Convex Sets

(1) $\{C_i\}_{i\in I}$ family of convex sets $\implies \bigcap_{i\in I} C_i$ is convex
(2) $C_i \subseteq \mathbb{R}^{n_i}$ convex for $i = 1,...,k \implies C_1 \times ... \times C_k \subseteq \mathbb{R}^{n_1} \times .... \times \mathbb{R}^{n_k}$ is convex
(3) $C_i \subseteq \mathbb{R}^n$ convex and $\alpha_i \in \mathbb{R}$ for $i = 1,...,k \implies \alpha_1 C_1 + ... + \alpha_k C_k \subseteq \mathbb{R}^n$ is convex
(4) $C \subseteq \mathbb{R}^n$ convex and T affine $\implies T(C)$ is convex
(5) $D \subseteq \mathbb{R}^n$ convex and T affine $\implies T^{-1}(D)$ is convex

Proposition 0.1. *Let $V \subseteq \mathbb{R}^n$ be an affine manifold and $S \subseteq V$ be given. Then:*
(a) $\mathrm{int}_V S \neq \emptyset \implies V = \operatorname{aff} S$ and hence $\operatorname{ri} S \neq \emptyset$.
(a) $\mathrm{int}\, S \neq \emptyset \implies \mathbb{R}^n = \operatorname{aff} S$ and hence $\operatorname{ri} S = \mathrm{int}\, S \neq \emptyset$.

Proposition 0.2. *If $\emptyset \neq C \subseteq \mathbb{R}^n$ convex, then $\operatorname{ri} C \neq \emptyset$.*

Proposition 0.3. *(resolution lemma) Let $\emptyset \neq C \subseteq \mathbb{R}^n$ convex, $x \in \operatorname{cl} C$ and $y \in \operatorname{ri} C$. Then $[y, x) \subseteq C$.*

Proposition 0.4. *Assume that $\bar{x} \in \operatorname{ri} C$. Then*
(a) $\exists \delta > 0$ such that $\bar{B}(\bar{x}; \delta) \cap \operatorname{aff} C \subseteq \operatorname{ri} C$
(b) Given any $x \in \operatorname{aff} C$, $\exists \varepsilon > 0$ s.t. $\bar{x} + t(x - \bar{x}) \in \operatorname{ri} C$, for all t s.t. $|t| \leq \varepsilon$
(c) Given any u lying in the subspace parallel to $\operatorname{aff} C$, $\exists \varepsilon > 0$ s.t. $\bar{x} + tu \in \operatorname{ri} C$, for all t s.t. $|t| < \varepsilon$.

Proposition 0.5. *Let $\emptyset \neq C \subseteq \mathbb{R}^n$ be convex. Then,*
(a) $\operatorname{aff}(\operatorname{ri} C) = \operatorname{aff} C = \operatorname{aff}(\operatorname{cl} C)$
(b) $\operatorname{ri}(\operatorname{ri} C) = \operatorname{ri} C = \operatorname{ri}(\operatorname{cl} C)$
(c) $\operatorname{cl}(\operatorname{ri} C) = \operatorname{cl} C = \operatorname{cl}(\operatorname{cl} C)$

Proposition 0.6. *The sets $\operatorname{ri} C$, C, and $\operatorname{cl} C$ all have the same* ri,cl*, and* aff.

Proposition 0.7. *Let C_1, C_2 convex. Then the following are equivalent:*
(1) $\operatorname{ri} C_1 = \operatorname{ri} C_2$,
(2) $\operatorname{cl} C_1 = \operatorname{cl} C_2$,
(3) $\operatorname{ri} C_1 \subseteq C_2 \subseteq \operatorname{cl} C_1$.

Proposition 0.8. *If $C \subseteq \mathbb{R}^m$ is convex and $A : \mathbb{R}^m \mapsto \mathbb{R}^n$ is affine, then*
(1) $\operatorname{ri} A(C) = A(\operatorname{ri} C)$
(2) $\operatorname{cl} A(C) \supseteq A(\operatorname{cl} C)$ (no need for convexity)
(3) $\operatorname{aff} A(C) = \operatorname{aff}(A(\operatorname{ri} C)) = \operatorname{aff}(A(\operatorname{cl} C)) = A(\operatorname{aff} C)$

Corollary 0.1. *If $\alpha_1, ..., \alpha_k \in \mathbb{R}$ and $C_1, ..., C_k \in \mathbb{R}^n$ convex. Then,*

$$\operatorname{ri}(\alpha_1 C_1 + ... + \alpha_k C_k) = \alpha_1 \operatorname{ri} C_1 + ... + \alpha_k \operatorname{ri} C_k.$$

Lemma 0.1. *For $S_i \subseteq \mathbb{R}^n$, $i = 1, ..., k$,*

$$\operatorname{ri}(S_1 \times ... \times S_k) = \operatorname{ri} S_1 \times ... \times \operatorname{ri} S_k.$$

Proposition 0.9. *Let $A : \mathbb{R}^n \mapsto \mathbb{R}^n$ be affine and $D \subseteq \mathbb{R}^n$ be convex. If $A^{-1}(\operatorname{ri} D) \neq \emptyset$ then*

$$\operatorname{ri} A^{-1}(D) = A^{-1}(\operatorname{ri} D)$$
$$\operatorname{cl} A^{-1}(D) = A^{-1}(\operatorname{cl} D).$$

The sets $A^{-1}(\operatorname{ri} D), A^{-1}(D), A^{-1}(\operatorname{cl} D)$ have the same affine hull, namely $A^{-1}(\operatorname{aff} D)$.

Proposition 0.10. *If $C_1, ..., C_k \subseteq \mathbb{R}^n$ are convex and $\bigcap_{i=1}^k \operatorname{ri} C_i \neq \emptyset$ then*

$$\operatorname{ri}\left(\bigcap_{i=1}^k C_i\right) = \bigcap_{i=1}^k \operatorname{ri} C_i$$
$$\operatorname{cl}\left(\bigcap_{i=1}^k C_i\right) = \bigcap_{i=1}^k \operatorname{cl} C_i.$$

Asymptotic or Recession Cone

Definition 0.1. Let $\emptyset \neq C \subseteq \mathbb{R}^n$ be closed and convex. Its **asymptotic cone**, denoted by C_∞, is defined as

$$C_\infty := \{d \in \mathbb{R}^n : x + td \in C, \forall t > 0, \forall x \in C\}.$$

Proposition 0.11. *C_∞ is a closed convex cone containing 0.*

Proposition 0.12. *If for source $x_0 \in C$ and $d \in \mathbb{R}^n$ we have*

$$\{x_0 + td : t > 0\} \subseteq C$$

then $d \in C_\infty$.

Lemma 0.2. *If $d = \lim_{k\to\infty} \alpha_k x^k$ where $\{x^k\} \subseteq C$ and $\{\alpha_k\} \subseteq \mathbb{R}_{++} \to 0$ then $d \in C_\infty$.*

Proposition 0.13. *C is bounded $\iff C_\infty = \{0\}$.*

Proposition 0.14. *(a) If $\{C_j\}_{j\in J}$ is a family of closed convex sets such that $\bigcap_{j\in J} C_j \neq \emptyset$ then*

$$\left(\bigcap_{j\in J} C_j\right)_\infty = \bigcap_{j\in J} (C_j)_\infty$$

(b) If $C_i \subseteq \mathbb{R}^{n_i}$ is a non-empty closed convex set for $i = 1, 2, ..., k$ then

$$(C_1 \times ... \times C_k)_\infty = (C_1)_\infty \times ... \times (C_k)_\infty.$$

(c) Let $A : \mathbb{R}^n \mapsto \mathbb{R}^m$ be linear. Then,
(i) If $\emptyset \neq C$ is closed convex and $A(C)$ is closed then $A(C_\infty) \subseteq [A(C)]_\infty$.
(ii) If $\emptyset \neq D$ is closed convex and $A^{-1}(D) \neq \emptyset$ then $A^{-1}(D_\infty) = [A^{-1}(D)]_\infty$.

Proposition 0.15. *Let $A : \mathbb{R}^n \mapsto \mathbb{R}^m$ be linear, $\emptyset \neq C \subseteq \mathbb{R}^n$ closed convex such that $A^{-1}(0) \cap C_\infty = \{0\}$ (or $\subseteq -C_\infty$) then:*
(i) $A(C)$ is closed
(ii) $A(C_\infty) = [A(C)]_\infty$

Definition 0.2. The **linearity space** of C is defined as $C_\infty \cap (-C_\infty)$ which you can prove is the largest subspace contained in C_∞.

