Probability Theory

- If $X$is an integer valued non-negative random variable then $E(X) = \sum_{k=0}^{\infty} P(X > k)$.

Properties of the expected value and variance

- If $X_1, ..., X_n$ are independent random variables and $f_1, ..., f_n$ are bounded functions, then $E\left[\prod_{i=1}^{n} f_i(X_i)\right] = \prod_{i=1}^{n} E\left[f_i(X_i)\right]$

Generating Functions

- If $\{p_k : k \geq 0\}$ then $P(s) = \sum_{k=0}^{\infty} p_k s^k = E[s^X]$. If $\sum_{k=0}^{\infty} p_k = 1$ then $P(1) = 1$.
- $P(X + Y = n) = \sum_{k=0}^{n} P(X = k, Y = n - k) = \sum_{k=0}^{n} a_k b_{n-k}$
- Let X have a probability mass function with $p_k = P(X = k)$ and $\sum_{k=0}^{\infty} p_k = 1$. Let $q_k = P(X > k)$ and define $Q(s) = \sum_{k=0}^{\infty} q_k s^k$. Then $Q(s) = \frac{1-P(s)}{1-s}, \forall s \in (0, 1)$.
- By the Monotone Convergence Theorem, $\lim_{s\to 1} Q(s) = \lim_{s\to 1} \sum_{k=0}^{\infty} q_k s^k = \sum_{k=0}^{\infty} \lim_{s\to 1} q_k s^k = \sum_{k=0}^{\infty} q_k = E[X]$
- By direct evaluation,

$$\frac{d}{ds} P(s)\Big|_{s=1} = \sum_{k=1}^{\infty} k p_k = E[X]$$

$$\frac{d^n}{ds^n} P(s)\Big|_{s=1} = E[X(X-1)...(X-n+1)]$$

- Note that $Var(X) = P''(1) + P'(1) - (P'(1))^2$.
- $P_{X_1+X_2}(s) = E\left[s^{X_1+X_2}\right] = E[s^{X_1}]E[s^{X_2}] = P_{X_1}(s)P_{X_2}(s)$
- (2) If $\{a_j\}$ and $\{b_j\}$ are two sequences with generating functions $A(s), B(s)$ then the generating functions of $\{a_n\} * \{b_n\}$ is $A(s)B(s)$.
- (Wald's identity) Note that $E[s_N] = \frac{d}{ds} P_N(P_{X_1}(s))\Big|_{s=1} = E[N]E[X_1]$
- Define $P_n(s) = E(s^{Z_n})$ for the branching process $\{Z_n = \sum_{i=1}^{Z_{n-1}} Z_{n,i}\}$. Then $P_n(s) = P_{n-1}(P(s)) = P(P_{n-1}(s))$.
  - If $m = E[Z_1] < 1$ then $\Pi = 1$. If $m > 1$ then $\Pi < 1$ and is the unique non-negative solution to the equation $s = P(s)$ which is less than 1. Π is the extinction probability.

Continuity Theorem

- Suppose for each $n = 1, 2, ...$ $\{p_k^{(n)} : k \geq 0\}$ is a probability mass function $\{0, 1, 2, ...\}$ so that $p_k^{(n)} \geq 0, \sum_{k=0}^{\infty} p_k^{(n)} = 1$. Then there exists a sequence $\{p_k^{(0)} : k \geq 0\}$ such that $\lim_{n\to\infty} p_k^{(n)} = p_k^{(0)}$ for all $k = 0, 1, ...$ if and only if there exists a function $P_0(s)$, $0 < s < 1$ such that $\lim_{n\to\infty} P_n(s) = P_0(s)$.

