(w)$$

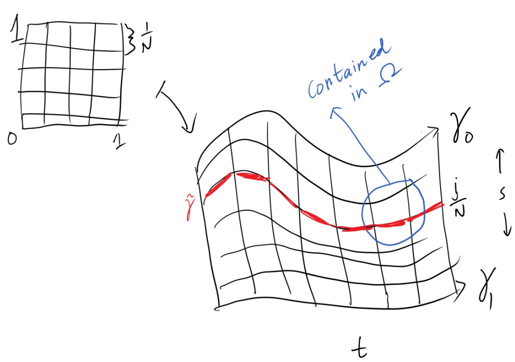
□

Theorem 6.3. *If γ_0, γ_1 are closed curves in Ω and $\gamma_0 \sim \gamma_1$ in Ω then $\gamma_0 \approx \gamma_1$. In particular, if γ_1 is homotopic to a point and $\gamma_0 \sim \gamma_1$ then $\gamma_0 \approx 0$.*

Proof. (By picture) The idea is to approximate $\gamma_s = \Gamma(s,t)$ by polygonal paths. Recall that $\Gamma : [0,1]\times[0,1] \mapsto \Omega$ is a homotopy from γ_0 to γ_1. Since Γ is continuous and $[0,1]\times[0,1]$ is compact, dist(Image(Γ), $\mathbb{C}\backslash\Omega$) $= \epsilon > 0$. Since Γ is continuous, there is some $\delta > 0$ such that

$$\max(|s_1 - s_2|, |t_1 - t_2|) < \delta \implies |\Gamma(s_1,t_1) - \Gamma(s_2,t_2)| < \frac{\epsilon}{2}$$

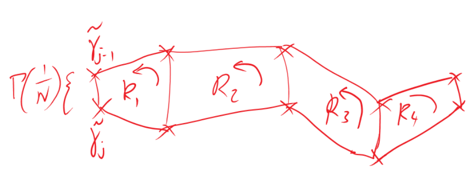
Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\epsilon}{2}$. Chop $[0,1]\times[0,1]$ into a grid $\left\{\left(\frac{j}{N}, \frac{k}{N}\right) : 0 \leq j,k \leq N\right\}$. For all $0 \leq j \leq N$ define $\tilde{\gamma}$ as a curve linking $\Gamma\left(\frac{j}{N}, \frac{0}{N}\right), \Gamma\left(\frac{j}{N}, \frac{1}{N}\right), ..., \Gamma\left(\frac{j}{N}, \frac{N}{N}\right)$. By the choices of $N(\delta, \epsilon, ...)$ we know that $\forall 1 \leq j,k \leq N-1$ the curve linking $\Gamma\left(\frac{j}{N}, \frac{k}{N}\right), \Gamma\left(\frac{j-1}{N}, \frac{k-1}{N}\right), \Gamma\left(\frac{j-1}{N}, \frac{k}{N}\right), \Gamma\left(\frac{j}{N}, \frac{k-1}{N}\right)$ lies in Ω, contained in a vall with radius $\frac{\epsilon}{2}$.



It is enough to show that

$$\begin{aligned}\text{Ind}_{\gamma_0}(w) = \frac{1}{2\pi}\oint_{\gamma_0}\frac{1}{z-w}dz = \frac{1}{2\pi}\oint_{\tilde{\gamma}_0}\frac{1}{z-w}dz &= \frac{1}{2\pi}\oint_{\tilde{\gamma}_j}\frac{1}{z-w}dz, 0 \leq j \leq N \\ &= \frac{1}{2\pi}\oint_{\gamma_1}\frac{1}{z-w}dz = \text{Ind}_{\gamma_1}(w)\end{aligned}$$

All of the proofs of equalities are the same as follows. Look at $\tilde{\gamma}_{j-1}$ and $\tilde{\gamma}_j$



This implies that

$$\sum_{i=1}^{4}\frac{1}{2\pi i}\oint_{\partial R_i}\frac{1}{z-w}dz=0=\frac{1}{2\pi i}\oint_{\tilde{\gamma}_j}\frac{1}{z-w}dz-\frac{1}{2\pi i}\oint_{\tilde{\gamma}_{j-1}}\frac{1}{z-w}dz \implies \frac{1}{2\pi i}\oint_{\tilde{\gamma}_j}\frac{1}{z-w}dz=\frac{1}{2\pi i}\oint_{\tilde{\gamma}_{j-1}}\frac{1}{z-w}dz$$

□

Theorem 6.4. *If $\gamma \approx 0$ then $\oint_\gamma f(z)dz = 0$.*

Proof. Since γ^* is compact, bounded by R, we may assume that Ω is bounded (or we can replace Ω by $\Omega \cap \mathcal{B}_R(0)$). Note that $\text{Ind}_\gamma(w) = 0$ for $|w| > R$. Let $\text{dist}(\gamma^*, \mathbb{C}\backslash\Omega) = \delta > 0$. We can cover $\mathbb{C}$ by a grid of horizontal and vertical lines separated by $\frac{\delta}{4}$. Take Q_i be the $\frac{\delta}{4} \times \frac{\delta}{4}$ closed square completely contained in Ω. For some index set I and $i \in I$ let $G = \bigsqcup_{i\in I} Q_i$ and $\Gamma = \sum_{i\in I} \partial Q_i = \partial G$ (directed boundaries). Note that $\gamma^* \subseteq G$.

We claim that $\gamma \approx 0$ in G. To see this, observe that $\forall w_0 \in \Omega\backslash G$ lies in a square Q not entirely contained in Ω. Thus, $\exists w_1 \in \mathbb{C}\backslash\Omega$ and $w_1 \in Q$. Then $\overline{w_0 w_1} \subseteq Q$ (line segment). Let $w_t = w_0(1-t) + w_1 t$. since $\text{Ind}_\gamma(w_1) = 0$ and $\text{Ind}_\gamma(w_t)$ is a continuous $\mathbb{Z}-$valued function, $\text{Ind}_\gamma(w_0) = 0$ and $\gamma \approx 0$ in G.

