AMATH 350 (Winter 2014 - 1141)

Partial Differential Equations for Finance

Prof. D. Harmsworth
University of Waterloo

LaTeXer: W. KONG
http://wwkong.github.io
Last Revision: April 30, 2014

Table of Contents

These notes are currently a work in progress, and as such may be incomplete or contain errors.

Acknowledgments:

Special thanks to *Michael Baker* and his LaTeX formatted notes. They were the inspiration for the structure of these notes.

Abstract

The purpose of these notes is to provide the reader with a secondary reference to the material covered in AMATH The formal prerequisite to this course is MATH 235, MATH 237, STAT 230 and ACTSC 371 but this author believes that the overlap between the two courses is less than 10%, the majority of which is in multivariate calculus. Readers should have a good background in advanced linear algebra, basic statistics, and calculus before enrolling in this course.

Errata

Prof: David Harmsworth

Office: MC 6441

Hours: DC 1350 (?) @ 2:30pm to 4:00pm

E-mail: dlharmsworth@uwaterloo.ca

1 Introduction

Let's begin with some basic concepts and terminology:

A **differential equation** (DE) is an equation relating an unknown function to its own derivative(s). An **ordinary differential equation** is one that involves normal (non-partial) derivatives.

Example 1.1. The following are (O)DEs:

$$\frac{dy}{dx} = x^2 \text{ OR } \frac{dy}{dx} = y$$

A **solution** to a a DE is a function which satisfies the inequality. In the above, the latter case has the solution $y = e^x$ and so is $y = Ce^x$ for any $C \in \mathbb{R}$. The **general solution** is an expression which represents all (or nearly all) of the solutions.

Occasionally, there will be **singular solutions**, which don't match the general pattern. In multivariate calculus, we'll encounter **partial differential equations** (PDEs) which involve partial derivatives.

Example 1.2. An example of a PDE is the **heat equation** $u_t = u_{xx}$ where $u = u(x,t)$.

The **order** of a DE is the order of the highest-order derivative that is present in the equation.

Example 1.3. The following is a third-order ODE:

$$y''' + 3y' - 2y = 0$$

and the following are first-order ODEs and PDEs respectively:

$$(y')^3 - xy = x^2, u_t = u_x + \sin x$$

In applications, we'll want one solution and not just in the general case. To find it, we'll need **initial condition** which are values of $y, y', ..., y^{(n-1)}$ at one point x_0. An ODE with initial conditions (ICs) is called an **initial value problem** (IVP).

Example 1.4. Solve the IVP

$$\frac{dy}{dx} = 2y, y(0) = 5$$

The general solution is $y = Ce^{2x}$ and applying the initial conditions, we get that $C = 5$ with the particular solution being $y = 5e^{2x}$.

We'll also encounter problems in which the data are given at *different* values of x. There are called **boundary value problems** (BVPs) .

Example 1.5. We might have $y'' = y$ with $y(0) = 0$ and $y(1) = 0$. Some general solutions are $y = C_1e^x + C_2e^{-x}$ and $y = C_1 \sin x + C_2 \cos x$ and this particular solution being $y = 0$ with $C_1 = C_2 = 0$.

A **linear** equation (in DEs) is one in which the function and its derivatives appear in separate terms with exponent 1. The general form for ODEs looks like

$$\sum_n y^{(n)} + \sum_n f_n(x)y^{(n)} = f(x)$$

Example 1.6. $y'' + x^2y' + xy = e^{x^2}$ is linear but $y' + \sin y = 0, (y')^2 = y, yy' = x$ are not linear.

Theorem 1.1. *(Existence and Uniqueness of 1st Order Solutions) The initial value problem $\frac{dy}{dx} = f(x,y), y(x_0) = y_0$ has a unique solution defined on some interval around x_0 of $f(x,y)$ and $f_y(x,y)$ are continuous within some rectangle containing the point (x_0,y_0).*

Example 1.7. The equation $\frac{dy}{dx} = x^2y + e^{xy}$ will have a unique solution for *any* initial condition.

Example 1.8. The equation $\frac{dy}{dx} = (xy)^{2/3}$ will have a unique solution provided that the initial condition is not on the x-axis.

1.1 Separable Equations

Suppose that $\frac{dy}{dx} = f(x,y)$. If $f(x,y)$ can be factored, so that $\frac{dy}{dx} = g(x)h(y)$, then we say that the DE is **separable**. We can solve these by dividing by $h(y)$ and integrating both sides with respect to x to get:

$$\int \frac{1}{h(y)} \cdot \frac{dy}{dx} dx = \int \frac{1}{h(y)} dy = \int g(x) dx$$

Example 1.9. Solve the IVP $\frac{dy}{dx} = e^{x+y}, y(0) = 0$. The solution is

$$\int \frac{dy}{e^y} = \int e^x dx \implies -e^{-y} = e^x + C_1 \implies y = -\ln(-e^x - C_1)$$

or if $C = -C_1$, then $y = -\ln(C - e^x)$ which is our general solution. Plugging in the initial condition, we can get $C = 2$ with $y = \ln\left(\frac{1}{2-e^x}\right)$ as our particular solution.

The only danger with this method is that we may lose certain "equilibrium" solutions when we separate the variables.

Example 1.10. Solving $\frac{dy}{dx} = -4xy^2$ we get $y = \frac{1}{2x^2 - C}$ if $y \neq 0$ and $y \equiv 0$ for the null case of $y = 0$.

Example 1.11. Consider the DE $\frac{dy}{dx} = (xy)^{2/3}$. For the case of $y \neq 0$, we have $y = \left(\frac{1}{5}x^{5/3} + C\right)^3$ and a case of $y \equiv 0$ as well.