Discrete Time Markov Chains

- We call the equation $p_{ij}^{(n+m)} = \sum_k p_{ik}^{(n)} p_{kj}^{(m)}$ the **Chapman-Komolgorov equation.**
- $\tau_B = \inf\{n \geq 0 : X_n \in B\}$ which we call the hitting time of B. We use $\tau_j = \tau_{\{j\}}$.
- For $i, j \in S$ we say state j is **accessible** from state i if $P(\tau_j < \infty | X_0 = i) > 0$ and we denote it as $i \to j$. Obviously $i \to i$.
- For $i \neq j$ we have $i \to j$ if and only if there exists $n > 0$ such that $p_{ij}^{(n)} > 0$. That is, $P(X_n = j | X_0 = i) > 0$.
- A Markov chain is **irreducible** if the state space consists of only one equivalence class. This means that $i \leftrightarrow j$ for all $i, j \in S$.
- A set of states $C \subset S$ is **closed** if for any $i \in C$ we have $P(\tau_{C^c} = \infty | X_0 = i) = 1$. If a singleton is closed then it is called an **absorbing state**.
  - (i) C is closed if and only if for all $i \in C$ and $j \in C^c$ we have $p_{ij} = 0$.
  - (ii) j is absorbing if and only if $p_{jj} = 1$.
- State i is **recurrent** if $P(\tau_i(1) < \infty | X_0 = i) = 1$
  - A recurrent state is **positive recurrent** if $E[\tau_i(1) | X_0 = i] < \infty$.
  - Otherwise if $E[\tau_i(1) | X_0 = i] = \infty$ then a recurrent state is **null recurrent**.
- State i is **transient** if $P(\tau_i(1) < \infty | X_0 = i) < 1 \implies P(\tau_i(1) = \infty | X_0 = i) > 0$
- We have for $i, j \in S$ and non-negative integer k we have $P(N_j = k | X_0 = i) = \begin{cases} 1 - f_{ii} & k = 0 \\ f_{ij} f_{jj}^{k-1}(1 - f_{jj}) & k \geq 1 \end{cases}$
- If j is transient, then for all states i we have $P(N_j < \infty | X_0 = i) = 1$and $E[N_j | X_0 = i] = f_{ij}/(1 - f_{jj})$ and $P(N_j = k | X_0 = j) = (1 - f_{jj}) f_{jj}^k$.
  - This implies that $\sum_n p_{ij}^{(n)} < \infty$
- If j is recurrent then $P(N_j = \infty | X_0 = j) = 1$.
  - This implies that $\sum_n p_{ij}^{(n)} = \infty$
- For $n \geq 1$ define:
  - $f_{jk}^{(0)} = 0, f_{jk}^{(n)} = P(\tau_k(1) = n | X_0 = j)$, $f_{jk} = \sum_{n=0}^{\infty} f_{jk}^{(n)} = P(\tau_k(1) < \infty | X_0 = j)$
  - Therefore, a state i is recurrent if and only if $f_{ii} = 1$ and a recurrent state i is positive recurrent if and only if $E[\tau_i(1) | X_0 = i] = \sum_{n=0}^{\infty} n f_{ii}^{(n)} < \infty$
- Define $F_{ij}(s) = \sum_{n=0}^{\infty} s^n f_{ij}^{(n)}$ and $P_{ij}(s) = \sum_{n=0}^{\infty} s^n p_{ij}^{(n)}$
  - a) We have for $i \in S$, $p_{ii}^{(n)} = \sum_{k=0}^{n} f_{ii}^{(k)} p_{ii}^{(n-k)}, \forall n \geq 1$ and for $0 < s < 1$ we have $P_{ii}(s) = \frac{1}{1 - F_{ii}(s)}$.

  - b) We have for $i \neq j$, $P_{ij}^{(n)} = \sum_{k=0}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)}, \forall n \geq 0$and for $0 < s < 1$ we have $P_{ij}(s) = F_{ij}(s)P_{jj}(s)$.