Convex Functions

Notation 1. Let us denote $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\} = [-\infty, \infty]$ and for $f : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ we denote

$$\operatorname{dom} f = \{x \in \mathbb{R}^n : f(x) < \infty\}$$
$$\operatorname{epi} f = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq r\}$$
$$\operatorname{epi}_S f = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : f(x) < r\}$$
$$f^{-1}(-\infty, r] = \{x \in \mathbb{R}^n : f(x) \leq r\}$$
$$f^{-1}(-\infty, r) = \{x \in \mathbb{R}^n : f(x) < r\}.$$

Definition 0.3. A **convex function** $f : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ is a function where its **epigraph** $\operatorname{epi} f$ is convex. We say such functions $f \in$ E-Conv $\mathbb{R}^n$.

Definition 0.4. $f : \mathbb{R}^n \mapsto \mathbb{R}$ is **proper convex** if $f \in$ E-Conv $\mathbb{R}^n$, $f(x) > -\infty$ for all $x \in \mathbb{R}^n$, and $f \neq \infty$ (or equivalently, $\exists x \in \mathbb{R}^n$ such that $f(x) < \infty$). We say that such that such functions $f \in$ Conv $\mathbb{R}^n$.

Proposition 0.16. *Let $f : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ be given. Then the following are equivalent:*

(a) $f \in$ E-Conv $\mathbb{R}^n$

(b) $\operatorname{epi}_S f$ is a convex set

(c) $f(\alpha x_0 + (1-\alpha)x_1) \leq \alpha f(x_0) + (1-\alpha) f(x_1)$ for all $\alpha \in (0,1)$ and $\forall x_0, x_1 \in \operatorname{dom} f$.

Proposition 0.17. *Let $f \in$ E-Conv $\mathbb{R}^n$. Then*

(a) $f^{-1}[-\infty, r)$ is convex for all $r \in \bar{\mathbb{R}}$

(b) $f^{-1}[-\infty, r]$ is convex for all $r \in \bar{\mathbb{R}}$

So $\operatorname{dom} f$ is convex.

Proposition 0.18. *(Jensen's inequality) If $f \in$ E-Conv $\mathbb{R}^n$ then*

$$f(\alpha_0 x_0 + \dots + \alpha_k x_k) \leq \sum_{i=1}^{k} \alpha_i f(x_i)$$

for all $(\alpha_0, \dots, \alpha_k) \in \Delta_k$ the k-dimensional probability simplex and $x_i \in \operatorname{dom} f$ for $i = 0, 1, \dots, k$.

Definition 0.5. A function $f : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ is **strictly convex** if f is proper and

$$f(\alpha x_0 + (1-\alpha)x_1) < \alpha f(x_0) + (1-\alpha) f(x_1)$$

for all $\alpha \in (0,1)$ and $x_0 \neq x_1 \in \operatorname{dom} f$.

Definition 0.6. A function $f : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ is **β-strongly convex** if f is proper and

$$f(\alpha x_0 + (1-\alpha)x_1) \leq \alpha f(x_0) + (1-\alpha) f(x_1) - \frac{\beta}{2}\alpha(1-\alpha)\|x_0 - x_1\|^2$$

for all $\alpha \in (0,1)$ and $x_0 \neq x_1 \in \operatorname{dom} f$.

Remark 0.1. We have f is $\beta-$strongly convex $\implies$ f is strictly convex $\implies$ f convex

Proposition 0.19. *f is β-strongly convex $\iff$ $f - \frac{\beta}{2}\|\cdot\|^2$ is convex.*

Proposition 0.20. *(a) If $f_1, \dots, f_k \in$ Conv $\mathbb{R}^n$ and $\alpha_1, \dots, \alpha_n \geq 0$ then*

$$\alpha_1 f_1 + \dots + \alpha_n f_k \in \begin{cases} \text{Conv } \mathbb{R}^n, & \text{if } \bigcap_{i=1}^{k} \operatorname{dom} f_i \neq \emptyset \\ \infty, & \text{otherwise.} \end{cases}$$

(b) If $\{f_i\}_{i\in I} \in$ E-Conv $\mathbb{R}^n$ then $\sup_{i \in I} f_i \in$ Conv $\mathbb{R}^n$ or $f = \infty$. Note that this can follow from $\operatorname{epi}(\sup_{i\in I} f_i) = \bigcap_{i \in I} \operatorname{epi} f$.

(c) If $f \in$ Conv $\mathbb{R}^n$ and $A : \mathbb{R}^n \mapsto \mathbb{R}^m$ is affine such that $A(\mathbb{R}^n) \cap \operatorname{dom} f \neq \emptyset$ then $f \circ A \in$ Conv $\mathbb{R}^n$.

Proposition 0.21. *If $f \in$ Conv $\mathbb{R}^n$ then $\forall x_0 \in \operatorname{ri}(\operatorname{dom} f)$, $\exists L = L(x_0) \geq 0$ and neighbourhood $N(x_0)$ of x_0 such that*

$$|f(x) - f(\bar{x})| \leq L\|x - \tilde{x}\|$$

for all $x, \tilde{x} \in N(x_0) \cap \operatorname{aff}(\operatorname{dom} f)$. In particular, this result implies that f is continuous on $\operatorname{ri}(\operatorname{dom} f)$.

<u>Continuity</u>

Proposition 0.22. *If $f \in$ Conv $\mathbb{R}^n$ then for all compact set $K \subseteq \operatorname{ri}(\operatorname{dom} f)$ there exists $L = L(K)$ such that*

$$|f(x) - f(\bar{x})| \leq L\|x - \tilde{x}\|.$$

Corollary 0.2. *If $f \in$ Conv $\mathbb{R}^n$ finite everywhere, then f is continuous on $\mathbb{R}^n$ and for every bounded set $C \subseteq \mathbb{R}^n$ there exists $L = L(C)$ such that*

$$|f(x) - f(\tilde{x})| \leq L\|x - \tilde{x}\|, \forall x, \tilde{x} \in C.$$

Definition 0.7. The **lower semi-continuous hull** of $f : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$, denoted by $\operatorname{lsc} f$ is defined as

$$\begin{aligned}(\operatorname{lsc} f)(x) &= \liminf_{y \to x} f(y) \\ &= \inf\left\{v : \exists \{y_k\} \to x \text{ s.t. } \lim_{n\to\infty} f(x_k) = v\right\} \leq f(x).\end{aligned}$$

Definition 0.8. A function $f : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ is lower semi-continuous (lsc) at $x \in \mathbb{R}^n$ if $f(x) = (\operatorname{lsc} f)(x)$. The function f is **lower semi-continuous** if $(\operatorname{lsc} f) = f$.

Proposition 0.23. *Let $f : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$. Then:*

(a) $\operatorname{epi}(\operatorname{lsc} f) = \operatorname{cl}(\operatorname{epi} f)$

(b) If $f \in$ E-Conv $\mathbb{R}^n$ then $\operatorname{lsc} f \in$ E-Conv $\mathbb{R}^n$

Proposition 0.24. *For $f : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ the following are equivalent:*

(a) $\operatorname{epi} f$ is closed

(b) $f^{-1}[-\infty, r]$ is (possibly empty) closed for all $\forall r \in \mathbb{R}$

(c) f is lsc

Proposition 0.25. *Let $f : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$. Then,*

(a) $\operatorname{lsc} f$ is lsc and $\operatorname{lsc} f \leq f$.

(b) $\operatorname{lsc} f = \sup\{g : g \leq f, g \text{ lsc}\} =: h$

(c) $\operatorname{lsc} f$ is the largest lsc function minorizing f, i.e. if g is lsc with $g \leq f$ then $g \leq \operatorname{lsc} f$.

Proposition 0.26. *Assume that $f : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ is lsc and $K \subseteq \mathbb{R}^n$ is compact and non-empty. Then $\exists x^* \in K$ such that*

$$f(x^*) = \inf\{f(x) : x \in K\}.$$

Definition 0.9. A function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is **0-coercive** if $\lim_{\|x\|\to\infty} f(x) = \infty$ or equivalently $\forall r \in \mathbb{R}$, $\exists M > 0$ such that $\|x\| > M \implies f(x) > r$. Also equivalently, $\forall r \in \mathbb{R}, \exists M > 0$ such that $x \in f^{-1}[-\infty, r] \implies \|x\| \leq M$ or equivalently $\forall r \in \mathbb{R}, \exists M > 0$ such that $f^{-1}[-\infty, r] \subseteq \bar{B}(0; M)$ or equivalently $\forall r \in \mathbb{R}$, $f^{-1}[-\infty, r]$ is bounded.