For $i \in I$, $\overline{Q_i} \subseteq \Omega$, by Cauchy's Integral Formula for squares,

$$\frac{1}{2\pi i}\oint_{\partial Q_i}\frac{f(z)}{z-w}dz = \begin{cases} f(w) & w \in Q_i \\ 0 & w \notin Q_i \end{cases}$$

Thus, let $\Gamma = \partial G = \sum_{i\in I} \partial Q_i$ and observe that $\forall w \in G, w \notin \Gamma^*$,

$$\frac{1}{2\pi i}\oint_{\Gamma}\frac{f(z)}{z-w}dz = \sum_{i\in I}\frac{1}{2\pi i}\oint_{\partial Q_i}\frac{f(z)}{z-w}dz = f(w)$$

for any $w \in \bigcup_{i\in I} Q_i^0$ where $Q_i^0 = Q_i\backslash\partial Q_i$. For $w \notin \Gamma$,

$$\frac{1}{2\pi i}\oint_{\Gamma}\frac{f(w)}{z-w}dz$$

is a continuous function in w. Thus,

$$\frac{1}{2\pi i}\oint_{\Gamma}\frac{f(z)}{z-w}dz = f(w)$$

for all $w \in G^0 = G\backslash\Gamma$. Now,

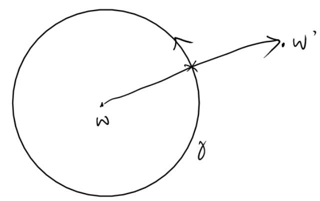
$$\begin{aligned}\oint_\gamma f(w)dw &= \oint_\gamma \left(\frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{z-w}dz\right)dw \\ &= \oint_\Gamma f(z)\left(\frac{1}{2\pi i}\oint_\gamma \frac{1}{z-w}dw\right)dz \\ &= \oint_\Gamma f(z)\left(\underbrace{-\text{Ind}_\gamma(z)}_{=0}\right)dz = 0\end{aligned}$$

□

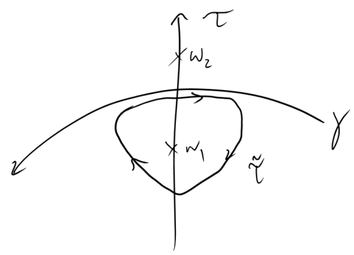
6.3 Computing Winding Numbers

Proposition 6.1. *Let γ be a regular piecewise $\mathcal{C}^1$ cycle in $\mathbb{C}$. That is, $\gamma'(t) \neq 0$ for all t. Let $w \in \mathbb{C}\backslash\gamma^*$ and τ be a piecewise $\mathcal{C}^1$ curve connecting w to a point w' which belongs to an unbounded component of $\mathbb{C}\backslash\gamma^*$. Then $\text{Ind}_\gamma(w) = n_+ - n_-$ where $n_+(n_-)$ ounts the number of crossings where γ crosses τ from right to left (left to right). These crossing must be transversal (not tangent to the curve)*

Example 6.1. Let $\gamma(t) = re^{it}$ with $0 \le t \le 2\pi$. Choose w and w' appropriately. Then following the orientation of w to w' gives us $\text{Ind}_\gamma(w) = 1$.



Remark 6.3. We know that $\text{Ind}_\gamma(\tau(t))$ only changes when τ crosses r. As in the picture, we create a "half-circle" closed curve $\tilde{\tau}$ which contains w_1 with the different orientation of γ on their common parts.



$\gamma + \tilde{\tau}$ is a new cycle such that w_1 and w_2 are on the same connected component of $\mathbb{C} \backslash (\gamma + \tilde{\tau})^*$. Thus,

$$\text{Ind}_{\gamma+\tilde{\tau}}(w_1) = \text{Ind}_{\gamma+\tilde{\tau}}(w_2)$$

and

$$\begin{aligned} \text{Ind}_{\gamma+\tilde{\tau}}(w_1) &= \text{Ind}_\gamma(w_1) + \text{Ind}_{\tilde{\tau}}(w_1) = \text{Ind}_\gamma(w_1) - 1 \\ \text{Ind}_{\gamma+\tilde{\tau}}(w_2) &= \text{Ind}_\gamma(w_2) + \text{Ind}_{\tilde{\tau}}(w_2) = \text{Ind}_\gamma(w_2) \end{aligned}$$

Thus $\text{Ind}_\gamma(w_2) + 1 = \text{Ind}_\gamma(w_1)$.

Example 6.2. Let $\Omega = \mathbb{C} \backslash \{0\}$, $\gamma_0(t) = \gamma(t)e^{i\theta(t)}$,$\gamma(0) = \gamma(1)$ with $\theta(1) = \theta(0) + 2\pi k$ for $k \in \mathbb{Z}$ and $0 \le t \le 1$.

We can see that $\gamma_0 \sim e^{i\theta(t)} = \gamma_1$ by $\Gamma(s,t) = \gamma(t)^{1-s}e^{i\theta(t)}$ and $\gamma_0 \sim e^{i(2\pi k)t + i\theta(0)} = \gamma_k$ by $\tilde{\Gamma}(s,t) = e^{i[\theta(0)+(1-s)(\theta(t)-\theta(0))+s2\pi kt]}$. It can be shown that $\text{Ind}_\gamma(0) = k$. Thus, in this case, homologous $\implies$ homotopic. However, it is not true in general.

Example 6.3. Let $\Omega = \mathbb{C} \backslash \{0, 1\}$. Then there is a closed curve $\gamma \approx 0$ but $\gamma \nsim 0$.

Theorem 6.5. *If Ω is simply connected and $f : \Omega \mapsto \mathbb{C}$ holomorphic then $\exists h : \Omega \mapsto \mathbb{C}$ such that $h' = f$.*

Example 6.4. Let $\Omega = \mathbb{C} \backslash \{x \in \mathbb{R} : x \le 0\}$. Then Ω is simpy connected. Consider $f : \Omega \mapsto \mathbb{C}$ with $z \mapsto 1/z$. Then $\exists h : \Omega \mapsto \Omega$ such that $h' = f$ which is $\ln z$ on Ω.

Corollary 6.1. *Let Ω be a simply connected subset of $\mathbb{C}$ and $f : \Omega \mapsto \mathbb{C}$ holomorphic and $f(z) \neq 0$ for all $z \in \Omega$. Then there exists a branch of $\ln(f(z))$. That is $\exists h : \Omega \mapsto \mathbb{C}$ holomorphic with*

$$h'(z) = \frac{f'(z)}{f(z)}, \forall z \in \Omega$$

Proof. Let $F(z) = \frac{f'(z)}{f(z)}, \forall z \in \Omega$. Since $f(z) \neq 0$ for all $z \in \Omega$. $F(z)$ is holomorphic on Ω. □

Remark 6.4. Suppose that $f : \Omega \mapsto \mathbb{C}$, $h'(z) = \frac{f'(z)}{f(z)}$ and $e^{h(a)} = f(a)$ for some $a \in \Omega$. We then claim that $e^{h(z)} = f(z)$.

Proof. To see this, define $H(z) = e^{-h(z)} f(z)$. $H(z)$ is holomorphic on Ω and

$$\begin{aligned} H'(z) &= e^{-h(z)}(-h'(z) \cdot f(z)) + e^{-h(z)} \cdot f'(z) \\ &= e^{-h(z)} \left[-\frac{f'(z)}{f(z)} \cdot f(z) + f(z) \right] = 0 \end{aligned}$$

So $H(z) = c$ for some constant $c \in \mathbb{C}$. So

$$c = H(a) = e^{-h(a)} f(a) = \frac{f(a)}{e^{h(a)}} = 1$$

So $e^{h(z)} = f(z)$ as required. □

Theorem 6.6. *(Argument Principle) Let Ω be an open subset of $\mathbb{C}$ with $\gamma \approx 0$, a cycle in Ω, and $f : \Omega \mapsto \mathbb{C}$ holomorphic. If $Z(f)$ is the set of zeroes of f then $Z(f) \cap \gamma^* = \emptyset$ and*

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)} dz = \sum_{a_i \in Z(f)} n_i Ind_\gamma(a_i)$$

where n_i is the multiplicity of a_i.