- A state i is **recurrent** if and only if $f_{ii} = 1$ if and only if $P_{ii}(1) = \sum p_{ii}^{(n)} = \infty$. Thus i is **transient** if and only if $f_{ii} < 1$ if and only $\sum p_{ii}^{(n)} < \infty$.
  - If i is **transient**, it also means $\sum p_{ij}^{(n)} < 0$
- Define the column vector $f^{(n)} = (f_{1j}^{(n)}, f_{2j}^{(n)}, ..., f_{ij}^{(n)}, ...f_{|S|j}^{(n)})^T$ and the matrix ${}^{(j)}P$ as the P matrix with the j^{th} column replaced by a column of zeroes. Then we can write $f^{(n)} = {}^{(j)}Pf^{(n-1)} = {}^{(j)}P^{(n-1)}f^{(1)}$.
- **Recurrence[1], transience[2], and periodicity[3] are equivalence class properties.**
- The state space S of a Markov chain can be decomposed as $S = T \cup C_1 \cup C_2 \cup ...$ where T consists of transient states (not necessarily in one class) and $C_1, C_2, ...$ are closed disjoint classes of recurrent states.
- If S is finite, not all states can be transient.
- If $P = \begin{pmatrix} Q & R \\ 0 & P_2 \end{pmatrix}$ and $u_{ik} = P(X_\tau = k | X_0 = i)$
  - Then $U = (I - Q)^{-1}R, (I - Q)^{-1} = \sum_{n=0}^{\infty} Q^n$
  - $(I - Q)_{ij}^{-1} = E\left[\sum_{n=0}^{\infty} 1(X_n = j) | X_0 = i\right]$

Stationary Distributions

- A stochastic process $\{Y_n : n \geq 0\}$ is **stationary** if of integers $m \geq 0$ and $k > 0$ we have $(Y_0, Y_1, ..., Y_m) \overset{d}{=} (Y_k, Y_{k+1}, ..., Y_{m+k})$
- Let $\pi = \{\pi_j : j \in S\}$ be a probability distribution. It is called a **stationary distribution** for the Markov chain with transition matrix P if $\pi^T = \pi^T P, \pi_j = \sum_{k \in S} \pi_k P_{kj}, \forall j \in S$
- Let $i \in S$ be recurrent and define for $j \in S$

$$\nu_j = E\left[\sum_{0 \leq n \leq \tau_i(1)-1} 1(X_n = j) | X_0 = i\right]$$
$$= \sum_{n=0}^{\infty} P(X_n = j, \tau_i(1) > n | X_0 = i)$$

  - Then ν is an invariant measure.
  - If i is positive recurrent, then $\pi_j = \frac{\nu_j}{E[\tau_i(1)|X_0=i]}$ is a stationary distribution.
- If the Markov chain is **irreducible and recurrent, then an invariant measure** ν **exists** and satisfies $0 < \nu_j < \infty, \forall j \in S$ and ν is unique up to a constant. If $\nu_1^T = \nu_1^T P$ and $\nu_2^T = \nu_2^T P$ then $\nu_1 = c\nu_2$.
  - Furthermore, if the Markov chain is **positive recurrent and irreducible**, there exists a **unique stationary distribution** π where $\pi_j = \frac{1}{E[\tau_j(1)|X_0=j]}$.
- Suppose $\{Y_n\}$ is a sequence of iid r.v.s with $E(|Y_i|) < \infty$. Then, $P\left(\lim_{n\to\infty} \frac{\sum_{i=1}^{n} Y_i}{n} = E[Y_1]\right) = 1$
- Suppose the Markov chain is **irreducible and positive recurrent**, and let π be the **unique stationary distribution**. Then $\lim_{N\to\infty} \frac{\sum_{n=0}^{N} f(X_n)}{N} = \sum_{j \in S} f(j)\pi_j$, a.s.
  - Note that if $f(k) = 1(k = i)$ then