Proposition 0.27. *Assume $f : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ is lsc and 0-coercive. Then $\exists x^* \in \mathbb{R}^n$ such that*

$$f(x^*) = \inf\{f(x) : x \in \mathbb{R}^n\}.$$

Closures of Convex Functions

Definition 0.10. For $f \in$ E-Conv $\mathbb{R}^n$ the closure of f, denoted by $\operatorname{cl} f$ is defined as

$$\operatorname{cl} f = \begin{cases} \operatorname{lsc} f, & \text{if } f \in \text{Conv}\,\mathbb{R}^n \text{ or } f = \infty \\ -\infty, & \text{otherwise.} \end{cases}$$

Definition 0.11. f is **closed** if $f = \operatorname{cl} f$.

Notation 2. E-$\overline{\text{Conv}}\,\mathbb{R}^n$ is the set of all **closed convex functions**. $\overline{\text{Conv}}\,\mathbb{R}^n$ is the set of all **proper closed convex functions**.

Lemma 0.3. *For $f \in$ E-Conv $\mathbb{R}^n$,*

$$\operatorname{ri}(\operatorname{epi} f) = \{(x,r) \in \mathbb{R}^n \times \mathbb{R} : x \in \operatorname{ri}(\operatorname{dom} f), r > f(x)\}$$

Proposition 0.28. *Suppose $f \in$ E-Conv $\mathbb{R}^n$ and $x_0 \in \operatorname{ri}(\operatorname{dom} f)$. Then $\forall x \in \mathbb{R}^n$ we have*

$$(\operatorname{lsc} f)(x) = \lim_{t \downarrow 0} f(x + t(x_0 - x)).$$

Proposition 0.29. *Suppose that $f \in$ E-Conv $\mathbb{R}^n$. Then:*
(a) $f(x) = (\operatorname{lsc} f)(x)$ *for all* $x \in \mathbb{R}^n \setminus \operatorname{rbd}(\operatorname{dom} f)$
(b) $\operatorname{dom} f \subseteq \operatorname{dom}(\operatorname{lsc} f) \subseteq \operatorname{cl}(\operatorname{dom} f)$

Corollary 0.3. *If $f \in$ Conv $\mathbb{R}^n$ then*
(a) $f(x) = (\operatorname{cl} f)(x)$ *for all* $x \in \mathbb{R}^n \setminus \operatorname{rbd}(\operatorname{dom} f)$
(b) $\operatorname{dom} f \subseteq \operatorname{dom}(\operatorname{cl} f) \subseteq \operatorname{cl}(\operatorname{dom} f)$

Corollary 0.4. *If $f \in$ Conv $\mathbb{R}^n$ and $\operatorname{dom} f$ is an affine manifold then $f \in \overline{\text{Conv}}\,\mathbb{R}^n$.*

Proposition 0.30. *Suppose $f \in$ E-Conv $\mathbb{R}^n$ and $(\operatorname{lsc} f)(x_0) = -\infty$ for some $x_0 \in \mathbb{R}^n$ (e.g. $f(x_0) = -\infty$ for some $x_0 \in \mathbb{R}^n$). Then,*
(a) $(\operatorname{lsc} f)(x) = -\infty$ *for all* $x \in \operatorname{cl}(\operatorname{dom} f)$ *and* $\operatorname{dom}(\operatorname{lsc} f) = \operatorname{cl}(\operatorname{dom} f)$
(b) $f(x) = -\infty$ *for all* $x \in \operatorname{ri}(\operatorname{dom} f)$
As a consequence of (a) and (b), $\operatorname{cl} f, \operatorname{lsc} f$ *agree on* $\operatorname{cl}(\operatorname{dom} f)$ *and* $f, \operatorname{cl} f$ *agree on* $\operatorname{ri}(\operatorname{dom} f)$.

Definition 0.12. The **convex hull** of denoted by $\operatorname{co} f$, is defined as

$$\operatorname{co} f = \sup\{g \in \text{E-Conv}\,\mathbb{R}^n : g \le f\}$$

Definition 0.13. The **closed convex hull** of $f : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$, denoted by $\overline{\operatorname{co}} f$, is defined as $\overline{\operatorname{co}} f = \operatorname{cl}(\operatorname{co} f)$.

Proposition 0.31. *(1)* $\operatorname{co} f \in$ *E-Conv* $\mathbb{R}^n$, $\operatorname{co} f \le f$
(2) if $g \in$ *E-Conv* $\mathbb{R}^n$, $g \le f$, *then* $g \le \operatorname{co} f$.

Proposition 0.32. *(1)* $\overline{\operatorname{co}} f \in$ *E-Conv* $\mathbb{R}^n$, $\overline{\operatorname{co}} f \le f$
(2) if $g \in$ *E-*$\overline{\textit{Conv}}\,\mathbb{R}^n$, $g \le f$, *then* $g \le \overline{\operatorname{co}} f$.

Proposition 0.33. *(1)* $\operatorname{cl} f \in$ *E-*$\overline{\textit{Conv}}\,\mathbb{R}^n$
(2) If $g \in$ *E-*$\overline{\textit{Conv}}\,\mathbb{R}^n$, $g \le f \implies g \le \operatorname{cl} f$.

Derivatives

Definition 0.14. Let $f : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ and $\bar{x} \in \mathbb{R}^n$ such that $f(\bar{x}) \in \mathbb{R}$. The **directional derivative** of f at $\bar{x}$ along d is

$$f'(x; d) = \lim_{t \downarrow 0} \frac{f(\bar{x} + td) - f(\bar{x})}{t}$$

whenever it exists where $\pm\infty$ is possible.

Definition 0.15. $f : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ is **differentiable** at $\bar{x}$ if $f(\bar{x}) \in \mathbb{R}$ and $\exists$ linear map $f'(\bar{x}) : \mathbb{R}^n \mapsto \mathbb{R}$ such that

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{R}^n}} \frac{f(\bar{x} + h) - [f(\bar{x}) + f'(\bar{x})h]}{\|h\|} = 0.$$

Remark 0.2. (1) $f'(\bar{x})$ is unique
(2) f is differentiable at $\bar{x} \implies \bar{x} \in \operatorname{int}(\operatorname{dom} f)$.
(3) f is differentiable at $\bar{x} \implies f'(\bar{x}; d) = f'(\bar{x})d$.

Remark 0.3. The gradient is $T : \mathbb{R}^n \mapsto \mathbb{R}$ over inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ where $\exists! a \in \mathbb{R}^n$ such that $T(\cdot) = \langle a, \cdot \rangle$. In particular, $T = f'(\bar{x})$ and $f'(\bar{x})d = \langle a, d \rangle$.

Proposition 0.34. *Let $f : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ and $\bar{x} \in \mathbb{R}^n$ be such that $f(\bar{x}) \in \mathbb{R}$. If $\bar{x}$ is a local minimum of $\inf\{f(x) : x \in \mathbb{R}^n\}$ then*

$$f'(\bar{x}; d) \ge 0, \forall d \in \mathbb{R}^n$$

whenever it exists. As a consequence, if f is differentiable at $\bar{x}$ then $f'(\bar{x}) = 0$.

Proposition 0.35. *Assume $f \in$ E-Conv $\mathbb{R}^n$ and $\bar{x}, d \in \mathbb{R}^n$ are such that $f(\bar{x}) \in \mathbb{R}$. Define*

$$\Delta f(\cdot; x, d) : \mathbb{R}_{++} \mapsto \bar{\mathbb{R}}$$

as

$$\Delta f(t; \bar{x}, d) = \frac{f(\bar{x} + td) - f(\bar{x})}{t}.$$

Then,
(1) $\Delta f(\cdot; \bar{x}, d)$ *is non-decreasing*
(2) if $f(\cdot)$ *is strictly convex and* $d \neq 0$ *then* $\Delta f(\cdot; \bar{x}, d)$ *is increasing*
(3) if f *is* β*-strongly convex, then for all* $0 < t_1 < t_2$,

$$\Delta f(t_1) \le \Delta f(t_2) - \frac{\beta}{2}(t_2 - t_1)\|d\|^2.$$

Proposition 0.36. *Assume that $f \in$ E-Conv $\mathbb{R}^n$ and $\bar{x} \in \mathbb{R}^n$ such that $f(\bar{x}) \in \mathbb{R}$. Then,*
(a) $\forall d \in \mathbb{R}^n$, $f'(\bar{x}; d)$ *exists and* $f'(\bar{x}; d) = \inf_{t>0} \Delta f(t; \bar{x}, d)$
(b) $f(x) - f(\bar{x}) \ge f'(\bar{x}; x - \bar{x})$, $\forall x \in \mathbb{R}^n$
(c) $f(x) - f(\bar{x}) > f'(\bar{x}; x - \bar{x})$, $\forall x \in \mathbb{R}^n \setminus \{\bar{x}\}$ *if f is strictly convex*
(d) $f(x) - f(\bar{x}) \ge f'(\bar{x}; x - \bar{x}) + \frac{\beta}{2}\|x - \bar{x}\|^2, \forall x \in \operatorname{dom} f$ *if f is β-strongly convex*