2 First-order Linear ODEs

These have the form $a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$ and to motivate the solution method, consider the equation $x\frac{dy}{dx} + y = e^x$. Observe that the LHS is just $\frac{d}{dx}(xy)$. This means that we can write $\frac{d}{dx}(xy) = e^x$ and then integrate to get

$$xy = e^x + C \implies y = \frac{e^x + C}{x}$$

We say that the expression $xy' + y$ is an **exact differential**. Go back and consider the general form $a_1(x)\frac{dy}{dx} + a_0(x)y$. This will be an exact differential if $a_0(x) = a_1'(x)$. But what if the LHS is not an exact differential? The following procedure can make it exact:

1. Write the equation in the **standard form**: $\frac{dy}{dx} + p(x)y = q(x)$
2. Multiply by an function $\mu(x)$, which we'll can an **integrating factor**, to get

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)q(x)$$

3. Now assume that the LHS is derivative of $\mu(x)y(x)$; that is

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)\frac{dy}{dx} + y\frac{d\mu}{dx} \implies y\frac{d\mu}{dx} = \mu(x)p(x)y$$

   Solving this separable equation gives us

$$\int \frac{d\mu}{\mu} = \int p(x)\ dx \implies \mu = \pm e^{\int p(x)\ dx}$$

and any such μ will work so we may choose " + " and set $C = 0$ in $\int p(x)\ dx$

4. The original DE is now

$$\frac{d}{dx}[\mu(x)y(x)] = \mu(x)q(x) \implies y = \frac{1}{\mu(x)} \int \mu(x)q(x)\ dx$$

where $\mu(x) = e^{\int p(x)\ dx}$

This is a formula for the solution. Usually we'll remember the formula for μ and reproduce step 4.

Example 2.1. Solve $\frac{dy}{dx} + xy = x$. To do this, we calculate the integrating factor as $\mu(x) = e^{\int x\ dx} = e^{x^2/2}$ and we get

$$\frac{dy}{dx}e^{x^2/2} + xe^{x^2/2}y = xe^{x^2/2} \implies \frac{d}{dx}(ye^{x^2/2}) = xe^{x^2/2} \implies y = 1 + Ce^{-x^2/2}$$

Example 2.2. Solve $xy' = 2y + x^3 \cos x$ for $x > 0$. Putting this in standard form gives us $y' - \frac{2}{x}y = x^2 \cos x$ and so with an integrating factor of $\mu(x) = e^{-\int \frac{2}{x} dx} = 1/x^2$ we have

$$\frac{y'}{x^2} - \frac{2}{x^3}y = \cos x \implies \frac{d}{dx}\left(\frac{y}{x^2}\right) = \cos x \implies y = x^2 \sin x + Cx^2$$

Example 2.3. Solve the IVP $\frac{dy}{dx} + y = \sqrt{1 + \cos^2 x}$ with $y(1) = 4$. Here, the integrating factor is $e^{\int dx} = e^x$ and so

$$\frac{d}{dx}(ye^x) = e^x\sqrt{1 + \cos^2 x} \implies ye^x = \int e^t\sqrt{1 + \cos^2 t}\ dt$$

We can turn the antiderivative into a definite integral with the IVT x-value as the starting point and proceed as follows:

$$\begin{aligned} ye^x = \int_1^x e^t\sqrt{1 + \cos^2 t}\ dt + C, y(1) = 4 \quad &\implies \quad 4 = 0 + Ce^{-1} \\ &\implies \quad C = 4e \end{aligned}$$

and hence

$$y = 4e^{1-x} + e^{-x}\int_1^x e^t\sqrt{1 + \cos^2 t}\ dt$$

2.1 Mathematical Modeling of Population Growth

Quantities which are measured in integers may occasionally be approximated by continuous functions. Suppose we wish to predict future values of a population, $P(t)$. We'll assume that $P(t)$ is differentiable. To find $P(t)$, we'll make assumptions about its **rate of change**.

In the **Malthusian Model**, we have the simple assumption that if the organisms have unlimited space and resources, then the *rate of change of P should be proportional to P itself.* That is, we should have:

$$\frac{dP}{dt} = rP$$

for some $r \in \mathbb{R}$. We also add the initial condition $P(0) = P_0$. Using separability, the solution to this DE and IVP is

$$P(t) = P_0e^{rt}$$

and hence we see exponential growth. Is this realistic, though?

- Works very well for simple organisms
- The same model applies to other phenomena:
  - Radioactive decay
  - Compound interest

Note 1. To determine r, we'd need to measure the population at one moment. That is, if $P(10) = 2.4P_0$, then $2.4 = e^{10r}$ and $r = \frac{1}{10}\ln 2.4$.

In the **Logistic Model**, Malthus adjusted his simple model by accounting for space and resources. To do this, he suggested the concept of a **maximum sustainable population**, a **carry capacity**. To incorporate this into the model, consider that we need $\frac{dP}{dt} = 0$ when $P = K$, $\frac{dP}{dt} < 0$ when $P > K$, and $\frac{dP}{dt} \approx rP$ when $P \ll K$. One way to achieve this is through the DE:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

This equation is separable with the solution

$$\int \frac{dP}{P\left(1-\frac{P}{K}\right)} = \int r\,dt \implies \int \frac{dP}{P(K-P)} = rt + C_1$$

and using the method of partial fractions (and some algebraic manipulation) we get

$$\left|\frac{P}{K-P}\right| = e^{rt+C_2} \implies \frac{P}{K-P} = C_3e^{rt}, C_3 = \pm e^{C_2}$$

and solving for P gives us

$$P = \frac{C_3Ke^{rt}}{1+C_3e^{rt}} = \frac{K}{1+Ce^{-rt}}, C = 1/C_3$$

If $P(0) = P_0$ then $C = (K - P_0)/P_0$.