Example 6.5. Let $f(z) = z^n$, $\gamma(t) = e^{it}$. The RHS is $1 \cdot n = n$ and the LHS is

$$\text{LHS} = \frac{1}{2\pi i} \oint_\gamma \frac{nz^{n-1}}{z^n} dz = n \cdot \left(\frac{1}{2\pi i} \oint_\gamma \frac{1}{z} dz \right) = n$$

Proof. (of Argument Principle) Since γ^* is compact, the bounded and connected components of $\mathbb{C} \backslash \gamma^*$ is bounded by say R. Since the zeros of $f(z)$ are isolated, the number of zeros in $\mathcal{B}_R(0)$ is finite. Let $a_1, a_2, ..., a_N$ be the zeros of $f(z)$ in $\mathcal{B}_R(0)$ with multiplicity $n_i, ..., n_N$ respectively. Now $\exists g(z) : \Omega \mapsto \mathbb{C}$ holomorphic, $g(z) \neq 0, \forall z \in \Omega$ such that

$$f(z) = \prod_{i=1}^{n} (z - a_i)^{n_i} g(z) \implies \frac{f'(z)}{f(z)} = \sum_{i=1}^{N} \frac{n_i}{z - a_i} + \frac{g'(z)}{g(z)}$$

and so

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)} = \left(\sum_{i=1}^{N} n_i \frac{1}{2\pi i} \oint_\gamma \frac{1}{z - a_i} dz \right) + \frac{1}{2\pi i} \oint_\gamma \frac{g'(z)}{g(z)} = \sum_{i=1}^{N} n_i \text{Ind}_\gamma(a_i) + 0$$

□

Corollary 6.2. *If $f : \Omega \mapsto \mathbb{C}$ holomorphic and $\overline{\mathcal{B}_R(p)} \subseteq \Omega$, $\gamma(t) = p + Re^{it}$, $0 \leq t \leq 2\pi$ and f is not zero at γ^*. Then*

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)} = n$$

the number of zeroes in $\mathcal{B}_R(p)$.

Remark 6.5. If $f(z) \neq w$ on γ^* then

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z) - w} dz$$

is the number of solutions $f(z) = w$ on $\mathcal{B}_R(p)$.

Theorem 6.7. *If $f(z)$ has a zero of order k at a then $\exists \epsilon > 0, \delta > 0$ such that $\forall 0 < |w| < \delta$, $f(z) = w$ has k single roots in $\mathcal{B}_\epsilon(a)$.*

Proof. $f(z)$ and $f'(z)$ are holomorphic and therefore, their zeros are isolated. Thus, we can choose $\epsilon > 0$ such that in $\mathcal{B}_\epsilon(a)$, $f(z)$ and $f'(z)$ has no zeros except at $z = a$. In particular, $f(z) = w$ has only single roots in $\mathcal{B}_\epsilon(a)$. Let

$$\delta = \frac{1}{2} \{|f(z)|, |z - a| = \epsilon\}$$

and $\gamma(t) = a + \epsilon e^{it}$, $0 \leq t \leq 2\pi$. Fix $|w| < \delta$. Define

$$h(s) = \frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(w) - sw} dz, 0 \leq s \leq 1$$

Since $|f(z)| \geq 2\delta \geq 2|w|$ on γ^*, $h(s)$ is well defined since $|f(z) - sw| \geq 2\delta - |w| \geq \delta > 0$. We know that $h(s)$ is continuous integral valued function and therefore $h(0) = h(1)$. Since

$$h(0) = \frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(w)} dz = k = \frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(w) - w} dz$$

and hence there are k solutions of $f(z) = w$ in $\mathcal{B}_\epsilon(a)$ and $f(z) = w$ has k single roots as required. □

Example 6.6. Consider $f(x) = x^2$ which has a double zero at 0. However $\forall \epsilon > 0$, $f(x) = -\epsilon$ has no solution.

Corollary 6.3. *If $\Omega \subseteq \mathbb{C}$ open and $f : \Omega \mapsto \mathbb{C}$ holomorphic then $\forall U \subseteq \Omega$ open $f(U)$ is open in $\mathbb{C}$ ($f(\Omega)$ is open).*

Remark 6.6. Remark that for $f(x) = x^2$ (in $\mathbb{R}$), $f(\mathbb{R}) = [0, \infty)$ is not open.

Proof. (of Corollary) Let $a \in U$ then $\mathcal{B}_\epsilon(a) \subseteq U$ and $\exists \delta > 0$, $\mathcal{B}_\delta(f(a)) \subseteq f(U)$. Therefore $f(U)$ is open. To see this, remark that $f(z) - f(a)$ has a zero at a. By the theorem, $\exists \epsilon > 0, \delta > 0$ such that $\forall z \in \mathcal{B}_\epsilon(z)$, $f(z) = w$ has a solution for $|w - f(a)| < \delta$. Hence $\mathcal{B}_\delta(f(a)) \subseteq f(U)$. □

Corollary 6.4. *If $f(z)$ is holomorphic on Ω and $a \in \Omega$, $f'(a) \neq 0$, then $\exists r > 0$ such that $f\Big|_{\mathcal{B}_r(a)}$ is 1-1 and onto on an open set U. Moreover, $f^{-1} : U \mapsto \mathcal{B}_r(a)$ is holomorphic and*

$$\left(f^{-1}(w)\right)' = \frac{1}{f'(f^{-1}(w))}$$

Proof. $f'(a) \neq 0 \implies \exists r > 0$ such that $f'(z)$ has no zero on $\mathcal{B}_r(a)$. Thus $f(z) = w$ has only single roots in $\mathcal{B}_r(a)$. We choose r such that $f(z)$ on $\mathcal{B}_r(a)$ is 1-1 since $f(z) = f(a)$ has only one root at a. Also since f is an open mapping, $f^{-1} : U \mapsto \mathcal{B}_r(a)$ is indeed a continuous function since $\forall V \subseteq \mathcal{B}_r(a)$ open we have

$$\left(f^{-1}\right)^{-1}(V) = f(V)$$

is open in U. Let $g = f^{-1}$,$w = f(b)$. Then

$$\begin{aligned} g'(w) &= \lim_{v \to w} \frac{g(v) - g(w)}{v - w} \\ &= \lim_{u \to b} \frac{g(f(u)) - g(f(b))}{f(u) - f(b)} \\ &= \lim_{u \to b} \frac{u - b}{f(u) - f(b)} \\ &= \lim_{u \to b} \frac{1}{\frac{f(u) - f(b)}{u - b}} = \frac{1}{f'(b)} = \frac{1}{f'(f^{-1}(w))} \end{aligned}$$

□

Corollary 6.5. *If $f : \Omega \mapsto f(\Omega)$ is 1-1 and onto holomorphic and Ω is open, then $f^{-1} : f(\Omega) \mapsto U$ is also 1-1 and onto, holomorphic.*

Theorem 6.8. *(Rouché's Theorem) Let f, g holomorphic on $\Omega \supseteq \overline{\mathcal{B}_R(p)}$ and $\gamma(t) = p + Re^{it}$ for $0 \leq t \leq 2\pi$. Suppose that*

$$|f(z) - g(z)| < |f(z)| + |g(z)|$$

for $|z - p| = R$. Then $f(z)$ and $g(z)$ have the same number of zeros in $\mathcal{B}_R(p)$.