Proposition 0.37. *Assume that $f \in$ E-Conv $\mathbb{R}^n$ and $\bar{x} \in \mathbb{R}^n$ such that $f(\bar{x}) \in \mathbb{R}$. Then the following are equivalent:*

(a) $\bar{x}$ is a global min of $f(x)$ on $\mathbb{R}^n$

(b) $\bar{x}$ is a local min of $f(x)$ on $\mathbb{R}^n$

(c) $f'(\bar{x}; d) \geq 0$ for all $d \in \mathbb{R}^n$

(d) $f'(\bar{x}; x - \bar{x}) \geq 0$ for all $x \in \operatorname{dom} f$

If f is differentiable at $\bar{x}$ then,

(e) $f'(\bar{x}) = 0$

Corollary 0.5. *Assume f is β-strongly convex and $\bar{x}$ is a global minimum of f over $\mathbb{R}^n$. Then:*

$$f(x) - f(\bar{x}) \geq \frac{\beta}{2}\|x - \bar{x}\|^2.$$

Definition 0.16. If $f : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ is proper and $\emptyset \neq C \subseteq \operatorname{dom} f$ is convex, we say f **is convex on** C if

$$f_C(x) = \begin{cases} f(x), & x \in C \\ +\infty, & \text{otherwise} \end{cases}$$

is convex.

Proposition 0.38. *Assume $f : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ is proper, $\emptyset \neq C \subseteq \operatorname{dom} f$ is convex, and f is convex on C. Then following are equivalent:*

(a) $\bar{x} \in C$ is a global minimum of f over C

(b) $\bar{x} \in C$ is a local minimum of f over C

(c) $f'(\bar{x}; d) \geq 0$ for all $d \in \mathbb{R}_+ \cdot (C - \bar{x})$

(d) $f'(\bar{x}; x - \bar{x}) \geq 0$ for all $x \in C$

Proposition 0.39. *Assume $f : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ is proper, $\emptyset \neq C \subseteq \operatorname{dom} f$ is convex, and f is strictly convex on C. Assume $\bar{x}$ is a global minimum of f over C. then $\bar{x}$ is the unique global minimum of f over C.*

Asymptotic Function

Definition 0.17. For $f \in \overline{\text{Conv}}\ \mathbb{R}^n$, its **asymptotic function** $f'_\infty : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ is defined as

$$f'_\infty(d) = \sup_{\substack{t>0 \\ x \in \operatorname{dom} f}} \frac{f(x+td) - f(x)}{t}.$$

Proposition 0.40. *For $f \in \overline{\text{Conv}}\ \mathbb{R}^n$, have:*

(a) $\operatorname{epi} f'_\infty = (\operatorname{epi} f)_\infty$

(b) If $x_0 \in \operatorname{dom} f$ then

$$f'_\infty(d) = \sup_{t>0} \underbrace{\frac{f'(x_0 + td) - f(x_0)}{t}}_{:=h_1(d)} \overset{(o)}{=} \sup_{x \in \operatorname{dom} f} \underbrace{f(x+d) - f(x)}_{:=h_2(d)}.$$

Proposition 0.41. *Let $f \in \overline{\text{Conv}}\ \mathbb{R}^n$. Then,*

(a) $f'_\infty \in \overline{\text{Conv}}\ \mathbb{R}^n$

(b) $f'_\infty(\alpha d) = \alpha f'_\infty(d)$ for all $\alpha \geq 0, d \in \mathbb{R}^n$

(c) $\forall r \in \mathbb{R}$ s.t. $f^{-1}[-\infty, r] \neq \emptyset$, we have $\left(f^{-1}[\infty, r]\right)_\infty = (f'_\infty)^{-1}[-\infty, 0]$.

Proposition 0.42. *Let $f \in \overline{\text{Conv}}\ \mathbb{R}^n$. Then the following are equivalent:*

(a) $\forall r \in \mathbb{R}$, $f^{-1}[-\infty, r]$ is bounded (i.e. f is coercive).

(b) $\exists r_0 \in \mathbb{R}$ s.t. $f^{-1}[-\infty, r_0] \neq \emptyset$ and bounded.

(c) the set of optimal solutions of $\min_{x \in \mathbb{R}^n} f(x) \neq \emptyset$ and bounded.

(d) $f'_\infty(d) > 0, \forall d \in \mathbb{R}^n \backslash \{0\}$.

Proposition 0.43. *(1) If $f_1, ..., f_k \in \overline{\text{Conv}}\ \mathbb{R}^n$ such that $\bigcap_{i=1}^k \operatorname{dom} f_i \neq \emptyset$ then for all $\alpha_1, ..., \alpha_k \geq 0$*

$$(\alpha_1 f_1 + ... + \alpha_k f_k)_\infty = \alpha_1 (f_1)'_\infty + ... + \alpha_k (f_k)'_\infty$$

and $\alpha_1 f_1 + ... + \alpha_k f_k \in \overline{\text{Conv}}\ \mathbb{R}^n$.

(2) If $\{f_i\}_{i \in I} \subseteq \overline{\text{Conv}}\ \mathbb{R}^n$ such that $\sup_{\underline{i \in I}} f_i(x_0) < \infty$ for some $x_0 \in \mathbb{R}^n$ then $f := \sup_{i \in I} f_i \in \overline{\text{Conv}}\ \mathbb{R}^n$ and $f'_\infty = \sup_{i \in I}(f_i)'_\infty$.

(3) If $f \in \overline{\text{Conv}}\ \mathbb{R}^n$, $A : \mathbb{R}^n \mapsto \mathbb{R}^m$ affine such that $A(\mathbb{R}^n) \cap \operatorname{dom} f \neq \emptyset$ then $f \circ A \in \overline{\text{Conv}}\ \mathbb{R}^n$ and

$$(f \circ A)'_\infty = f'_\infty \circ (A_0) \text{ where } A_0(\cdot) = A(\cdot) - A(0).$$

Corollary 0.6. *We have*

$$(f_C)'_\infty(d) = (f + I_C)'_\infty(d) = f'_\infty(d) + (I_C)'_\infty = f'_\infty(d) + I_{C_\infty}(d).$$

Differentiable Functions

Proposition 0.44. *Let $f : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ be differentiable on a nonempty convex set $C \subseteq \operatorname{dom} f$. Then the following are equivalent:*

(a) f is convex on C, i.e.

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y), \forall x, y \in C, \alpha \in (0,1)$$

(b) $f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle$, $\forall x, y \in C$

(c) $[f'(y) - f'(x)](y - x) \geq 0$, $\forall x, y \in C$.

Corollary 0.7. *Assume $f : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ is differentiable on a nonempty convex set $C \subseteq \operatorname{dom} f$. Then for all $\forall \beta \in \mathbb{R}$, the following are equivalent,*

(a) $\forall x, y \in C, \forall \alpha \in (0,1)$ we have

$$f(\alpha x + (1-\alpha)y) + \frac{\beta}{2}\alpha(1-\alpha) \leq \alpha f(x) + (1-\alpha) f(y)$$

(b) $f - \frac{\beta}{2}\|\cdot\|^2$ is convex.

(c) $\forall x, y \in C$, $f(y) \geq f(x) + f'(x)(y-x) + \frac{\beta}{2}\|y - x\|^2$

(d) $\forall x, y \in C$, $[f'(y) - f'(x)](y - x) \geq \beta\|y - x\|^2$.

Corollary 0.8. *Assume $f : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ is differentiable on a nonempty convex set $C \subseteq \operatorname{dom} f$. Then $\forall L \in \mathbb{R}$ the following are equivalent:*

(a) $\forall x, y \in C, \forall \alpha \in (0,1)$ we have

$$f(\alpha x + (1-\alpha)y) + \frac{L}{2}\alpha(1-\alpha) \geq \alpha f(x) + (1-\alpha) f(y)$$

(b) $\frac{L}{2}\|\cdot\|^2 - f$ is convex.

(c) $\forall x, y \in C$, $f(y) \leq f(x) + f'(x)(y-x) + \frac{L}{2}\|y - x\|^2$

(d) $\forall x, y \in C$, $[f'(y) - f'(x)](y - x) \leq L\|y - x\|^2$.

Separation Theory

Proposition 0.45. $\bar{c} = \Pi_C(x) \iff \langle c - \bar{c}, x - \bar{c}\rangle \le 0$ *for all* $c \in C$.

Proposition 0.46. *For every* $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\|\Pi_C(x) - \Pi_C(y)\|^2 \le \langle x - y, \Pi_C(x) - \Pi_C(y)\rangle$$

and as a consequence,

$$\|\Pi_C(x) - \Pi(y)\| \le \|x - y\|.$$

Definition 0.18. Let $C_1, C_2 \subseteq \mathbb{R}^n$ be nonempty an H be a hyperplane.