2.2 Graphing Families of Solutions

We can determine a lot about the solutions to a DE from the DE itself (the DE tells us about slopes). The general algorithm is:

1. Examine how $\frac{dy}{dx}$ behaves
   (a) Set $\frac{dy}{dx} = 0$ and find the equilibrium solutions
   (b) Determine its behaviour when moving away these equilibrium solutions
2. Examine how $\frac{d^2y}{dx^2}$ behaves
   (a) Set $\frac{d^2y}{dx^2} = 0$ and check the inflection points
3. If the DE is solved, examine the constants for vertical//horizontal asymptotes

The algorithm described in class is:

- Solve DE if possible
- Identify any **exceptional solutions** which behave differently from the rest (usually set $C = 0$)
- Consider the behaviour of the other solutions as $x \to \pm\infty$ or near vertical asymptotes
- Set $\frac{dy}{dx} = 0$ in the DE to find the **horizontal isocline**
- Determine how $\frac{dy}{dx}$ behaves outside of the horizontal isocline

Example 2.4. Consider the DE

$$\frac{dy}{dx} = y - x^2$$

which has the solution

$$y = Ce^x + x^2 + 2x + 2$$

The exceptional solution is $y = x^2 + 2x + 2 = (x+1)^2 + 1$. As $x \to \infty$, $Ce^x \to \pm\infty$ and as $x \to -\infty$, $Ce^x \to 0$ so the exceptional solution attracts the other solutions as $x \to -\infty$ and repels them as $x \to \infty$. We can also see that if $y' = 0$ then $y = x^2$ which is our horizontal isocline.

Example 2.5. Recall the equation

$$\frac{dy}{dx} = (xy)^{2/3}$$

which has the solution

$$y = \left(\frac{1}{5}x^3 + C\right)^3$$

The exceptional solution is $y = x^5/125$ with isoclines $x = 0$ or $y = 0$ when $y' = 0$. Note that $y' \geq 0$ always.

2.3 Change of Variables in ODEs

Some equations which are neither linear nor separable may be converted into linear or separable equations via a **change of variable**. We'll discuss 3 special classes of these.

1. The form $y' = f(ax + by)$

   (a) We can replace $y(x)$ with $u(x)$ where $u = ax + by$.

   (b) E.g. Solve $\frac{dy}{dx} = \sin(x + y)$. Let $u = x + y$. Then $\frac{du}{dx} = 1 + \frac{dy}{dx}$ and the DE becomes $\frac{du}{dx} = 1 + \sin(u)$ which is separable. We then get

   $$\int \frac{du}{1 + \sin(u)} = \int dx$$

   and since

   $$\frac{1}{1 + \sin(u)} = \frac{1 - \sin(u)}{1 - \sin^2(u)} = \sec^2 u - \sec u \tan u$$

   then

   $$\int (\sec^2 u - \sec u \tan u) du = \int dx \implies \tan(x + y) - \sec(x + y) = x + C$$

   where we can't solve for y but we do have an implicit form.

   (c) E.g. Solve $\frac{dy}{dx} = \sqrt{x + y} - 1$. Let $u = x + y \implies \frac{du}{dx} = 1 + \frac{dy}{dx}$. The DE becomes

   $$\frac{du}{dx} = \sqrt{u} \implies \int u^{-1/2} du = \int dx \implies 2\sqrt{u} = x + C_1$$

   and solving for u and y thereafter, we get:

   $$u = \left(\frac{x + C_1}{2}\right)^2 \implies y = \frac{1}{4}(x + C)^2 - x$$

   Remark that

   $$\sqrt{x + y} - 1 = \frac{1}{2}\sqrt{(x + C)^2} - 1 = \frac{1}{2}(x + C) - 1 = \frac{d}{dx}\left(\frac{1}{4}(x + C)^2 - x\right)$$

   if $x + C > 0$ and that this is not a restriction on x but rather helps us determine C. For example, suppose that we add the initial condition $y(1) = 0$. Then $0 = \frac{1}{4}(1 + C)^2 - 1 \implies C = -3, 1$. We must have $x + C \geq 0$ near $x = 1$ so we choose $C = 1$ (this makes the DE valid for this particular initial value).

2. "Homogeneous" Equations: if $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ or $\frac{dy}{dx} = f\left(\frac{x}{y}\right)$

   (a) We can replace y with $u = \frac{y}{x} \implies y = ux$ (vice-versa for $\frac{x}{y}$). This gives us

   $$\frac{dy}{dx} = x\frac{du}{dx} + u$$

   which is a separable equation.

(b) E.g. Solve $\frac{dy}{dx} = \frac{x^2-y^2}{3xy} = \frac{x}{3y} - \frac{y}{3x}$. Letting $u = \frac{y}{x}$ we have

$$u + x\frac{du}{dx} = \frac{1}{3}\left(\frac{1}{u} - u\right)$$

which is a separable equation that can be written as

$$\int \frac{3du}{1-4u^2} = \int \frac{dx}{x} \implies -\frac{3}{8}\int \frac{dZ}{Z} = \int \frac{dx}{x}, Z = 1-4u^2 \implies \ln|z|^{-3} = \ln|x|^8 + C_1$$

and substituting the original variables, we get

$$\ln|1-4u^2|^{-3} = C_2 \ln|x|^8 \implies ... \implies x^2\left[x^2 - 4y^2\right]^3$$

3. Bernoulli Equations

(a) A Bernoulli equation has the form

$$\frac{dy}{dx} + p(x)y = q(x)y^n, n \in \mathbb{Z}$$

These can be converted to linear equations by letting $v = y^{1-n} = y^{-(n-1)}$

(b) E.g. Solve $\frac{dy}{dx} - 5y = -\frac{5}{2}xy^3$. Let $v = y^{-2} \implies \frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$ and notice that multiplying the original equation by y^{-3} gives us

$$y^{-3}\frac{dy}{dx} - 5y^{-2} = -\frac{5}{2}x \implies -\frac{1}{2}\frac{dv}{dx} - 5v = -\frac{5}{2}x$$

which is linear. Set $\mu(x) = e^{10x}$ and this gives us, via integrating factors,

$$e^{10x}v = \frac{1}{2}xe^{10x} - \frac{1}{20}e^{10x} + C \implies v = \frac{1}{2}x - \frac{1}{20} + Ce^{-10x}$$

$$\implies y = \pm\frac{1}{\sqrt{\frac{1}{2}x - \frac{1}{20} + Ce^{-10x}}} \text{ OR } y = 0$$

3 Linear ODEs of All Orders

An n^{th} order ODE is **linear** if it has the form

$$(1)\ a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x)$$

and if $F(x) \equiv 0$, then the ODE is **homogeneous**. Otherwise, it is **inhomogeneous** where $F(x)$ may be called the **inhomogeneous term** or the **forcing term**. If $a_0, a_1, ..., a_n$ are constants, then we have a **constant-coefficient equation**.