Example 6.7. Compute the number of zeros of $f(z) = z^7 - 2z^5 - 6z^3 - z + 1$ in $\mathcal{B}_1(0)$.

(Solution) Set $g(z) = -6z^3$.Then

$$|f(z) - g(z)| = |z^7 - 2z^5 - z + 1| \leq 1 + 2 + 1 + 1 = 5 < 6 = |g(z| \leq |f(z)| + |g(z)|$$

By Rouché's Theorem, $f(z)$ and $g(z)$ have the same number of zeros in $\mathcal{B}_1(0)$. Since $g(z)$ has 3 zeros in $\mathcal{B}_1(0)$ so does $f(z)$.

Proof. (of Rouche's Theorem) Note that if $f(z)$ then $|f(z) - g(z)| = |g(z)| = |f(z)| + |g(z)|$. So $f(z) \neq 0$ for $z \in \gamma^*$. Similarly, $g(z)$ has no zero in γ^*. So

$$\begin{aligned} |f(z) - g(z)| < |f(z)| + |g(z)|, z \in \gamma^* \quad &\iff \quad \left|\frac{f(z)}{g(z)} - 1\right| < \left|\frac{f(z)}{g(z)}\right| + 1, z \in \gamma^* \\ &\iff \quad \frac{f(z)}{g(z)} \notin (-\infty, 0) \end{aligned}$$

Define $f_s(z) = sf(z) + (1-s)g(z), 0 \leq s \leq 1$. $f_s(z)$ is holomorphic and $f_0(z) = g(z)$, $f_1(z) = f(z)$. If $f_s(z) = 0$ then

$$\frac{f(z)}{g(z)} = -\frac{1-s}{s} < 0$$

and thus $f_s(z) \neq 0$ for $z \in \gamma^*$. Define

$$I_s := \frac{1}{2\pi i} \oint_\gamma \frac{f_s'(z)}{f_s(z)} dz$$

and remark that I_s is an integral valued continuous function in s. Thus, I_s is a constant and I_0, I_1 are the number of zeros of $g(z)$ and $f(z)$ respectively in $\mathcal{B}_R(p)$. □

7 Singularities

Definition 7.1. If $f(z)$ is holomorphic on a punctured disk, $\mathcal{B}_\epsilon(a)\backslash\{a\}$ we say $f(z)$ has an *isolated singularity* at a or a is an isolated singularity of $f(z)$. There are three types of isolated singularities:

- a is called a *removable singularity* if there exists a holomorphic function $g(z) : \mathcal{B}_\epsilon(z) \mapsto \mathbb{C}$ and $f(z) = g(z)$ on $\mathcal{B}_\epsilon(a)\backslash\{a\}$
- a is a *pole* if $\lim_{z\to a} |f(z)| = \infty$
- a is an *essential singularity* otherwise

Notation 3. Define

$$\begin{aligned} A_r(a) : &= \{z \in \mathbb{C} : 0 < |z-a| < r\} \\ A_{r_1,r_2}(a) : &= \{z \in \mathbb{C} : r_1 < |z-a| < r_2\} \end{aligned}$$

where $A_{0,r}(a) = A_r(a)$.

Theorem 7.1. *Let $a \in \mathbb{C}$ be an isolated singularity of $f(z)$ and $|f(z)| < \infty$ on $A_r(a)$ for some $r > 0$. Then, a is a removable singularity of $f(z)$.*

Proof. Consider

$$h(z) = \begin{cases} (z-a)f(z) & z \in A_r(a) \\ 0 & z = a \end{cases}$$

We know that $h(z)$ is holomorphic on $A_r(a)$ and $h(z)$ is continuous at a. This implies that $h(z)$ is continuous on $\mathcal{B}_r(a)$ and holomorphic on $A_r(a)$. So $h(z)$ is holomorhpic on $\mathcal{B}_r(a)$. By a theorem before, $h(z) = (z-a)g(z)$ for some holomorphic function $g(z)$ on $\mathcal{B}_r(a)$. Then $g(z) = f(z)$ on $A_r(a)$ and $g(z)$ is holomorphic on $\mathcal{B}_r(a)$. Thus, a is a removable singularity. □

Remark 7.1. If a is a removable singularity of $f(z)$, $f(z)$ will be bounded on $A_r(a)$ for some $r > 0$ since $f(z)$ is continuous at a. Therefore, a is a removable singularity of $f(z)$ if and only if $f(z)$ is bounded on $A_r(a)$ for some $r > 0$.

Theorem 7.2. *Let a be an isolated singularity of $f(z)$. Then a is a pole if and only if*

$$f(z) = \frac{g(z)}{(z-a)^m}$$

for some $m \geq 1$, $m \in \mathbb{Z}$ and $g(z)$ is holomorphic on $\mathcal{B}_r(a)$ for some $r > 0$ and $g(a) \neq 0$.

Proof. ($\Longleftarrow$) Trivial

($\Longrightarrow$) Since $\lim_{z\to a} |f(z)| = \infty$, $\exists r > 0$ such that $|f(z)| \geq 1$ for $0 < |z-a| < r$. In $A_r(a)$, we define $h(z) = \frac{1}{f(z)}$. Then $h(z)$ is holomorphic on $A_r(a)$ and $\lim_{z\to a} h(z) = 0$. In particular, $h(z)$ is bounded on $A_r(a)$. Therefore, a is a removable singularity of $h(z)$. So $h(z)$ has a zero at a and holomorphic on $\mathcal{B}_r(a)$. Thus,

$$h(z) = \tilde{h}(z) \cdot (z-a)^m$$

where $\tilde{h}(z)$ is holomorphic on $\mathcal{B}_r(a)$ with $\tilde{h}(z) \neq 0$ on $\mathcal{B}_r(a)$ with $m \geq 1, m \in \mathbb{Z}$. It then implies that

$$f(z) = \frac{1}{\tilde{h}(z)(z-a)^m}$$

If we set $g(z) = \frac{1}{\tilde{h}(z)}$ then $g(a) = \frac{1}{\tilde{h}(a)} \neq 0$ and $g(z)$ is holomorphic on $\mathcal{B}_r(a)$. □

Theorem 7.3. *If a is an essential singularity then $\forall \epsilon > 0$, $f(A_\epsilon(a))$ is dense in $\mathbb{C}$.*

Proof. If not, $\exists \epsilon > 0$ such that $f(A_\epsilon(a))$ is not dense in $\mathbb{C}$. That is $\exists c \in \mathbb{C}$ such that $|f(z) - c| > \delta$ for some $\delta > 0$, $\forall z \in A_\epsilon(a)$. Consider $g(z) = \frac{1}{f(z))-c}$. Then $g(z)$ is holomorphic on $A_\epsilon(a)$. Moreover, $\forall z \in A_\epsilon(a)$, $|g(z)| = \frac{1}{|f(z)-c|} < \frac{1}{\delta}$ is bounded. Therefore, a is a removable singularity and $g(z)$ is holomorphic on $\mathcal{B}_r(a)$.