$$\lim_{N\to\infty} \frac{\sum_{n=0}^{N} f(X_n)}{N} = \pi_i$$

  - If f is bounded then $\lim_{N\to\infty} \frac{\sum_{n=0}^{N} E[f(X_n)|X_0=i]}{N} = \sum_{j \in S} f(j)\pi_j$
- A **limit distribution is a stationary distribution**.
- Suppose the Markov chain is **irreducible and aperiodic** and that a **stationary distribution** π **exists** with $\pi^T = \pi^T P$ and $\sum_{j \in S} \pi_j = 1$ with $\pi_j \geq 0$. Then:
  - (1) The Markov chain is **positive recurrent**
  - (2) π is a limit distribution with $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j, \forall i, j \in S$
  - (3) For all $j \in S$, $\pi_j > 0$
  - (4) **The stationary distribution is unique**
- <u>REMARK</u>: If **irreducible** then **stationary distribution exists** if and only if it is **positive recurrent**
- Let the chain be **irreducible and aperiodic**. Then for $i, j \in S$ there exists $n_0(i, j)$ such that for all $n \geq n_0(i, j)$ we have $p_{ij}^{(n)} > 0$.
- If a Markov chain is **null recurrent**, then $\lim_{n\to\infty} p_{ij}^{(n)} = 0$.
- <u>REMARK</u>: Assume that a Markov chain is **irreducible and aperiodic**. A **stationary distribution exists** if and only if the chain is **positive recurrent** if and only if **a limit distribution (defined through** $\lim_{n\to\infty} P^n$**) exists**.
- If the chain is **irreducible and periodic**, existence of a stationary distribution is **equivalent to positive recurrent states**.
- If the Markov chain is **irreducible and aperiodic** and **either null recurrent or transient**, then

$$\lim_{n\to\infty} p_{ij}^{(n)} = 0, \text{ for all } i, j \in S$$

We can conclude that in **a finite state irreducible Markov chain, no state can be null recurrent**.