Theorem 3.1. *(Existence and Uniqueness Theorem) Consider the IVP of (1) and the n initial conditions*

$$y(x_0) = p_0, y'(x_0) = p_1, ..., y^{(n-1)}(x_0) = p_{n-1}$$

Then there exists a unique solution if there is an open interval I containing x_0 such that

1. The functions $a_n(x), a_{n-1}(x), ..., a_0(x), F(x)$ are continuous

2. $a_n(x) \neq 0$ on I

(Alternatively, if we put the equation in standard form

$$y^{(n)}(x) + b_{n-1}(x)y^{(n-1)}(x) + ... + b_0(x)y(x) = G(x)$$

then we just need $b_0, ..., b_{n-1}, G$ to be continuous)

Example 3.1. Consider the equation $(x^2-1)y'' + xy' - y = \sin x$ with ICs $y(0) = 1, y'(0) = 0$. This has a unique solution on $(-1, 1)$.

3.1 Operator Notation

Definition 3.1. An **operator** is a transformation which maps functions to functions. For example, the **differential operator** transforms f into f'. We may write this as D_f and the **identity operator** as $I_f = f$. We can also combine operators (E.g. $D^2 = D \circ D$).

Remark 3.1. If we let

$$\Phi = a_n(x)D^n + a_{n-1}(x)D^{n-1} + ... + a_1(x)D + a_0(x) = F(x)$$

then (1) can be written as $\Phi(y) = F(x)$.

Theorem 3.2. *(Principle of Superposition [I]) Let Φ be a linear differential operator. If y_1 is solution to $\Phi(y) = F_1(x)$ and y_2 is a solution to $\Phi(y) = F_2(x)$ then $y_1 + y_2$ is a solution to $\Phi(y) = F_1(x) + F_2(x)$.*

Proof. (1st-order case) If y_1 and y_2 are as described and $y = y_1 + y_2$ then

$$\begin{aligned} y' + p(x)y &= [y_1' + y_2'] + p(x)[y_1 + y_2] \\ &= f_1(x) + f_2(x) \end{aligned}$$

□

Example 3.2. Suppose we wish to solve the DE

$$y' + 2y = 6 + 3e^x$$

If we happen to notice that $y_1 = 3$ is a solution to the problem $y' + 2y = 6$ and $y_2 = e^x$ is a solution to $y' + 2y = 3e^x$ then we can conclude that $y = 3 + e^x$ is a solution to $y' + 2y = 6 + 3e^x$.

Corollary 3.1. *If y_h is a solution to $\Phi(y) = 0$ and a solution y_p to $\Phi(y) = F(x)$, then $y_h + y_p$ is also a solution to $\Phi(y) = F(x)$.*

Example 3.3. Consider again the DE,

$$y' + 2y = 6 + 3e^x$$

The homogeneous problem $y' + 2y = 0$ has general solution $y = Ce^{-2x}$ and therefore, the function $y = Ce^{-2x} + 3 + e^x$ is a solution to the original DE. It is also, in fact, the general solution which we will prove later.

Theorem 3.3. *(Principle of Superposition [II]) If y_1 and y_2 are both solutions to $\Phi(y) = 0$, then so is $y = c_1y_1 + c_2y_2$ for any $c_1, c_2 \in \mathbb{R}$.*

Proof. (1st-order case) If $Y = c_1y_1 + c_2y_2$, where y_1 and y_2 are solutions to $y' + p(x)y = 0$, then

$$\begin{aligned} Y' + p(x)Y &= c_1y_1' + c_2y_2' + p(x)(c_1y_1 + c_2y_2) \\ &= c_1\underbrace{(y_1' + p(x)y_1)}_{=0} + c_2\underbrace{(y' + p(x)y_2)}_{=0} \\ &= 0 \end{aligned}$$

□

3.2 Linear Independence of Functions

Definition 3.2. The functions $f_1, ..., f_n$ are **linearly dependent** if there exists constants $c_1, c_2, ..., c_n$ not all zero such that

$$\sum_{i=1}^{n} c_i f_i(x) = 0, \forall x \in I$$

Otherwise, they are **linearly independent**. Essentially, this is analogous to the linear algebraic definition. Usually, independence will be obvious.

Definition 3.3. The **Wronskian** of the n functions $f_1, ..., f_n$ is defined as the determinant

$$W(x) = W(f_1, ..., f_n) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

Theorem 3.4. *If $W(x_0) \neq 0$ for some $x_0 \in I$ then $f_1, f_2, ..., f_n$ are linearly independent on I.*

Proof. (for $n = 2$) Suppose $(f_1, f_2) \neq 0$ for some $x_0 \in I$. Suppose $c_1 f_1(x) + c_2 f_2(x) = 0$ for all $x \in I$. Then $c_1 f_1'(x) + c_2 f_2'(x) = 0$ and in matrix form,

$$\begin{bmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since $W(x_0) \neq 0$, this matrix is invertible at x_2 and so $c_1 = c_2 = 0$. Therefore f_1 and f_2 are linearly independent. □

Example 3.4. For $f_1 = x, f_2 = x^2$,

$$W(x, x^2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2 \neq 0$$

Remark 3.2. In general, the converse of the above statement is not true. A famous counterexample is $f(x) = x^2|x|$ and $g(x) = x^3$. You can show that $W(f, g) = 0$ but f and g are clearly independent. However, we can add one more condition to make this true.

Theorem 3.5. *Let $p(x)$ and $q(x)$ be continuous on an interval I and suppose that $y_1(x)$ and $y_2(x)$ are solutions to the homogeneous linear equation*

$$y'' + p(x)y' + q(x)y = 0$$

on I. If $W(y_1, y_2) = 0$ for some $x_0 \in I$, then y_1 and y_2 are linearly dependent.