Write $g(z) = h(z) \cdot (z-a)^m$ for some holomorphic function on $\mathcal{B}_r(a)$, $h(a) \neq 0$ and $m \geq 0$, $m \in \mathbb{Z}$. So

$$f(z) = c + \frac{1}{g(z)} = c + \frac{1}{h(z)(z-a)^m}$$

and as $z \to a$,

$$\lim_{z\to a} |f(z)| = \begin{cases} \infty & m \geq 1 \\ c + \frac{1}{h(a)} & m = 0 \end{cases}$$

and in either case, a is a removable singularity ($m = 0$) or a pole ($m \geq 1$). This then contradicts that a is essential. □

Example 7.1. Let $f(z) = e^{1/z}$. Then $z = 0$ is an essential singularity of $f(z)$. Remark that

$$\lim_{r\to 0^+} e^{1/r} = \infty, \lim_{r\to 0^-} e^{-1/r} = 0$$

and

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{z}\right)^n}{n!} = \sum_{n=0}^{n} \frac{1}{n!} z^{-n} = \sum_{k=-\infty}^{0} \frac{1}{|k|!} z^k$$

7.1 Laurent Series

Theorem 7.4. *(Laurent Series) Suppose that $f(z)$ is holomorphic on $A_{R_1,R_2}(a) = \{z \in \mathbb{C} | R_1 < |z-a| < R_2\}$. Then, there are unique scalars $a_n, n \in \mathbb{Z}$ such that*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

This converges absolutely and uniformly on any $A_{r_1,r_2}(a)$ for $R_1 < r_1 < r_2 < R_2$.

Proof. Consider $R_1 < p_1 < r_1 < r_2 < p_2 < R_2$ and let $z \in A_{r_1,r_2}(a)$. Let $\gamma_1(t) = a + p_1 e^{it}, \gamma_2(t) = a + p_2 e^{it}$ for $0 \le t \le 2\pi$. Then for any $z \in A_{r_1,r_2}(a)$ and $\text{Ind}_{\gamma_2,\gamma_1}(z) = 1 - 0 = 1$. Note that if $\gamma = \gamma_2 - \gamma_1$ then $\gamma \approx 0$ on $A_{R_1,R_2}(a)$. By Cauchy's Integral formula,

$$f(w) = \frac{1}{2\pi i}\oint_{\gamma_2-\gamma_1}\frac{f(z)}{z-w}dz = \frac{1}{2\pi i}\oint_{\gamma_2}\frac{f(z)}{z-w}dz - \frac{1}{2\pi i}\oint_{\gamma_1}\frac{f(z)}{z-w}dz$$

for $w \in A_{r_1,r_2}$ and $p_1 < r_1 < |w-a| < r_2 < p_2$. For $z \in \gamma_2^*$, $|w-a| < r_2 < p_2 = |z-a|$ we have

$$\frac{1}{z-w} = \frac{1}{(z-a)-(w-a)} = \frac{1}{(z-a)}\cdot\frac{1}{1-\frac{w-a}{z-a}} = \frac{1}{z-a}\sum_{n=0}^{\infty}\left(\frac{w-a}{z-a}\right)^n = \sum_{n=0}^{\infty}(w-a)^n(z-a)^{-n-1}$$

and for $z \in \gamma_1^*$, $|z-a| < p_2 < r_2 = |w-a|$ we have

$$\frac{1}{z-w} = \frac{1}{(z-a)-(w-a)} = -\frac{1}{(w-a)}\cdot\frac{1}{1-\frac{z-a}{w-a}} = -\frac{1}{w-a}\sum_{n=0}^{\infty}\left(\frac{z-a}{w-a}\right)^n = -\sum_{n=0}^{\infty}(w-a)^{-n-1}(z-a)^n$$

So we can rewrite the above integral formula as

$$\begin{aligned} f(w) &= \frac{1}{2\pi i}\left(\oint_{\gamma_2} f(z)\sum_{n=0}^{\infty}(w-a)^n(z-a)^{-n-1}dz\right) + \frac{1}{2\pi i}\left(\oint_{\gamma_1} f(z)\sum_{n=-\infty}^{-1}(w-a)^n(z-a)^{-n-1}dz\right) \\ &= \sum_{n=-\infty}^{\infty} a_n(w-a)^n \end{aligned}$$

where

$$a_n = \begin{cases} \frac{1}{2\pi i}\oint_{\gamma_2} f(z)(z-a)^{-n-1}dz = \frac{1}{2\pi i}\int_0^{2\pi} f(a+p_2e^{it})e^{-i(n+1)}p_2^{-(n+1)}e^{it} & n \ge 0 \\ \frac{1}{2\pi i}\oint_{\gamma_1} f(z)(z-a)^{-n-1}dz = \frac{1}{2\pi i}\int_0^{2\pi} f(a+p_1e^{it})e^{-i(n+1)}p_1^{-(n+1)}e^{it} & n < 0 \end{cases}$$

and simplifying gives us

$$a_n = \begin{cases} \frac{1}{2\pi p_2^n}\int_0^{2\pi} f(a+p_2e^{it})e^{-int} & n \ge 0 \\ \frac{1}{2\pi p_1^n}\int_0^{2\pi} f(a+p_1e^{it})e^{-int} & n < 0 \end{cases}$$

Now if $M = \sup_{z \in A_{p_1,p_2}(a)}|f(z)|$ then

$$|a_n| \le \begin{cases} \frac{M}{p_2^n} & n \ge 0 \\ \frac{M}{p_1^n} & n < 0 \end{cases}$$

and

$$\begin{aligned} \sum_{n=-\infty}^{\infty}|a_n(w-a)^n| &= \sum_{n=0}^{\infty}|a_n||(w-a)|^n + \sum_{n=-\infty}^{-1}|a_n||(w-a)|^n \\ &\le \sum_{n=0}^{\infty}M\cdot\left(\frac{r_2}{p_2}\right) + \sum_{n=-\infty}^{-1}M\cdot\left(\frac{p_1}{r_1}\right) < \infty \end{aligned}$$