Renewal Theory

- (Cauchy product) $\left(\sum_{i=0}^{\infty} a_i\right)\left(\sum_{j=0}^{\infty} b_j\right) = \sum_{k=0}^{\infty}\sum_{l=0}^{k} a_l b_{k-l}$
- If $F(\infty) = 1$ then the process is called a **proper renewal process**. If $F(\infty) < 1$ then the process is called **terminating** or **transient**.
- **Wald's Lemma:** $E[S_{N(t)}] = E[N(t)]E[Y_1]$
- $F * g(t) = \int_0^t g(t-x)F(dx), \text{ for } t \geq 0$
- $\hat{F}(\lambda) = E[e^{-\lambda X}] = \int_0^{\infty} e^{-\lambda x} F(dx), \lambda \geq 0$
  - The Laplace transform uniquely determines the distribution function
  - $(\widehat{F_1 * F_2})(\lambda) = \hat{F}_1(\lambda)\hat{F}_2(\lambda)$
  - $E[X] = -\hat{F}(\lambda), E[X^2] = \hat{F}''(0)$
  - $\int_0^{\infty} e^{-\lambda x} F(x)dx = \frac{1}{\lambda}\hat{F}(\lambda)$
- **Binomial distribution**
  - $f(x) = \binom{n}{x} p^x (1-p)^{n-x}$, $\mu = np, \sigma = np(1-p)$
  - $P(s) = (1 - p + ps)^n$
  - Bernoulli is the case of $n = 1$ for above
- **Poisson distribution**
  - $f(x) = \lambda^x e^{-\lambda}/x!$, $P(s) = e^{\lambda(s-1)}, s > 0$
- **Geometric distribution**
  - $f(x) = 1 - q^{x+1}$, $F(x) = pq^x$, $\mu = q/p, \sigma^2 = q/p^2$
  - $P(s) = p/(1-qs), s < 1/q$
- **Erlang distribution**
  - $g(x) = \frac{\alpha(\alpha x)^n e^{-\alpha x}}{n!} 1_{[0,\infty)}(x), \quad G(x) = 1 - \sum_{n=0}^{k-1} \frac{e^{-\alpha x}(\alpha x)^n}{n!}$
  - $\hat{G}(\lambda) = \left(\frac{\alpha}{\alpha+\lambda}\right)^{n+1} \implies G(x) = F^{(n+1)*}(x)$
    - $F(x)$ is the exponential distribution
- **Exponential distribution**
  - $f(x) = \alpha e^{-\alpha x}$, $F(x) = 1 - e^{-\alpha x}$
  - $\hat{F}(\lambda) = \frac{\alpha}{\lambda+\alpha}$, $U(t) = 1 + \alpha t$ (point mass at $t = 0$)
  - $\mu = 1/\alpha, \sigma^2 = 1/\alpha^2$
- **Uniform distribution**
  - $f(x) = I\{x \in [a,b]\}/(b-a)$, $F(x) = \frac{x-a}{b-a}$
  - $\hat{F}(\lambda) = \frac{1-e^{-\lambda}}{\lambda}$, $U(t) = e^t$, $\mu = (a+b)/2, \sigma^2 = (b-a)^2/12$
- Suppose that $\mu = E[Y_1] = \int_0^{\infty} xF(dx) < \infty$.
  - If $P(Y_0 < \infty) = 1$ then as $t \to \infty$ we have $N(t)/t \to 1/\mu$ almost surely.
  - Suppose that $\sigma^2 = Var(Y_1) < \infty$. Then as $t \to \infty$, $N(t)$ has a normal distribution with mean t/μ and variance $t\sigma^2/\mu^3$