Proof. Assume that $W(y_1, y_2) = 0$ for some $x_0 \in I$. Assume also that $c_1 y_1 + c_2 y_2 = 0$ for all $x \in I$. We then may write:

$$\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and since $W(x_0) = 0$, there exists a nonzero solution $\bar{c}_1, \bar{c}_2$ such that $\bar{c}_1 y_1(x_0) + \bar{c}_2 y_2(x_0) = 0$ and $\bar{c}_1 y_1'(x_0) + \bar{c}_2 y_2'(x_0) = 0$. Now let $u(x) = \bar{c}_1 y_1(x) + \bar{c}_2 y_2(x)$ and $u'(x) = \bar{c}_1 y_1'(x) + \bar{c}_2 y_2'(x)$ such that $u(x_0) = u'(x_0) = 0$. Consider these to be initial conditions for the DE $y'' + p(x)y' + q(x)y = 0$. Since p and q are continuous, the conditions of the Existence and Uniqueness theorem are met and so the IVP

$$y'' + py' + qy = 0$$

$$y(x_0) = 0, y'(x_0) = 0$$

has a unique solution. We can see that $u(x)$ is a solution (by the principle of superposition) and we can also see that $y \equiv 0$ is a solution. Therefore $u(x_0) = \bar{c}_1 y_1(x_0) + \bar{c}_2 y_2(x_0) = 0$ for all $x_0 \in I$ which means that y_1 and y_2 are linearly dependent. □

Proposition 3.1. *(Abel's Formula) If y_1 and y_2 are solutions to $y'' + p(x)y' + q(x)y = 0$ then*

$$W(y_1, y_2) = W(x_0)e^{-\int_{x_0}^{x} p(x)\ dx}$$

Proof. If (1) $y_1'' + p(x)y_1' + q(x)y_1 = 0$ and (2) $y_2'' + p(x)y_2' + q(x)y_2 = 0$. Now $y_2(1) - y_1(2)$ gives us

$$\underbrace{(y_1 y_2'' - y_1'' y_2)}_{\frac{d}{dx}W(x)} + p(x) \underbrace{(y_1 y_2' + y_1' y_2)}_{W(x)} = 0 \implies W'(x) + p(x)W(x) = 0$$

and the solution of this new DE is

$$W(x) = Ce^{-\int p(x)\ dx}$$

If we then apply the initial condition at x_0, we get $C = W(x_0)$. □

Theorem 3.6. *Let $p(x)$ and $q(x)$ be continuous on an interval I and consider the DE*

$$y'' + p(x)y' + q(x)y = 0$$

If y_1 and y_2 are two linearly independent solutions on I, then the general solution is $y = c_1y_1 + c_2y_2$. There will be no singular solutions.

Proof. Let $\psi(x)$ be a solution to the DE on I. Consider the IVP consisting of the DE $y'' + p(x)y' + q(x)y = 0$ and the ICs $y(x_0) = \psi(x_0)$, $y'(x_0) = \psi'(x_0)$. Note that $\psi(x)$ is the unique solution to this IVP. Now, let $y = c_1y_1 + c_2y_2$ where c_1 and c_2 are chosen such that

$$\begin{aligned} c_1y_1(x_0) + c_2y_2(x_0) &= \psi(x_0) \\ c_1y_1'(x_0) + c_2y_2'(x_0) &= \psi'(x_0) \end{aligned}$$

This is true because $W(x) \neq 0$ and explicitly, if $c = [c_0\ c_1]^t$, then $W(y_1, y_2)c = W(\psi)$ can be solved for c. Now y also satisfies the DE by the principle of superposition. Hence $y(x)$ is a solution to the IVP. But this means that $y(x) = \psi(x)$ and that is $\psi(x) = c_1y_1(x) + c_2y_2(x)$. □

3.3 Characteristic Equations

So how do we find y_1 and y_2? We'll restrict ourselves to constant coefficient equations for now: $y'' + ay' + by = f(x)$.

Example 3.5. Consider $y'' - y = 0$. By observation, two solutions are e^x and e^{-x} and so $y = c_1e^x + c_2e^{-x}$ is the general solution. Similarly with $y'' + y = 0$, we have $y = c_1 \cos x + c_2 \sin x$.

Remark 3.3. We can relate $\sin x, \cos x$ using Euler's formula as $e^{ix} = \cos x + i \sin x$. In fact, the general solution to $y'' + y' = 0$ can be expressed as $y = c_1e^{ix} + c_2e^{-ix}$. We can perhaps assume that every similar DE has exponential solutions.

Assume that the function $y = e^{mx}$ is a solution to the DE $y'' + ay' + by = 0$. Plug it in and we get

$$m^2e^{mx} + ame^{mx} + be^{mx} = 0 \iff k(m) = m^2 + am + b = 0$$

where we can call $k(m)$ the **characteristic equation** of the DE. So e^{mx} is a solution if m is a solution to the characteristic equation

We examine the various cases for the roots of the characteristic equation.

1. (Case I) If the characteristic equation has distinct real roots, then there are two independent solutions.

   (a) E.g. $y'' - y' - 2y = 0$ has the characteristic equation $m^2 - m - 2 = 0 \iff (m-2)(m+1) \iff m = 2, -1$ and hence the general solution is
   $$y = c_1e^{2x} + c_2e^{-x}$$

2. (Case II) If the characteristic equation has complex conjugate roots, $m = \alpha \pm i\beta$ and so we *could* write $y = c_1e^{(\alpha+i\beta)x} + c_2e^{(\alpha-i\beta)x}$. However, we know that there are two linearly independent *real-valued* solutions. To find them, we'll rewrite this as

$$\begin{aligned} y = e^{\alpha x}\left[c_1e^{i\beta x} + c_2e^{-i\beta x}\right] &\implies y = e^{\alpha x}\left[c_1\left(\cos\beta x + i\sin\beta x\right) + c_2\left(\cos\beta x - i\sin\beta x\right)\right] \\ &\implies y = e^{\alpha x}\left[(c_1 + c_2)\cos\beta x + (c_1 - c_2)\, i\sin\beta x\right] \\ &\implies y = e^{\alpha x}\left[A\cos\beta x + B\sin\beta x\right] \end{aligned}$$

   (a) E.g. $y'' + 2y' + 5 = 0$ has the characteristic equation $m^2 + 2m + 5 = 0 \iff (m+1)^2 + 4 \implies m = -1 \pm 2i$ and the general solution is
   $$y = e^{-x}\left[c_1\cos 2x + c_2 \sin 2x\right]$$

3. (Case III) Repeated Roots

   (a) In this case, our guess has yielded only one exponential solution e^{mx}. We need another solution.