By the Weierstrass $M-$Test, the series converges absolutely and uniformly on $A_{r_1,r_2}(a)$. This proves the existence of the Laurent series. We will show uniqueness later. □

Corollary 7.1. *If a is an isolated singularity of $f(z)$ on $A_r(a)$ and $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^m$ as its Laurent series expansion. Then*

i) a is removable $\iff \forall n \le -1$, $a_n = 0$

ii) a is a pole $\iff \exists m \ge 1$ such that $\forall n < -m$, $a_n = 0, a_m \ne 0$. Thus, m is called the order of the pole.

iii) a is essential otherwise

7.2 Residues

Definition 7.2. Let a be an isolated singularity of $f(z)a$ and $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n$. The *residue* of $f(z)$ at a is

$$\text{Res}(f, a) := a_{-1}$$

Theorem 7.5. *(Residue Theorem) Suppose that f is holomorphic on $\Omega \backslash \{P_1, ..., P_l\}$ where the $P_i's$ are singularities of $f(z)$. $\forall\gamma$ which are circles such that $\gamma \approx 0$ in Ω we have*

$$\oint_\gamma f(z)dz = 2\pi i \sum_{i=1}^{l} \mathit{Ind}(P_i) \cdot \mathit{Res}(f, P_i)$$

Proof. Let $\epsilon = \frac{1}{2}\min_i(\text{dist}(P_i, \gamma^*\mathbb{C}\backslash\Omega))$ and $\gamma_i(t) = P_i + \epsilon e^{it}$ for $0 \le t \le 2\pi$. Let $n_i = \text{Ind}_\gamma(P_i)$. Then

$$\gamma - \sum_{i=1}^{l} n_i\gamma_i \approx 0$$

in Ω. thus,

$$\oint_{\gamma - \sum_{i=1}^{l} n_i\gamma_i} f(z)dz = 0 \implies \oint_\gamma f(z)dz = \sum_{i=1}^{l} n_i \int_{\gamma_i} f(z)dz$$

Now for any i, P_i is a singularity and $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-P_i)^n$. Recall that

$$\oint_{\gamma_i} \frac{1}{(z-a)^m} = \begin{cases} 0 & m \ne 1 \\ 2\pi i & m = 1 \end{cases} \implies \oint_{\gamma_i} f(z)dz = \sum_{n=-\infty}^{\infty} a_n \oint_{\gamma_i} (z-P_i)^n dz = 2\pi i \cdot a_{-1} = 2\pi i \cdot \text{Res}(f, P_i)$$

To sum up,

$$\oint_\gamma f(z)dz = \sum_{i=1}^{l} n_i \oint_{\gamma_i} f(z)dz = 2\pi i \sum_{i=1}^{l} \text{Ind}(P_i)\text{Res}(f, P_i)$$

□

Problem 7.1. How to compute residues?

Answer. (1) Find the Laurent series.

(2) If $f(z)$ has a pole of order m at a, then $\text{Res}(f, a) = \frac{1}{(m-1)!}g^{(m-1)}(a)$ where $f(z) = (z-a)^{-m}g(z)$. To see this, note that $g(z) = \sum_{n=0}^{\infty} b_n(z-a)^n \implies$

$$f(z) = (z-a)^{-m}\sum_{n=0}^{\infty} b_n(z-a)^n = \sum_{n=-m}^{\infty} b_{n-m}(z-a)^n = \sum_{n=-m}^{\infty} a_n(z-a)^n$$

which implies that $a_{-1} = b_{m-1} = \frac{1}{(m-1)!}g^{(m-1)}(a)$.

(3) If $f(z)$ has a simple pole at a then $\text{Res}(f, a) = \lim_{z\to a}(z-a)f(z)$.

(4) If $f(z) = \frac{p(z)}{q(z)}$ where $p(z)$ and $q(z)$ are holomorphic on $\mathcal{B}_r(a)$, $r > 0$, $p(a) \ne 0$ and $q(z)$ has a simple zero at a then $\text{Res}(f, a) = \frac{p(a)}{q'(a)}$. To see this, note that the residue is equal to

$$\lim_{z\to a} \frac{p(z)}{q(z)} \cdot (z-a) = \lim_{z\to a} \frac{p(z)}{\frac{q(z)-q(a)}{z-a}} = \frac{p(z)}{q'(a)}$$

Theorem 7.6. *(Type I) Compute $I = \int_0^{2\pi} R(\cos\theta, \sin\theta)d\theta$ where $R(x, y)$ is a rational function in x. For example, $I = \int_0^{\pi} \frac{1}{a+\cos\theta}d\theta$.*

Proof. (of Type I) To compute, let

$$\gamma(\theta) = e^{i\theta}, 0 \leq t \leq 2\pi$$

Then

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{zi} = \frac{z - \frac{1}{z}}{zi} = \frac{z^2 - 1}{2zi}$$
$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{z} = \frac{z + \frac{1}{z}}{z} = \frac{z^2 + 1}{2z}$$

Let $z = e^{i\theta}$. Then $dz = ie^{i\theta}d\theta = izd\theta$. Then

$$I = \oint_\gamma R\left(\frac{z^2+1}{2z}, \frac{z^2-1}{2zi}\right)\frac{dz}{iz} = 2\pi i \sum_{|a|<1} \text{Res}\left(\frac{1}{iz}R\left(\frac{z^2+1}{2z}, \frac{z^2-1}{2zi}\right), z\right)$$

□

Example 7.2. Compute $\int_0^\pi \frac{1}{a+\cos\theta}d\theta$ for $a > 1$.

Solution. We have

$$I = \int_0^\pi \frac{1}{a+\cos\theta}d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1}{a+\cos\theta}d\theta = \frac{1}{2}\left(2\pi \sum_{|b|<1} \text{Res}\left(\frac{1}{z}\cdot\frac{1}{a+\frac{z^2+1}{2z}}, b\right)\right) = \pi \sum_{|b|<1} \text{Res}\left(\frac{2}{z^2+2az+1}, b\right)$$

Now, $f(z) = \frac{2}{z^2+2az+1}$ for $a > 1$ has two single poles at the roots of $z^2 + 2az + 1$ which are $z = -a \pm \sqrt{a^2-1}$. Only $-a + \sqrt{a^2-1}$ inside $\mathcal{B}_1(0)$. So

$$I = \pi \text{Res}\left(\frac{2}{z^2+2az+1}, -a + \sqrt{a^2-1}\right) = \pi\frac{2}{z(z+a)} = \frac{\pi}{\sqrt{a^2-1}}$$

Theorem 7.7. *(Type II) Let $R(x)$ be a function such that $\exists\epsilon > 0$ with $\lim_{z\to\infty}|z|^{1+\epsilon}|R(z)| = 0$ and $R(z)$ has only finitely many simple poles on $\mathbb{R}$. For example $R(z) = \frac{p(z)}{q(z)}$ where $p(z)$ and $q(z)$ are polynomials in z with deg$q(z) \geq$ deg$p(z) + 2$. Then we can compute*

$$I = \int_{-\infty}^{\infty} R(x)\ dx$$

which will be shown later.