(a) H **separates** C_1, C_2 if $C_1 \subseteq H^{\le}$ and $C_2 \subseteq H^{\ge}$.

(b) H **properly separates** C_1, C_2 if H separates them and $C_1 \cup C_2 \subseteq H$.

(c) H **strongly separates** C_1, C_2 if H separates $C_1 + \bar{B}(0;\delta_1), C_2 + \bar{B}(0;\delta_2)$ for some $\delta_1, \delta_2 > 0$.

Proposition 0.47. *Let* $\emptyset \ne C_1, C_2 \subseteq \mathbb{R}^n$ *be given*

(a) $\exists$ *hyperplane separating* $C_1, C_2 \iff \exists 0 \ne s \in \mathbb{R}^n$ *s.t.* $\sup_{x_1 \in C} \langle s, x_1\rangle \le \inf_{x_2 \in C} \langle s, x_2\rangle \quad (*)$.

(b) $\exists$ *hyperplane properly separating* $C_1, C_2 \iff \exists s \in \mathbb{R}^n$ *s.t. (*) holds and* $\inf_{x_1 \in C} \langle s, x_1\rangle < \sup_{x_2 \in C} \langle s, x_2\rangle$.

(c) $\exists$ *hyperplane strongly separating* $C_1, C_2 \iff \exists s \in \mathbb{R}^n$ *s.t. (*) holds strictly.*

Proposition 0.48. *Let* $\emptyset \ne C_1, C_2 \subseteq \mathbb{R}^n$ *be given. Then* C_1, C_2 *can be separated* $\iff \{0\}, C = C_1 - C_2$ *can be separated.*

Proposition 0.49. *Let* $\emptyset \ne C \subseteq \mathbb{R}^n$ *be a convex set and* $x \in \mathbb{R}^n$. *Then,*

(a) x, C (C_1, C_2) *can be strongly separated* $\iff x \notin \operatorname{cl} C$ $(0 \notin \operatorname{cl}(C_1 - C_2))$

(b) x, C (C_1, C_2) *can be properly separated* $\iff x \notin \operatorname{ri} C$ $(0 \notin \operatorname{ri}(C_1 - C_2))$.

Proposition 0.50. *Let* $\emptyset \ne C \subseteq \mathbb{R}^n$ *be a convex set and* $x \in \mathbb{R}^n$. *Then,*

$$\operatorname{cl} C = \bigcap \left\{H^{\le} : H \text{ is a hyperplane}, C \subseteq H^{\le}\right\}.$$

Corollary 0.9. *If* $f \in$ *E-Conv* $\mathbb{R}^n$ *then*

$$\operatorname{epi}(\operatorname{lsc} f) = \operatorname{cl}(\operatorname{epi} f) = \bigcap \left\{H^{\le} : H \text{ is a hyperplane}, \operatorname{epi} f \subseteq H^{\le}\right\}.$$

Remark 0.4. A closed halfspace has one of the following representations:

(1) $H^+(s, \beta) = \{(x, t) : \langle s, t\rangle + t \le \beta\}$

(2) $H^-(s, \beta) = \{(x, t) : \langle s, t\rangle - t \le \beta\}$

(3) $H^0(s, \beta) = \{(x, t) : \langle s, t\rangle \le \beta\}$

Observe that

(1) $H^+(s, \beta)$ is **not** an epigraph

(2) $H^-(s, \beta) = \operatorname{epi}(\langle s, \cdot\rangle - \beta)$

(3) $H^0(s, \beta) = H^{\le}_{s,\beta} \times \mathbb{R}$

Proposition 0.51. *If* $f \in$ *E-Conv* $\mathbb{R}^n$ *then*

$$\begin{aligned}\operatorname{cl} f &= \sup\{A : A \text{ is affine}, A \le f\} \\ &= \sup_{(s,\beta)} \{\langle s, \cdot\rangle - \beta : \langle s, \cdot\rangle - \beta \le f\}.\end{aligned}$$

Also if $f \in$ *Conv* $\mathbb{R}^n$ *then* $\exists$ *affine function minorizing* f.

Conjugate Functions

Definition 0.19. The **conjugate** of $f : \mathbb{R}^n \mapsto \mathbb{R}$, denoted by f^*, is defined as $f^* : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ where

$$s \mapsto f^*(s) = \sup_{x \in \mathbb{R}^n} \langle x, s\rangle - f(x).$$

Observe that $\forall s \in \mathbb{R}^n$ we have

$$f^*(s) = \sup_{x \in \operatorname{dom} f} \langle x, s\rangle - f(x) = \sup_{(x,t) \in \operatorname{epi} f} \langle x, s\rangle - t.$$

Proposition 0.52. *We have:*

(a) if $f = \infty$ *then* $f^* = -\infty$

(b) if $f(x_0) = -\infty$ *for some* x_0 *then* $f^* = \infty$

(c) $\operatorname{epi} f^* = \{(s, \beta) : \langle s, \cdot\rangle - \beta \le f\}$

(d) $f^*(s) = \inf\{\beta : \langle s, \cdot\rangle - \beta \le f\}$

(e) $-f^*(0) = \inf\{f(x) : x \in \mathbb{R}^n\}$

(f) $\forall x, s \in \mathbb{R}^n$, $f^*(s) \ge \langle x, s\rangle - f(x)$

Proposition 0.53. *For any* $f \in$ *E-Conv* $\mathbb{R}^n$,

$$f^* = (\operatorname{cl} f)^* = (\operatorname{lsc} f)^*.$$

Proof. Let $A = \langle s, \cdot\rangle - \beta$. Then $A \le f \iff A \le \operatorname{lsc} f \iff A \le \operatorname{cl} f$. □

Definition 0.20. **Fenchel's inequality** is

$$f^*(x) \ge \langle x, s\rangle - f(x).$$

Proposition 0.54. *Let* $f : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ *be such that*

(1) $f \ne \infty$

(2) f *is minorized by an affine function*

Then, $f^* \in \overline{\text{Conv}}\, \mathbb{R}^n$. *As a consequence, if* $f \in$ *Conv* $\mathbb{R}^n$ *then* $f^* \in \overline{\text{Conv}}\, \mathbb{R}^n$.

Proposition 0.55. *Assume that* $f \in$ *E-Conv* $\mathbb{R}^n$. *Then*

$$\operatorname{cl} f = f^{**} = (f^*)^*.$$

Subgradients

Definition 0.21. We say $s \in \partial f(\bar{x})$ where ∂f is the subgradient of f if and only if

$$f(x) \ge f(\bar{x}) + \langle s, x - \bar{x}\rangle, \forall x \in \mathbb{R}^n.$$

Remark 0.5. We have

- $f(\bar{x}) = +\infty \implies \partial f(\bar{x}) = \mathbb{R}^n$
- $f(\bar{x}) = -\infty$ then $\partial f(\bar{x}) \ne \emptyset \iff f = +\infty$ in which case $\partial f(\bar{x}) = \mathbb{R}^n$.

Assumption. *(A) $f : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ and $\bar{x} \in \mathbb{R}^n$ such that $f(\bar{x}) \in \mathbb{R}$.*

Proposition 0.56. *If (A) holds then*

(a) $\bar{x}$ is a global minimum of f over $\mathbb{R}^n \iff 0 \in \partial f(\bar{x})$.

(b) $\partial f(\bar{x})$ is a (possibly empty) closed convex set.

Proposition 0.57. *Assume that $f \in$ E-Conv $\mathbb{R}^n$ and $\bar{x} \in \mathbb{R}^n$ such that $f(\bar{x}) \in \mathbb{R}$. Then,*

$$\partial f(\bar{x}) = \{s \in \mathbb{R}^n : \langle s, \cdot\rangle \leq f'(\bar{x}; \cdot)\}$$

and also

$$\operatorname{cl} f'(\bar{x}; \cdot) = \sigma_{\partial f(\bar{x})} = \sup_{s \in \partial f(\bar{x})} \langle s, \cdot\rangle .$$

Proposition 0.58. *Let $f : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ and $\bar{x} \in \mathbb{R}^n$ be given. Then $s \in \partial f(x) \iff f^*(s) \leq \langle x, s\rangle - f(x)$.*

Definition 0.22. or a multivalued map $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, define $A^{-1}(y) = \{x : y \in A(x)\}$.

Lemma 0.4. *Let $f : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ and $\bar{x} \in \mathbb{R}^n$ such that $\partial f(\bar{x}) \neq \emptyset$ be given. Then:*

(a) $(\operatorname{lsc} f)(\bar{x}) = f(\bar{x})$, i.e. f is lsc at $\bar{x}$

(b) If $f \in$ E-Conv $\mathbb{R}^n$ then $(\operatorname{cl} f)(\bar{x}) = f(\bar{x})$.