(b) Here's how it was found (D'Alembert): Having two roots should not be much different from having two *nearly* identical roots. The general solution is

$$y = c_1 e^{mx} + c_2 e^{(m+\epsilon)x}$$

So what happens when $\epsilon \to 0$? For most values of c_1 and c_2, the solutions merge. However, if $c_1 = -1/\epsilon$ and $c_2 = 1/\epsilon$. We then have the particular solution

$$y = \frac{1}{\epsilon} e^{(m+\epsilon)x} - \frac{1}{\epsilon} e^{mx} = e^{mx} \left[\frac{e^{\epsilon x} - 1}{\epsilon} \right]$$

By l'Hopital's rule,

$$\lim_{\epsilon \to 0} \frac{e^{\epsilon x} - 1}{\epsilon} \overset{(H)}{=} \lim_{\epsilon \to 0} x e^{\epsilon x} = x$$

and so $y \to x e^{mx}$. Therefore, the general solution for this case is

$$y = c_1 e^{mx} + c_2 x e^{mx}$$

(c) E.g. $y'' + 6y' + 9y = 0$ has the characteristic equation $m^2 + 6m + 9 = 0 \iff (m+3)^2 = 0 \implies m = -3$ and the general solution is

$$y = c_1 e^{-3x} + c_2 x e^{-3x}$$

In general, for an n^{th} order homogeneous equation

$$y^{(n)} + b_{n-1} y^{(n-1)} + ... + b_1 y' + b_0 y = 0$$

the characteristic equation is

$$m^n + b_{n-1} m^{n-1} + ... + b_1 m + b_0 = 0$$

For every distinct real root m, e^{mx} will be a solution. For every pair of complex roots $\alpha \pm i\beta$, $e^{\alpha x} \cos \beta x$ and $e^{\alpha x} \sin \beta x$ will be solutions. For any root which is repeated, we multiply the above solutions by x (repeatedly, if necessary).

Example 3.6. 1) Given $y''' - 4y'' + 7y' - 6y = 0$, the characteristic equation is $m^3 - 4m^2 + 7m - 6 = 0$. We can see that $m = 2$ is a root so factor it out:

$$(m-2)(m^2 - 2m + 3) = 0 \implies (m-2)(m - (1+2i))(m - (1-2i))$$

and hence the general solution is

$$y = c_1 e^{2x} + c_2 e^x \cos 2x + c_3 e^x \sin 2x$$

2) Given $y''' + 3y'' + 3y' + y = 0$, where the characteristic equation is $m^3 + 3m^2 + 3m + 1 = 0 \iff (m+1)^3 = 0 \implies m = -1$ and the general solution is

$$y = e^{-x} + x e^{-x} + x^2 e^{-x}$$

3) Given $y^{(4)} + 2y'' + y = 0$, the characteristic equation is $m^4 + 2m^2 + 1 = 0 \implies (m^2 + 1)^2 = 0$ and hence the general solution is

$$y = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x$$

3.4 Inhomogeneous Equations

To solve $\Phi(y) = f(x)$ we just need to find one particular solution, and find the general solution to the homogeneous equation $\Phi(y) = 0$. The general solution is

$$y = y_h + y_p$$

To find y_p, we will use a method called the **method of undetermined coefficients**. Essentially, we guess the particular solution using a general form based on the inhomogeneous term - with unknown coefficients - and solve for the coefficients, using the original DE and higher derivatives of the DE.

Example 3.7. Consider $y'' + y' - 6y = 6x^2$. If $y_p = Ax^2 + Bx + C$, then $y_p' = 2Ax + B$ and $y_p'' = 2A$. Plugging this in, we get

$$\begin{aligned} 2A + (2Ax + b) - (6Ax^2 + Bx + C) = 6x^2 &\implies A = -1, 2A - B = 0 \\ &\implies A = -1, B = -1/3, 2A + B - 6C = 0 \\ &\implies A = -1, B = -1/3, C = -7/18 \end{aligned}$$

Therefore, $y_p = -x^2 - \frac{x}{3} - \frac{7}{18}$. It is easy to see that the roots of the characteristic equation are $m = -3, 2$ and so the general solution is

$$y = c_1 e^{-3x} + c_2 e^{2x} - x^2 - \frac{x}{3} - \frac{7}{18}$$

Example 3.8. Consider $y'' + y' - 4y = \cos 2x$. Here, we try $y_p = A\cos 2x + B\sin 2x$ where $y_p' = -2A\sin 2x + 2B\cos 2x$ and $y_p'' = -4A\cos 2x - 4B\sin 2x$. After some arithmetic, it can be shown that

$$\begin{aligned} -8A + 2B = 1, -2A - 8B = 0 &\implies 34B = 1, A + 4B = 0 \\ &\implies B = 1/34, A = -2/17 \end{aligned}$$

So $y_p = -\frac{2}{17}\cos 2x + \frac{1}{34}\sin 2x$.

Summary 1. We have the following table for estimates

Forcing Term	Trial Function
e^{kx}	Ae^{kx}
$\sin kx$,$\cos kx$	$A\cos kx + B\sin kx$
x^n	$\sum_{k=0}^{n} A_k x^k$
xe^x	$(Ax + B)e^x$
$x^2\cos x$	$(Ax^2 + Bx + C)\cos x + (Dx^2 + Ex + F)\sin x$

Remark 3.4. Note that if our trial function is a solution to the homogeneous DE of the inhomogeneous DE, then this method fails. To fix this, it turns out that we generally just need to multiply our usual guess by x.