- (**Elementary Renewal Theorem**) Let $\mu = E[Y_1] < \infty$ and $P(Y_0 < \infty) = 1$. Then, $\lim_{t\to\infty} \frac{V(t)}{t} = \lim_{t\to} \frac{U(t)}{t} = \frac{1}{\mu}$.
- Suppose we have a renewal sequence $\{S_n\}$ and suppose that at each epoch $S_n$we receive a **random reward** R_n. Suppose that $\{R_n : n \geq 1\}$ is a sequence of i.i.d. r.vs and define $R(t) = \sum_{i=0}^{\infty} R_i 1(S_i \leq t) = \sum_{i=1}^{N(t)-1} R_i$.
  - If $E[|R_j|] < \infty$ for all $j = 0, 1, ...$ and $E[Y_1] < \infty$ with $P(Y_0 < \infty) = 1$ then $\lim_{t\to\infty} \frac{R(t)}{t} = \frac{E[R_1]}{\mu}$.
- (**Renewal Equation**) $Z = z + F * Z \implies Z(t) = z(t) + \int_0^t Z(t-s)F(ds)$
  - $U(t) = F^{0*}(t) + F * U(t)$ where $F^{0*}(x) = 1_{[0,\infty)}^{(x)}$
  - A locally bounded **solution of the renewal equation** is $U * z(t) = \int_0^t z(t-s)U(ds)$.
  - $E[A(t)] = t[1 - F(t)] + \int_0^t E[A(t-s)]F(ds)$
  - $E[B(t)] = \int_t^{\infty}(s-t)F(ds) + \int_0^t E[B(t-s)]F(ds)$
  - $E[A_D(t)] = t\bar{G}(t) + \int_0^t (t-s)\bar{F}(t-s)dG(s) + \int_0^t\int_0^{t-s}(t-s-x)\bar{F}(t-s-x)dU(x)\,dG(s)$
  - $E[B_D(t)] = \int_t^{\infty}(x-t)dG(x) + \int_0^t\int_{t-s}^{\infty}(x-t+s)dF(x)\,dV(x)$
  - $P(A(t) \leq x) = 1_{[0,x]}(t)[1-F(t)] + P(A(t) \leq x) * F(t)$
  - $P(B(t) > x) = [1 - F(t+x)] + P(B(t) > x) * F(t)$
  - $\lim_{t\to\infty} P(A(t) \leq x) = \frac{1}{\mu}\int_0^x (1 - F(s))\,ds = F_0(x)$
  - $\lim_{t\to\infty} P(B(t) > x) = 1 - F_0(x)$
  - Use the notation $U(t) = m(t), V(t) = m_D(t)$
- (**Blackwell's Theorem**) If $V(t, t+a] = E[N(t+u)] - E[N(t)]$ then $\frac{V(t,t+a]}{t} \to \frac{a}{\mu}$.
- (**Key Renewal Theorem**) Suppose $z(t)$ is directly Riemann integrable. We have $\lim_{t\to\infty} Z(t) = \lim_{t\to\infty} z * U(t) = \frac{1}{\mu}\int_0^{\infty} z(s)\,ds$.
- (**Direct Riemann Integrability**)
  - If z has a compact support then Riemann integrability is the same as direct Riemann integrability.
  - If z is directly Riemann integrable then it is Riemann integrable.
  - If $z \geq 0$ and z is non-increasing then z is directly Riemann integrable if and only if it is Riemann integrable.
  - If z is Riemann integrable on $[0, a]$ for all $a > 0$ and $\sigma(1) < \infty$ then z is directly Riemann integrable.
  - If z is Riemann integrable on $[0, \infty)$ and $z \leq g$ where g is directly Riemann integrable then z is directly Riemann integrable.