Proposition 0.59. *Let $f \in$ E-Conv $\mathbb{R}^n$ and $x \in \mathbb{R}^n$ be given. Then for $s \in \mathbb{R}^n$ the following are equivalent*

(a) $s \in \partial f(x)$

(b) $s \in \partial(\operatorname{cl} f)(x)$ and $(\operatorname{cl} f)(x) = f(x)$

(c) $x \in \partial f^*(x)$ and $(\operatorname{cl} f)(x) = f(x)$.

Corollary 0.10. *If $f \in$ E-Conv $\mathbb{R}^n$ then $s \in \partial f(x) \iff x \in \partial f^*(s)$.*

Corollary 0.11. *If $f \in \overline{\text{Conv}}$ $\mathbb{R}^n$ then $\partial f^*(0) = \operatorname{argmin}_{x\in\mathbb{R}^n} f(x)$.*

Sublinear Functions

Definition 0.23. $\sigma : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ is **sublinear** if $\operatorname{epi} \sigma$ is a convex cone.

Definition 0.24. $\sigma : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ is **subadditive** if $\sigma(x_0 + x_1) \leq \sigma(x_0) + \sigma(x_1)$ and is **positively homogeneous** (of degree 1) if $\sigma(tx) = t\sigma(x)$ for all $t > 0$ and for all $x \in \mathbb{R}^n$.

Proposition 0.60. *Let $\sigma : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$. Then the following are equivalent:*

(a) σ is sublinear

(b) σ is convex and positively homogeneous

(c) σ is subadditive and positively homogeneous

(d) $\sigma(t_0x_0 + t_1x_1) \leq t_0\sigma(x_0) + t_1\sigma(t_1)$ for all $t_0, t_1 > 0$ and for all $x_0, x_1 \in \operatorname{dom} \sigma$

Proposition 0.61. *Let $\sigma : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ be sublinear. Then,*

(a) $\operatorname{dom} \sigma$ is a convex cone

(b) $\sigma(0) \in \{-\infty, 0, +\infty\}$

(c) if σ is proper then $\sigma(x) + \sigma(-x) \geq \sigma(0) \geq 0$

(d) if σ is proper closed then $\sigma(0) = 0$

Remark 0.6. $\sigma_C(s) = \sup_{x\in C} \langle s, x\rangle = (I_C)^*$.

Proposition 0.62. *For any $C \subseteq \mathbb{R}^n$ we have*

$$\operatorname{lsc} I_C = I_{\operatorname{cl} C}$$
$$\operatorname{co} I_C = I_{\operatorname{co} C}$$
$$\overline{\operatorname{co}} I_C = I_{\overline{\operatorname{co}} C}.$$

Proposition 0.63. *For any $C \subseteq \mathbb{R}^n$ we have*

$$\sigma_C = \sigma_{\operatorname{cl} C} = \sigma_{\operatorname{co} C} = \sigma_{\overline{\operatorname{co}} C}.$$

Proposition 0.64. *Let $C_1, C_2 \subseteq \mathbb{R}^n$ be closed convex. Then,*

$$C_1 \subseteq C_2 \iff \sigma_{C_1} \leq \sigma_{C_2}$$

and in particular, $C_1 = C_2 \iff \sigma_{C_1} = \sigma_{C_2}$.

Corollary 0.12. *Σ is one-to-one.*

Corollary 0.13. *For any $C \subseteq \mathbb{R}^n$,*

$$\overline{\operatorname{co}} C = \{x \in \mathbb{R}^n : \langle x, \cdot\rangle \leq \sigma_C(\cdot)\}.$$

Proposition 0.65. *(Σ is onto) If σ is a closed sublinear function such that $\sigma \neq \infty$ then $\sigma = \sigma_C$ where*

$$C = C(\sigma) = \{x \in \mathbb{R}^n : \langle x, \cdot\rangle \leq \sigma\} .$$

By the previous result,

$$C = \overline{\operatorname{co}} C = \{x \in \mathbb{R}^n : \langle x, \cdot\rangle \leq \sigma_C\} = \{x \in \mathbb{R}^n : \langle x, \cdot\rangle \leq \sigma\} = C(\sigma).$$

Proposition 0.66. *Assume $f \in$ E-Conv $\mathbb{R}^n$ and $\bar{x} \in \mathbb{R}^n$ is such that $f(\bar{x}) \in \mathbb{R}$. Then,*

(a) $\operatorname{dom}(f'(\bar{x}; \cdot)) = \mathbb{R}_{++} \cdot (\operatorname{dom} f - \bar{x})$

(b) $f'(\bar{x}; \cdot)$ is sublinear.

Proposition 0.67. *Assume $f \in$ E-Conv $\mathbb{R}^n$ and $\bar{x} \in \mathbb{R}^n$ is such that $f(\bar{x}) \in \mathbb{R}$. Then,*

$$\operatorname{cl} f'(\bar{x}; \cdot) = \sigma_{\partial f(\bar{x})}.$$

Proposition 0.68. *Assume $f \in$ E-Conv $\mathbb{R}^n$ and $\bar{x} \in \mathbb{R}^n$ is such that $f(\bar{x}) \in \mathbb{R}$. Then,*

$$\partial f(\bar{x}) = \emptyset \iff \exists d_0 \in \mathbb{R}^n \text{ s.t. } f'(\bar{x}; d_0) = -\infty$$

in which case

$$f'(\bar{x}; d) = -\infty, \forall d \in \operatorname{ri}(\operatorname{dom} f - \bar{x}).$$

Proposition 0.69. *Assume $f \in$ E-Conv $\mathbb{R}^n$ and $\bar{x} \in \mathbb{R}^n$ is s.t. $f(\bar{x}) \in \mathbb{R}$. Then:*

(a) if $\bar{x} \in \operatorname{ri}(\operatorname{dom} f)$, then $\partial f(\bar{x}) \neq \emptyset$ and $f'(\bar{x}; \cdot) = \sigma_{\partial f(\bar{x})}$.

(b) $\bar{x} \in$ int$(\operatorname{dom} f)$ iff $\partial f(\bar{x}) \neq \emptyset$ and bounded, in which case $f'(\bar{x}; d) = \max\{\langle d, s\rangle : s \in \partial f(\bar{x})\}$.

Duality [ECP]

Definition 0.25. Define the **Lagrangian function** for (ECP) $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^E \mapsto (-\infty, +\infty]$ by

$$(x, \lambda) \mapsto \begin{cases} f(x) + \sum_{i\in E} \lambda_i g_i(x), & \text{if } x \in X \\ +\infty, & \text{otherwise} \end{cases} = \begin{cases} f(x) + \langle \lambda, g_E(x)\rangle , \\ +\infty, \end{cases}$$

Note that (ECP) $\iff \inf_x \sup_\lambda \mathcal{L}(x,\lambda) \geq \sup_\lambda \inf_x \mathcal{L}(x,\lambda)$ which we call the dual. Also,

$$\sup_{\lambda\in\mathbb{R}^E} \mathcal{L}(x,\lambda) = \begin{cases} f(x), & \text{if } g_E(x)=0, x\in X \\ +\infty, & \text{otherwise} \end{cases}$$

and so (ECP) $\leftrightarrow \inf_{x\in\mathbb{R}^n}\sup_{\lambda\in\mathbb{R}^E}\mathcal{L}(x,\lambda)$.

Definition 0.26. The **dual function** $\theta : \mathbb{R}^E \mapsto [-\infty,\infty)$ is defined as $\theta(\lambda) = \inf_{x\in\mathbb{R}^n}\mathcal{L}(x,\lambda)$. The dual (ECP) is

$$\text{(DECP)} \qquad \theta^* = \sup_{\lambda\in\mathbb{R}^E}\theta(\lambda) = \sup_{\lambda\in\mathbb{R}^E}\inf_{x\in\mathbb{R}^n}\mathcal{L}(x,\lambda).$$

Note that $-\theta \in \overline{\text{Conv}}\,\mathbb{R}^n$.