Example 3.9. Consider $y'' - y' - 2y = e^{2x}$. The usual guess is Ae^{2x} which will result in a contradiction $0 = 1$. Instead, we try $y = Axe^{2x}$ with $y' = Ae^{2x} + 2Axe^{2x}$ and $y'' = 4Ae^{2x} - 4xe^{2x}$ and so with some arithmetic, we can get

$$3A = 1 \implies A = \frac{1}{3}$$

and thus $y_p = \frac{1}{3}xe^{2x}$ is a solution.

Example 3.10. Consider $y' - y = e^x + x^2$. Here, $y_h = Ce^x$ and we could try $Axe^x + Bx^2 + Cx + D$

Example 3.11. Consider $y'' + 2y' + y = e^{-x}$ where we have repeated roots and $y_h = c_1 e^{-x} + c_2 xe^{-x}$. In our estimate for y_p, we will need to multiply x^2 to guess $y_p = Ax^2 e^{-x}$

Example 3.12. Consider $y^{(4)} + 2y'' + y = x\cos x$. We previously showed that $y_h = c_1\cos x + c_2\sin x + c_3 x\cos x + c_4 x\sin x$. In this case, we need to guess $y_p = (Ax^3 + Bx^2)\cos x + (Cx^3 + Dx^2)\sin x$.

An alternate method to find a particular solution is to use something called the **variation of parameters**. For first-order equations, this is equal to the integrating factor method. However, it can be *generalized* to solve problems of the form $\Phi(y) = f(x)$.

Example 3.13. (1st-Order) Consider the DE $y' + xy = \frac{1}{x}$ and using integrating factors, we can find y_h to be

$$y_h = C_1 e^{-x^2/2}$$

We use the form y_h to guess at y_p and guess it to be $y_p = u(x)e^{-x^2/2}$. That is, we replace constants with arbitrary functions to *allow our parameters to vary*. If we differentiate, we get

$$y = u(x)e^{-x^2/2} \implies y' = u'(x)e^{-x^2/2} - xe^{-x^2/2}u(x)$$

and placing this in the DE gives us

$$u'(x)e^{-x^2/2} - xe^{-x^2/2}u(x) + xe^{-x^2/2}u(x) = 1/x \implies u'(x) = \frac{1}{x}e^{x^2/2}$$
$$\implies u(x) = \int_{x_0}^{x} \frac{1}{t}e^{\frac{1}{2}t^2 dt} + C_2$$

and hence

$$y_p = e^{-x^2/2}\left(\int_{x_0}^{x} \frac{1}{t}e^{\frac{1}{2}t^2 dt} + C_2\right)$$

with the general solution being

$$y = C_3 e^{-x^2/2} + e^{-x^2/2}\int_{x_0}^{x} \frac{1}{t}e^{\frac{1}{2}t^2 dt}$$

Keeping the C_2 in y_p allows to find the general solution directly, but keeping $C_2 = 0$ and adding y_h does the same.

Example 3.14. (2nd-Order) Consider the general equation

$$\underbrace{y'' + p(x)y' + q(x)y}_{G(y)} = F(x)$$

and suppose that we know $y_h = c_1 y_1 + c_2 y_2$. We vary *both* parameters and try

$$y = u_1(x)y_1 + u_2(x)y_2 \implies y' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$$

We now have 2 unknown functions and only 1 condition, so we can impose a second condition. We'll require that $(1)\ u_1'y_1 + u_2'y_2 = 0$ so that u_1'' and u_2'' will *not* appear. Then,

$$y'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''$$

Plugging this into our DE, we have

$$u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' + p(x)\left[u_1y_1' + u_2y_2'\right] + q(x)\left[u_1y_1 + u_2y_2\right] = f(x)$$

and rearrange to get

$$u_1'y_1' + u_2'y_2' + u_1\underbrace{G(y_1)}_{=0} + u_2\underbrace{G(y_2)}_{=0} = F(x) \implies (2)\ u_1'y_1' + u_2'y_2' = F(x)$$

We can then solve (1) and (2) for u_1' and u_2' and integrate.

Example 3.15. Solve the DE $y'' + 9y = 9\sec^2 3x$. We can see that

$$y_h = c_1\cos 3x + c_2\sin 3x$$

so we try $y = u_1\cos 3x + u_2\sin 3x$. We can find u_1' and u_2' by solving

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= F(x) \end{aligned}$$

This gives us

$$\begin{aligned} u_1'\cos 3x + u_2'\sin 3x &= 0 \\ -3u_1'\cos 3x + 3u_2'\sin 3x &= 9\sec^2 3x \end{aligned}$$

after some pesky algebra and arithmetic, we can get

$$\begin{aligned} u_2' &= 3\sec 3x \\ u_1' &= -3\sec 3x\tan 3x \end{aligned}$$

and hence

$$\begin{aligned} u_2 &= \ln|\sec 3x + \tan 3x| + C_2 \\ u_1 &= -\sec 3x + C_1 \end{aligned}$$

Therefore, the general solution is

$$\begin{aligned} y &= u_1y_1 + u_2y_2 \\ &= -1 + C_1\cos 3x + \sin 3x \cdot \ln|\sec 3x + \tan 3x| + C_2 \sin 3x \end{aligned}$$

The idea behind the variation of parameters also helps in another situation. It turns out that if we know one solution to a 2^{nd}-order homogeneous linear equation, we can find another one. More generally, knowing one solution to an n^{th}-order equation can allow us to reduce it to an $(n-1)^{th}$ -order equation and it even works on inhomogeneous equations. This is a **reduction in order** technique.