Definition 7.3. (Principle value for the line integral) Let $\gamma(t) = p + tq$ for some $p, q \in \mathbb{C}$. Let $f(z)$ be some holomorphic function on γ^* except for finitely many poles at $\gamma_1, ..., \gamma_l$ at $t_1, .., t_l$. That is, $\gamma_i = p + t_i q$. Define the *principle value* as

$$PV\oint_\gamma f(z)dz := \lim_{\epsilon\to 0}\left(\int_{-\frac{1}{\epsilon}}^{t_!-\epsilon} f(p+tq)dt + \int_{t_1+\epsilon}^{t_2-\epsilon} f(p+tq)dt + ... + \int_{t_l+\epsilon}^{\frac{1}{\epsilon}} f(p+tq)dt\right)$$

Example 7.3. $\int_{-\infty}^{\infty}\frac{1}{x}$ does not exist, but

$$PV\int_{-\infty}^{\infty}\frac{1}{x} = \lim_{\epsilon\to 0}\left(\int_{-\frac{1}{\epsilon}}^{-\epsilon}\frac{1}{x}dx + \int_{\epsilon}^{\frac{1}{\epsilon}}\frac{1}{x}dx\right) = 0$$

Summary 2. Coming back to the Type II evaluation,

$$I = \int_{-\infty}^{\infty} R(x)\ dx = 2\pi i \sum_{a\in A} \text{Res}\,(R(z), a) + \pi i \sum_{b\in B} \text{Res}\,(R(z), b)$$

where $A = \{z \in \mathbb{C} | \Im(z) > 0\}$ and B is the set of all poles in $\mathbb{R}$.

Lemma 7.1. *If $z = a$ is a simple pole of $f(z)$ and $\gamma_\epsilon(t) = a + \epsilon e^{-it}$ for $\alpha \leq t \leq \beta$. Then*

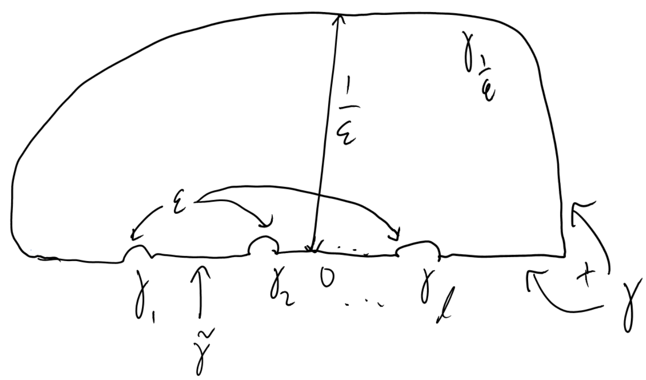
$$\lim_{\epsilon\to 0}\int_{\gamma_\epsilon} f(z)dz = (\beta - \alpha)i\mathit{Res}(f, a)$$

Proof. (of Lemma) We have $f(z) = \frac{a-1}{z-a} + g(z)$ for $g(z)$ holomorphic. Then

$$\begin{aligned}\oint_{\gamma_\epsilon} f(z)dz = \int_\alpha^\beta f(a+\epsilon e^{it}) \cdot \epsilon i \cdot e^{it} dt &= \left(\int_\alpha^\beta \frac{a-1}{\epsilon e^{it}} \epsilon e^{it} dt\right) + \left(\int_\alpha^\beta g(a+\epsilon e^{it}) \epsilon e^{it} dt\right) \\ &= (\beta - \alpha)\, i a_{-1} + 0 \\ &= (\beta-\alpha) i \text{Res}(f,a)\end{aligned}$$

□

Proof. (of Type II) Consider the following curve:



Here, $\gamma = \gamma_\epsilon + \gamma_{\frac{1}{\epsilon}}$ where $\gamma_\epsilon = \sum_{j=1}^{l} \gamma_j + \tilde{\gamma}\epsilon$. By the Residue Theorem,

$$\begin{aligned}2\pi i \sum_{\Im(a)} \text{Res}(f,a) &= \lim_{\epsilon\to 0} \oint_\gamma R(z)dz \\ &= I + \lim_{\epsilon \to 0} \sum_{j=1}^{l} \oint_{\gamma_j} R(z)dz + \lim_{\epsilon\to 0} \oint_{\gamma_{\frac{1}{\epsilon}}} R(z)dz \\ &= I - \pi i \sum_{j=1}^{l} \text{Res}(f,p_j) + 0\end{aligned}$$

since

$$\left|\oint_{\gamma_{\frac{1}{\epsilon}}} R(z)dz\right| \leq \pi |z||R(z)|, |z| = \frac{1}{\epsilon} \implies \lim_{\epsilon\to 0} \left|\oint_{\gamma_{\frac{1}{\epsilon}}} R(z)dz\right| \leq \lim_{z\to\infty} \pi\,|z|\,|R(z)| = 0$$

and hence

$$I = 2\pi i \sum_{\Im(a)} \text{Res}(f,a) + \pi i \sum_{j=1}^{l} \text{Res}(f,p_j)$$

□

Example 7.4. Compute $I = \int_0^\infty \frac{1}{1+x^6} dx$.

Solution. We have

$$\begin{aligned} I &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{1+x^6}dx \\ &= \frac{1}{2}\left(2\pi i \text{Res}\left(\frac{1}{1+z^6}a_i\right)\right) \\ &= \frac{1}{2}\left(2\pi i \sum_{i=1}^{3} \frac{1}{6a_i^5}\right) \\ &= \pi i \sum_{i=1}^{3} \frac{-a_i}{6} \\ &= \frac{\pi i}{6}\left(\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)+i\left(-\frac{\sqrt{3}}{2}+\frac{i}{2}\right)\right) \\ &= \frac{\pi}{3} \end{aligned}$$

Theorem 7.8. *(Type III) If* $I = \int_{-\infty}^{\infty} f(x)e^{ix}dx$ *with* $\lim_{|z|\to\infty, \Im(z)>0} f(z) = 0$ *and* $f(z)$ *has only simple poles on* $\mathbb{R}$*. Then*

$$I = 2\pi i \sum_{\Im(a)>0} Res(f(z)e^{iz}, a) + \pi i \sum_{\Im(b)=0} Res(f(z)e^{iz}, b)$$

Proof. It is enough to show that

$$\lim_{\epsilon\to 0}\left|\oint_{\gamma_{\frac{1}{\epsilon}}} f(z)e^{iz}dz\right| = 0$$

To do this, remark that

$$\oint_{\gamma_{\frac{1}{\epsilon}}} f(z)e^{iz}dz = \int_0^{\pi} f\left(\frac{1}{\epsilon}e^{i\theta}\right) e^{i\frac{1}{\epsilon}(\cos\theta + i\sin\theta)}\frac{1}{\epsilon}ie^{i\theta}d\theta$$

which implies that

$$\left|\oint_{\gamma_{\frac{1}{\epsilon}}} f(z)dz\right| \le \int_0^{\pi} \left|f\left(\frac{1}{\epsilon}e^{i\theta}\right)\right| e^{-\frac{1}{\epsilon}i\sin\theta}\frac{1}{\epsilon}d\theta$$

Let $R = \frac{1}{\epsilon}$ and $M(R) = \sup_{|z|=R} f(z) \to 0$ as $R \to \infty$ with $\frac{2\theta}{\pi} \le \sin\theta \le \theta$. So the above is less than or equal to

$$\pi M(R)\int_0^{\infty} e^{-t}dt = \pi M(R) \to 0$$

as $R \to \infty$ with $t = \frac{2R\theta}{\pi}$. □

Example 7.5. Compute $I = \int_0^{\infty} \frac{\cos x}{1+x^2}dx$.