- The following are equivalent:
  - (i) The Blackwell Theorem
  - (ii) The Key Renewal Theorem

Regenerative Process

- Suppose $\{X(t)\}$ is a **regenerative process** with state space E. For fixed A, assume that $K(t, A)$ is Riemann integrable. Set $\mu \in E[S_1]$ and $S_0 = 0$.
  - $Z(t) = P(X(t) \in A) = K(t, A) + \int_0^t Z(t-s)F(ds)$
  - (Smith's Theorem)
    - a) If $\mu < \infty$, then $\lim_{t\to\infty} P(X(t) \in A) = \frac{E[\text{time spent in } A \text{ in a cycle}]}{E[\text{cycle length}]}$
    - b) If $\mu = \infty$, then $\lim_{t\to\infty} P(X(t) \in A) = 0$.
- For **alternating renewal processes**, Z_i for on time, Y_i for off time :
  - $X(t) = \begin{cases} 1 & \text{if the system is on at time } t \\ 0 & \text{otherwise} \end{cases}$
  - $\lim_{t\to\infty} P(X(t) = 1) = \frac{E[Z_1]}{E[Z_1]+E[Y_1]}$

Poisson Process

- **Distribution analysis**
  - (Law of Small Numbers) If $n \to \infty$ and $p \to 0$ in such a way that $np \to \infty$, then $Bin(n, p) \to Pois(\alpha)$.
  - Let $T_n \sim Geo(p_n)$ where $P(T_n > k) = (1 - p_n)^k$ for $k = 0, 1, \ldots$. If $np_n \to \alpha$ as $n \to \infty$ then $T_n/n \xrightarrow{D} exp(\alpha)$
  - Suppose that $N \sim Pois(\alpha)$ and $X_1, X_2, \ldots$ are i.i.d. $Bernoulli(p)$ independent of N. Let $S_n = \sum_{i=1}^n X_i$. Then, $S_N \sim Pois(\alpha p)$.
  - If $X_1, X_2, \ldots, X_n$ are respectively i.i.d. $\exp(\lambda_1), \exp(\lambda_2), \ldots, \exp(\lambda_n)$ then
    - $P(\max\{X_1, \ldots, X_n\} \le k) = \prod_{i=1}^n P(X_i \le k) = \exp(\sum_{i=1}^n \lambda_i)$
    - $P(\min\{X_1, \ldots, X_n\} \ge k) = \prod_{i=1}^n P(X_i \ge k)$
- A **point process** on the timeline $[0, \infty)$ is a mapping $J \mapsto N_j = N(j)$ that assigns to each subset $J \subset [0, \infty)$ a non-negative integer value random variable N_j in such a way that if $J_1, J_2, \ldots$ are pairwise disjoint then $N(\cup_i J_i) = \sum_i N(J_i)$.We will interchangeably use $N(t) = N([0, t])$.
- A **Poisson point process** of intensity $\alpha > 0$ is a point process $N(J)$ with the following properties:
  - a) If $J_1, J_2, \ldots$ are non-overlapping intervals of $[0, \infty)$ then the random variables $N(J_1), N(J_2), \ldots$ are mutually independent. (***Independent Increments***)
  - b) For every interval J, we have $P(N(J) = k) = \frac{e^{-\alpha|J|}(\alpha|J|)^k}{k!}$, $k = 0, 1, \ldots$where $|J|$ is the length of the interval J.
- If $N(t)$ is a Poisson process with rate 1 then $N(\lambda t)$ is a Poisson with rate λ.
- (**Generalized Thinning Theorem**) Suppose N is a Poisson random variable with parameter α and the $X_1, X_2, \ldots$ are i.i.d. multinomial random variables with parameters $(p_1, p_2, \ldots, p_m)$. That is, $P(X_i = k) = p_k$ for each $k = 1, 2, \ldots, m$. Then the random variables $N_1, N_2, \ldots, N_m$ defined as $N_k = \sum_{i=1}^N 1\{X_i = k\}$ are i.i.d. Poisson random variables with $E[N_k] = \alpha p_k$.
- Define $0 = S_0 \le S_1 \le S_2 \le \ldots$ as the **successive times** that the process $N(t)$ has jumps. Define the interarrival times as $Y_n = S_n - S_{n-1}$.
  - (a) The interarrival times $Y_1, Y_2, \ldots$ of a Poisson process with rate α are i.i.d. $\exp(\alpha)$.
  - (b) Conversely let $X_1, X_2, \ldots$be i.i.d. $\exp(\alpha)$ and define$N(t) = \max\{n : \sum_{i=1}^n X_i \le t\}$. Then $\{N(t) : t \ge 0\}$ is a Poisson process with rate α.
- The (stationary) counting process $\{N(t) : t \ge 0\}$ is said to be a **Poisson process** with intensity $\alpha > 0$ if:
  - (i) the process has **independent increments**
  - (ii) $P(N(h) = 1) = \alpha h + o(h)$
  - (iii) $P(N(h) \ge 2) = o(h)$
    - Recall that a function f is $o(h)$ if $\lim_{h\to\infty}(f(h)/h) = 0$.
- For each $m \ge 1$, let $\{X_r^m : r \in N/m\}$ be a **Bernoulli process** indexed by the integer multiples of $1/m$ with probability of success p_m. Let $\{N^m(t)\}$ be the corresponding counting process that is $N^m(t) = \sum_{r \le t} X_r^m$If $\lim_{m\to\infty} mp_m = \alpha > 0$. Then for any finite set of points $0 = t_0 < t_1 < \ldots < t_n$

$$(N^m(t_1), N^m(t_2), \ldots, N^m(t_n)) \xrightarrow{D} (N(t_1), N(t_2), \ldots, N(t_n))$$

- Given that $N[0, 1] = k$, the k points are uniformly distributed on the unit interval $[0, 1]$, that is for any partition $J_1, J_2, \ldots, J_m$ of $[0, 1]$ into non-overlapping intervals $P(N(J_i) = k_i, i = 1, 2, \ldots, m | N[0, 1] = k)$ is equal to $\frac{k!}{k_1!k_2!\ldots k_m!} \prod_{i=1}^m |J_i|^{k_i}$ for all non-negative integers $k_1, \ldots, k_m$ with $\sum_{i=1}^m k_i = k$.
- Let $S_1, S_2, \ldots$ be the arrival times of a Poisson process $\{N(t) : t \ge 0\}$ with rate α. Then conditional on the event that $N[0, t] = k$, the variables $S_1, S_2, \ldots, S_k$ are distributed in the same manner as the order statistics of i.i.d. uniform $[0, t]$ random variables.
- If $N_1(t)$ and $N_2(t)$ represent the type I and type II events, respectively by time t, then $N_1(t)$ and $N_2(t)$ are independent Poisson random variables with intensities $\lambda_1 = \alpha \int_0^t p(s)\, ds$ and $\lambda_2 = \alpha \int_0^t (1 - p(s))\, ds$.