Notation 3. For $\lambda\in\mathbb{R}^E$, denote $X(\lambda) = \{x\in\mathbb{R}^n : \mathcal{L}(x,\lambda) = \theta(\lambda)\}$. Observe that:

(1) if $\theta(\lambda) = -\infty$ then $X(\lambda)=\emptyset$

(2) $\theta(\lambda) < \infty$ for all $\lambda\in\mathbb{R}^E$

(3) $X(\lambda) = \{x\in X : \theta(\lambda) = f(x) + \langle\lambda, g_E(x)\rangle\rangle$

Proposition 0.70. *(Everett) Assume $x_\lambda \in X(\lambda)$ for some $\lambda\in\mathbb{R}^E$. Then x_λ is an optimal solution of*

$$\begin{aligned} (P_\lambda) \qquad & \inf f(x) \\ & \text{s.t. } g_E(x) = g_E(x_\lambda) \\ & \quad x\in X. \end{aligned}$$

Definition 0.27. $\lambda^*\in\mathbb{R}^E$ is a **Lagrange multiplier** (LM) of (ECP) if $f_*\in\mathbb{R}$ and $f_*\in\theta(\lambda^*)$ ($\iff f_* = \inf_{x\in X} f(x) + \langle\lambda^*, g_E(x)\rangle$).

Remark 0.7. Consider the set

$$S = \left\{ \begin{pmatrix} g_E(x) \\ f(x) \end{pmatrix} \in \mathbb{R}^E\times\mathbb{R} : x\in X \right\}$$

and let $\eta^* = \begin{pmatrix} \lambda^* \\ 1 \end{pmatrix}$, $s^* = \begin{pmatrix} 0 \\ f^* \end{pmatrix}$. Let $H^{\geq} = \{s : (\eta^*)^T(s-s^*)\geq 0\}$. Then $S\subseteq H^{\geq}$ since $f_* \leq f(x) + \langle\lambda^*, g_E(x)\rangle$ for all $x\in X$ or equivalently,

$$\begin{pmatrix} \lambda^* \\ 1 \end{pmatrix}^T \begin{pmatrix} g_E(x) - 0 \\ f(x) - f_* \end{pmatrix} \geq 0.$$

Proposition 0.71. *For a given $(x^*,\lambda^*)\in\mathbb{R}^n\times\mathbb{R}^E$, then following are equivalent:*

(a) x^* is an optimal solution and λ^* is a Lagrange multiplier for (ECP)

(b) $x^*\in X(\lambda^*)$, $g_E(x^*)=0$.

Proposition 0.72. *(Weak Duality) For every feasible x of (ECP) and $\lambda\in\mathbb{R}^E$, we have $f(x)\geq\theta(\lambda)$. As a consequence, $f_*\geq\theta_*$.*

Proof. $f(x) = \mathcal{L}(x,\lambda) \geq \inf_u \mathcal{L}(u,\lambda) = \theta$. □

Proposition 0.73. *λ^* is a LM of (ECP) $\iff f_* = \theta_*\in\mathbb{R}$ and λ^* is an optimal solution of (DECP).*

Proof. Follows from $f_*\geq\theta_*\geq\theta(\lambda^*)$. So

$$\mathbb{R}\ni f_* = \theta(\lambda^*) \iff f_* = \theta_* \text{ and } \theta_* = \theta(\lambda^*).$$

□

Corollary 0.14. *Assume $f_* = \theta_*\in\mathbb{R}$. Then the set of LM's is equal to the set of dual optimal solutions.*

Definition 0.28. The **value function** for (ECP) is defined as

$$\begin{aligned} v(b) = & \inf f(x) \\ & \text{s.t. } g_E(x) + b = 0 \ (\iff g_E(x) = -b) \\ & \quad x\in X. \end{aligned}$$

Observe that $f_* = v(0)$.

Proposition 0.74. *For all $\lambda\in\mathbb{R}^E$, $v^*(\lambda) = (-\theta)(\lambda)$.*

Corollary 0.15. $(-\theta)^* = \overline{\text{co}}v$ *using the fact that* $v^{**} = \overline{\text{co}}v$.

Proposition 0.75. $\theta_* = (\overline{\text{co}}v)(0)$.

Corollary 0.16. $f_* = \theta_* \iff v(0) = (\overline{\text{co}}v)(0)$.

Proposition 0.76. *The set of dual optimal solutions is equal to $\partial(\overline{\text{co}}v)(0)$.*

Remark 0.8. Observe that $(-\theta)^*(0) = \theta_*$. Also if Λ^* is the set of optimal solutions of (DECP), then $\Lambda^* = \partial(-\theta)^*(0)$.

Corollary 0.17. $\overline{\text{co}}v(0) = \theta_*$ *and* $\partial(\overline{\text{co}}v)(0) = \Lambda^*$.

Proposition 0.77. *λ^* is a Lagrange multiplier (L.M.) of (ECP) $\iff v(0)\in\mathbb{R}$ and $\lambda^*\in\partial v(0)$ (or $f_*\in\mathbb{R}$).*

Proposition 0.78. *Assume $f_*\in\mathbb{R}$, $v\in$ E-Conv $\mathbb{R}^n$, and $0\in \text{ri}(\text{dom}\, v)$. Then (ECP) has a LM.*

Duality [ICP]

Definition 0.29. The **Lagrangian function** for (ICP) is defined as

$$\mathcal{L}(x,\lambda) = \begin{cases} f(x) + \langle\lambda, g_I(x)\rangle, & \text{if } x\in X, \lambda\geq 0 \\ -\infty & \text{if } x\in X, \lambda\not\geq 0 \\ +\infty & \text{if } x\notin X. \end{cases}$$

Define

$$\begin{aligned} (\widetilde{ECP}) \qquad f_* = & \inf f(x) \\ & \text{s.t. } g_i(x) + s = 0, i\in I \\ & \quad x\in X, s\in\mathbb{R}^I_+. \end{aligned}$$

Proposition 0.79. *We relate $(\widetilde{ECP})$ to (ICP):*

(a) $f_* = \tilde{f}_*$ and $v = \tilde{v}$ (i.e. x^* is an optimal solution of (ICP) $\iff (x^*, -g_I(x^*))$ is an optimal solution of $(\widetilde{ECP})$)

(b) $\theta = \tilde{\theta}$ and

$$\tilde{XS}(\lambda) = \begin{cases} X(\lambda)\times\{s\in\mathbb{R}^I, s\geq 0, \langle s,\lambda\rangle\geq 0\}, & \text{if } \lambda\geq 0 \\ \emptyset, & \text{otherwise.} \end{cases}$$

(c) $\theta^* = \tilde{\theta}^*$ and $\Lambda^* = \tilde{\Lambda}^*$.

Proposition 0.80. *For $(x^*, \lambda^*) \in \mathbb{R}^n \times \mathbb{R}^I$, we have:*

$$x^* \text{ is an optimal solution of (ICP)} \iff \begin{array}{r} \lambda^* \geq 0, g(x^*) \leq 0 \\ \langle \lambda^*, g(x^*)\rangle = 0 \\ x \in X(\lambda^*) \end{array}$$
$$\lambda^* \text{ is a LM of (ICP)}$$

and $x^* \in X(\lambda^*) \iff x^* \in \operatorname{argmin}_{x \in X} f(x) + \langle \lambda^*, g_I(x)\rangle$.

Proposition 0.81. *The following are equivalent:*
(a) $f_* = \theta_* \in \mathbb{R}$ and $\lambda^* \in \Lambda^*$
(b) λ^* is a LM of ICP
(c) $v(0) \in \mathbb{R}$ and $\lambda^* \in \partial v(0)$

Proposition 0.82. *Assume that $f_* \in \mathbb{R}$, $v \in$ E-Conv $\mathbb{R}^n$ and $0 \in \operatorname{ri}(\operatorname{dom} v)$. Then (ICP) has a LM.*

Assumption 1. *Suppose X is convex and f, g_i are convex for $i \in I$.*

Proposition 0.83. *Under assumption **??** and assumption 1, the value function v is convex and*

$$\operatorname{ri}(\operatorname{dom} v) = \left\{ b \in \mathbb{R}^I : \begin{array}{c} \exists x \in \operatorname{ri}(X) \text{ s.t.} \\ g_I(x) + b < 0 \end{array} \right\}.$$

Proposition 0.84. *Let $f_1, ..., f_m : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ and convex set $X \subseteq \mathbb{R}^n$ such that $\emptyset \neq X \subseteq \bigcap_{i=1}^m \operatorname{dom} f_i$ be given. If each f_i is convex on X then*

$$U = \{(x, r) \in X \times \mathbb{R}^m : f_i(x) \leq r_i, i = 1, 2, ..., m\}$$

is convex and

$$\operatorname{ri} U = \{(x, r) \in \operatorname{ri} X \times \mathbb{R}^m : f_i(x) < r_i, i = 1, 2, ..., m\}.$$

Theorem 0.1. *Consider the problem*

$$\begin{aligned} (NLP) \qquad f_* = \inf\ & f(x) \\ \text{s.t. } & g_I(x) \leq 0, \quad g_i, i \in I \text{ convex} \\ & g_E(x) = 0, \quad g_i, i \in I \text{ affine} \\ & x \in X. \end{aligned}$$

and define

$$\begin{aligned} I_a &= \{i \in : g_i \text{ is affine}\} \\ I_c &= I \backslash I_a. \end{aligned}$$

If $f_* \in \mathbb{R}$ and $\exists x^0 \in \operatorname{ri} X$ such that $g_E(x^0) = 0$, $g_{I_a}(x^0) \leq 0$, $g_{I_c}(x^0) < 0$ then (NLP) has a LM.