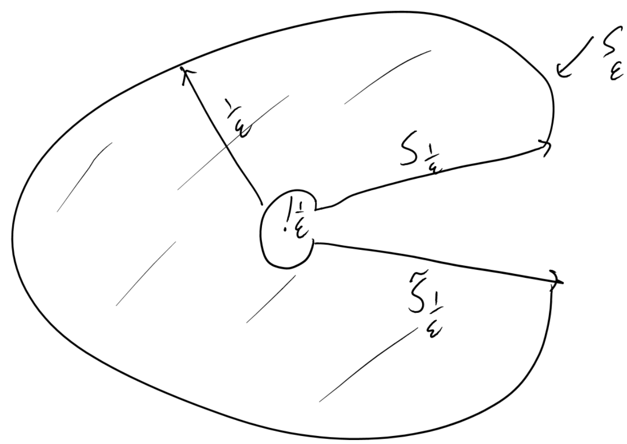
Solution. We have

$$\begin{aligned} I &= \Re\left(\int_0^{\infty} \frac{e^{ix}}{1+x^2}dx\right) \\ &= \frac{1}{2}\Re\left(\int_{\infty}^{\infty} \frac{e^{ix}}{1+x^2}dx\right) \\ &= \frac{1}{2}\Re\left(2\pi i \text{Res}\left(\frac{e^{iz}}{1+z^2}, i\right)\right) \\ &= \Re\left(\pi i \text{Res}\left(\frac{e^{-1}}{2i}, i\right)\right) = \Re\left(\frac{\pi}{2e}\right) = \frac{\pi}{2e} \end{aligned}$$

Theorem 7.9. *(Type IV) Suppose that $I = \int_0^\infty R(x)x^\alpha dx$ for $0 < \alpha < 1$, $R(x) = \frac{p(x)}{q(x)}$ a rational function with deg$(q) \geq$ deg$(p+2)$ or $\lim_{z\to\infty} |z^2 R(z)| = 0$. Then*

$$I = \frac{2\pi i}{1 - e^{2\pi\alpha i}} \sum_{a\in\mathbb{C}} \textit{Res}(R(z)z^\alpha, a)$$

Proof. Consider the curve



We have

$$\sum_{a\in\mathbb{C}} \text{Res}(R(z)z^\alpha, a) = \lim_{\epsilon\to 0} \oint_{\gamma_\epsilon} R(z)z^2 dz$$

and so

$$I = \lim_{\epsilon\to -} \oint_{S_{\frac{1}{\epsilon}}} R(z)z^\alpha dz$$

and

$$\oint_{\tilde{S}_{\frac{1}{\epsilon}}} R(z)z^\alpha dz = \oint_{\tilde{S}_{\frac{1}{\epsilon}}} R(z)e^{\alpha \ln z} dz = \int_\epsilon^{\frac{1}{\epsilon}} R(e^{(2\pi-\epsilon)i}t)t^\alpha e^{(2\pi-t)i\alpha} dz = -Ie^{2\pi i\alpha}$$

with $z = e^{(2\pi-\epsilon)i}t \ln z = (2\pi - t)i + \ln t$ for $\epsilon \leq t \leq \frac{1}{\epsilon}$. So

$$I - Ie^{2\pi i\alpha} = 2\pi i \sum_{\alpha\in\mathbb{C}} \text{Res}(f, a) \implies I = \frac{2\pi i}{1 - e^{2\pi i\alpha}} \sum_{a\in\mathbb{C}} \text{Res}(f, a)$$

□

Theorem 7.10. *(Type V) Given $I = \int_0^\infty R(x) \ln x \, dx$ with $\lim_{z\to\infty} |zR(z)| = 0$. Then*

$$I = -\frac{1}{2}\Re\left(\sum_{a\in\mathbb{C}} \textit{Res}\left(R(z)\ln^2 z\right), a\right) \implies I' = \int_0^\infty R(x) = -\frac{1}{2\pi}\Im\left(\sum_{a\in\mathbb{C}} \textit{Res}\left(R(z)\ln^2 z, a\right)\right)$$

Proof. This is the same γ as Type IV □

Theorem 7.11. *(Type VI) Let $f(z)$ be a meromorphic function on $\mathbb{C}$ with*

$$\sum_{k=-\infty}^{\infty} f(k) = -\pi \sum_{\textit{poles of } f=a} \textit{Res}\left(\cot(\pi z)f(z), a\right)$$

where we say that a function f is meromorphic if all of f's singularities are poles.

Proof. Remark that

$$\text{Res}(\pi\cot(\pi z), n) = 1, \forall n \in \mathbb{Z}$$

Consider the rectangle bounded by $z = n + \frac{1}{2}$, $z = -n + \frac{1}{2}$, $j = i(n + \frac{1}{2})$, $j = -i\left(n + \frac{1}{2}\right)$ □

Example 7.6. (Random Example) Compute Res $\left(\frac{1}{z^2 \tan z}, 0\right)$.

Solution. We know that

$$\begin{aligned}
(\tan z)^{-1} = \frac{\cos z}{\sin z} &= \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + ...}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - ...} \\
&= \frac{1}{z}\left(\frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + ...}{1 - (\frac{z^2}{3!} - \frac{z^4}{5!} + ...)}\right) \\
&= \frac{1}{z}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + ...\right) \underbrace{\left(1 + \frac{z^2}{3!} + \frac{z^4}{?} + ...\right)}_{\text{using } \frac{1}{1-x} = 1 + x + x^2 + ...} \\
&= \frac{1}{z}\left(1 - \frac{1}{3}z^2 + \frac{z^4}{?} + ...\right)
\end{aligned}$$

Now

$$\frac{1}{z^2}(\tan z)^{-1} = \frac{1}{z^2}\left[\frac{1}{z}\left(1 - \frac{1}{3}z^2 + \frac{z^4}{?}\right)\right] = \frac{1}{z^2} - \frac{1}{z}\cdot\frac{1}{3} + \frac{z}{?} + ...$$

So for $|z| < 1$ we have $\text{Res}(f(z), 0) = -\frac{1}{3}$.

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