Calculus of Conjugate Functions

Definition 0.30. Let $A : \mathbb{R}^n \mapsto \mathbb{R}^m$ affine and $f : \mathbb{R}^n \mapsto [-\infty, +\infty]$. Define $Af : \mathbb{R}^m \mapsto [-\infty, +\infty]$ as

$$\begin{aligned} y \mapsto (Af)(y) = \inf\ & f(x) \\ \text{s.t. } & Ax = y \end{aligned}$$

Proposition. *(1) $f \in$ E-Conv $\mathbb{R}^n \implies Af \in$ E-Conv $\mathbb{R}^n$*
(2) $\operatorname{dom}(Af) = A(\operatorname{dom} f)$

Proposition 0.85. $(Af)^* = f^* \circ A^*$

Proposition 0.86. *For any $g \in$ E-Conv $\mathbb{R}^n$ and $B : \mathbb{R}^n \mapsto \mathbb{R}^m$ linear, we have*

$$(\operatorname{cl} g \circ B)^* = \operatorname{cl}(B^* g^*).$$

Proposition 0.87. *Let $g \in$ E-Conv $\mathbb{R}^m$ and $B : \mathbb{R}^n \mapsto \mathbb{R}^m$ linear be such that*

$$(*) \qquad \text{Im } B \cap \operatorname{ri}(\operatorname{dom} g) \neq \emptyset.$$

Then $(g \circ B)^* = B^* g^*$ and for every $s \in \mathbb{R}^n$ such that $B^* g^*(s)$ is finite, the infimum

$$\begin{aligned} (B^* g^*)(s) = \inf\ & g^*(y) \\ \text{s.t. } & B^* y = s \end{aligned}$$

is achieved.

Proposition 0.88. *Let $g \in$ E-Conv $\mathbb{R}^m$ and $B : \mathbb{R}^n \mapsto \mathbb{R}^m$ be linear. Then,*

$$B^*(\partial g(Bx)) \subseteq \partial (g \circ B)(x), \forall x.$$

If, in addition, Im $B \cap \operatorname{ri}(\operatorname{dom} g) \neq \emptyset$ then equality holds.

Definition 0.31. The **ϵ-subgradient** is defined as

$$s \in \partial_\epsilon f(x) \iff f(x') \geq f(x) + \langle s, x' - x\rangle - \epsilon, \forall x'.$$

An equivalent characterization is

$$s \in \partial_\epsilon f(x) \iff f^*(s) \leq \langle x, s\rangle - f(x) + \epsilon.$$

Corollary 0.18. *Let $\epsilon > 0$, $g \in$ E-Conv $\mathbb{R}^m$, and $B : \mathbb{R}^n \mapsto \mathbb{R}^m$ be linear. Then,*

$$B^*(\partial g_\epsilon(Bx)) \subseteq \partial_\epsilon (g \circ B)(x), \forall x.$$

If, in addition, Im $B \cap \operatorname{ri}(\operatorname{dom} g) \neq \emptyset$ then equality holds.

Infimal Convolution

Definition 0.32. For $f_1, ..., f_m : \mathbb{R}^n \mapsto (-\infty, +\infty]$, their **infimal convolution** is defined as

$$(f_1 \Box ... \Box f_m)(x) = \begin{bmatrix} \inf f_1(x_1) + ... + f_2(x_m) \\ \text{s.t. } x_1 + ... + x_m = m \end{bmatrix}.$$

Proposition 0.89. *$f_1, ..., f_m \in$ Conv $\mathbb{R}^n$ implies that $f_1 \Box ... \Box f_m \in$ E-Conv $\mathbb{R}^n$ and*

$$\operatorname{dom}(f_1 \Box ... \Box f_m) = \operatorname{dom} f_1 + ... + \operatorname{dom} f_m.$$

Remark 0.9. Let $f(x_1, ..., x_m) = f_1(x_1) + ... + f_2(x_m)$ and $A(x_1, ..., x_m) = x$. Then $f_1 \Box ... \Box f_m = Af$ and $f \circ A^* = (f_1 + ... + f_m)$.

Proposition 0.90. *Let $f_i : \mathbb{R}^m \mapsto (-\infty, \infty]$, $i = 1, 2, ..., m$ be given. Then:*
(i) $(f_1 \Box ... \Box f_m)^* = f_1^* + ... + f_m^*$
(ii) If $f_i \in$ Conv $\mathbb{R}^n$ for $i = 1, 2, ..., m$ then $(\operatorname{cl}[f_1 + ... + f_m])^* = \operatorname{cl}(f_1^* \Box ... \Box f_m^*)$.
(iii) If $f_i \in$ Conv $\mathbb{R}^n$ for $i = 1, 2, ..., m$ and

$$\bigcap_{i=1}^m \operatorname{ri}(\operatorname{dom} f_i) \neq \emptyset$$

then

$$(f_1 + ... + f_m)^* = (f_1^* \Box ... \Box f_m^*).$$

Corollary 0.19. *We have*

$$\begin{aligned}\partial(f_1 + ... + f_m)(x) &= \partial(f \circ A^*)(x) \\ &= A\left[\partial f(A^* x)\right] \\ &= A\left(\partial f_1(x) \times ... \times \partial f_m(x)\right) \\ &\overset{(*)}{=} \partial f_1(x) + ... + \partial f_m(x)\end{aligned}$$

if the standard constraint qualification holds, where $(*)$ *is left as an exercise. Note that* $\supseteq$ *always holds regardless of the constraint set.*

Corollary 0.20. *If* $0 \leq \epsilon_1 + ... + \epsilon_m \leq \epsilon$ *then*

$$\partial_\epsilon(f_1 + ... + f_m)(x) = \partial_{\epsilon_1} f_1(x) + ... + \partial_{\epsilon_m} f_m(x)$$

when the standard constraint qualification holds. Note that $\supseteq$ *always holds regardless of the constraint set.*

<u>Applications</u>

(1) Consider the problem

$$\begin{aligned}&\min\ f(x) \\ &\text{s.t. } x \in C\end{aligned}$$

where $f : \mathbb{R}^n \mapsto (-\infty, \infty]$ and $C \subseteq \mathbb{R}$. This is equivalent to

$$(*) \qquad \begin{aligned}&\min\ f(x) + I_C(x) = (f + I_C)(x) \\ &\text{s.t. } x \in \mathbb{R}^n.\end{aligned}$$

Now x^* is a global min of $(*)$ $\iff$ $0 \in \partial(f+I_C)(x^*)$ $\Longleftarrow$ $0 \in \partial f(x^*)+\partial I_C(x^*)$ $\iff$ $0 \in \partial f(x^*)+N_C(x^*)$ $\iff$ $-\partial f(x^*) \cap N_C(x^*) \neq \emptyset$. All the statements are equivalent if f is convex, C is convex, $\operatorname{ri}(\operatorname{dom} f) \cap \operatorname{ri} C \neq \emptyset$. The last expression is a generalization of the requirement $-\nabla f(x^*) \in N_C(x^*)$.

Proposition 0.91. *Consider ICP with* $\emptyset \neq X \subseteq \operatorname{dom} f \cap \bigcap_{i\in I} \operatorname{dom} g_i$. *Let* $\bar{x}$ *be a feasible point of* $(*)$, *i.e.* $g_I(x) \leq 0$, $x \in X$. *If* $\exists \bar{\lambda} \in \mathbb{R}^m_+$ *s.t.*

$$\begin{cases}\partial f(\bar{x}) + \sum_{i\in I} \bar{\lambda}_i \partial g_i(\bar{x}) + N_X(\bar{x}), & \\ \bar{\lambda}^T g_I(\bar{x}) = 0 & (a)\end{cases} \quad (*)$$

then $\bar{x}$ *is an optimal solution and* $\bar{\lambda}$ *is a Lagrange multiplier of* $(**)$.

Conversely suppose that $f, \{g_i\}_{i\in I}$ *are convex,* X *is convex and* $\exists x^0 \in \operatorname{ri}(\operatorname{dom} f) \cap \bigcap_{i\in I} \operatorname{ri}(\operatorname{dom} g_i) \cap \operatorname{ri} X$ *such that* $g_I(x_0) < 0$. *Then if* $\bar{x}$ *is a global minimum of* (2), $\exists \bar{\lambda} \in \mathbb{R}^m_+$ *satisfying* $(*)$.