Consider the equation

$$x^2y'' + 2xy' - 2y = 0, x > 0$$

While we have no general techniques for finding solutions to variable coefficient equations, we *might* be able to guess that $y = x$ is a solution here. We then assume that $y = u(x)x$ where

$$\begin{aligned} y' &= u'x + u \\ y'' &= u''x + 2u' \end{aligned}$$

and putting this into the DE gives us

$$(x^3u'' + 2x^2u') + (2x^2u' + 2xu) - 2xu = 0 \implies xu'' + 4u' = 0$$

and this can be viewed as a *first-order equation* for u'. For convenience, let $v = u'$ with

$$\begin{aligned} \frac{dv}{dx}x + 4x = 0 &\implies \int \frac{dv}{v} = -4\int \frac{dx}{x} \\ &\implies \ln|v| = -4\ln|x| + C_0 \\ &\implies v = C_0x^{-4} \end{aligned}$$

That is, $u' = C_0x^{-4}$ so $u(x) = C_1x^{-3} + C_2$. Therefore,

$$y = \frac{C_1}{x^2} + C_2x$$

Example 3.16. Consider the equation

$$y^{(4)} + 4y''' + 6y'' + 4y' + y = 0$$

Note that the characteristic equation is $(m+1)^4 = 0$, so one solution is e^{-x}. Suppose that $y = ue^{-x}$. Then

$$\begin{aligned} y' &= u'e^{-x} - ue^{-x} \\ y'' &= u''e^{-x} - 2u'e^{-x} + ue^{-x} \\ y''' &= u'''e^{-x} - 3u''e^{-x} + 3u'e^{-x} - ue^{-x} \\ y^{(4)} &= u^{(4)}e^{-x} - 4u'''e^{-x} + 6u''e^{-x} - 4u'e^{-x} + ue^{-x} \end{aligned}$$

The DE becomes

$$\begin{aligned} u^{(4)}e^{-x} + u'''e^{-x}(-4+4) + u''e^{-x}(6-12+6) \\ +u'(-4+12-12+4) + ue^{-x}(1-4+6-4+1) = 0 \end{aligned}$$

which reduces to

$$u^{(4)} = 0 \implies u''' = C_1 \implies u'' = C_1x + C_2 \implies u' = C_1\frac{x^2}{2} + C_2x + C_3$$

and finally

$$u(x) = C_1\frac{x^2}{6} + \frac{1}{2}C_2x^2 + C_3x + C_4$$

Discarding the $1/6$ and $1/2$ we have

$$y = C_1x^3e^{-x} + C_2x^2e^{-x} + C_3xe^{-x} + C_4e^{-x}$$

Example 3.17. Consider the equation

$$x^2y'' + xy' - y = 72x^5$$

Observe that $y_1 = x$ is a solution to the homogeneous equation

$$x^2y'' + xy' - y = 0$$

We could guess that

$$y = u(x)x \implies y' = u'x + u, y'' = u''x + 2u'$$

The DE becomes

$$(x^3y'' + 2x^2u') + (x^2u' + xu) - xu = 72x^5 \implies xu'' + 3u' = 72x^3$$

Letting $v = u'$, this is

$$x\frac{dv}{dx} + 3v = 72x^3 \implies \frac{dv}{dx} + \frac{3}{x}v = 72x^2$$

Using the integrating factor $I(x) = e^{\int \frac{3}{x}\,dx} = x^3$, we get that

$$\begin{aligned} x^3\frac{dv}{dx} + 3x^2v = 72x^5 \quad &\implies \quad \frac{d}{dx}\left(x^3v\right) = 72x^5 \\ &\implies \quad v = 12x^3 + Cx^{-3} \\ &\implies \quad u = 3x^4 + C_1x^{-2} + C_2 \\ &\implies \quad y = C_1x^{-1} + C_2x + 3x^5 \end{aligned}$$

Alternatively, we could look for y_2 first and then find y_p. For the same system above, we can find $y_h = C_1x + C_2x^{-1}$. For the particular solution, we try

$$y = u_1x + u_2x^{-1}$$

and we must solve

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1 + u_2'y_2 &= 72x^3 \end{aligned}$$

Note that we have x^3 in the forcing term because we must standardize the original equation by dividing by x^2 on both sides.

3.5 Boundary Value Problems

A **boundary value problem** involves a DE with side conditions specified at different points. For example, we might have

$$y'' + y = 0, y(0) = 0, y(\pi) = 1$$

There are no Existence and Uniqueness (E/U) theorems for these.

Example 3.18. (1) Consider $y'' + \pi^2y = 0, y(0) = 0, y(1) = 1$. The general solution to the DE is

$$y = c_1\cos\pi x + c_2\sin\pi x$$

If $y(0) = 0$ then $c_1 = 0$ but if $y(1) = 1$ then $1 = -c_1$. Therefore, this problem has no solutions.

(2) Next, consider $y'' + \pi^2y = 0, y(0) = 0, y(1) = 0$. The general solution to the DE is

$$y = c_1\cos\pi x + c_2\sin\pi x$$

If $y(0) = 0$ then $c_1 = 0$ but if $y(1) = 0$ then $0 = -c_1$. Therefore, c_2 is free and *any* multiple of $\sin \pi x$ is a a solution.

Since we usually have no solutions, we'll usually ask a different question.

Example 3.19. Here is a typical BVP question: '

For which values of $k \in \mathbb{R}$ does the BVP

$$y'' + ky = 0, y(0) = 0, y(1) = 0$$

have solutions?

Solution: The characteristic equation is $m^2 + k = 0$.

(Case I) If $k < 0$ and we let $\lambda^2 = -k$, we consider $m^2 - \lambda^2 = 0 \implies m \pm \lambda$ and the general solution is

$$y = c_1 e^{\lambda x} - c_2 e^{-\lambda x}$$

If $y(0) = 0$ then $c_1 + c_2 = 0$. If $y(1) = 0$ then

$$\begin{aligned} c_1 e^{\lambda} + c_2 e^{-\lambda} = 0 &\implies c_1 e^{2\lambda} + c_2 = 0 \\ &\implies c_1(1 - e^{2\lambda}) = 0 \\ &\implies c_1 = c_2 = 0 \\ &\implies y = 0 \text{ is the only solution} \end{aligned}$$

(Case II) If $k = 0$ we have $y = C_1 + C_2 x, y(0) = 0, y(1) = 0 \implies c_1 = c_2 = 0$ and $y = 0$ is the only solution again.

(Case III) If $k > 0$ we can let $k = \lambda^2$ with $m^2 + \lambda^2 = 0$ giving us the general equation

$$y = c_1 \cos \lambda x + c_2 \sin \lambda x$$

If $y(0) = 0 \implies c_1$ and $y(1) = 0 \implies c_2 \sin \lambda = 0 \implies c_2 = 0$ giving $y = 0$ OR $\sin \lambda = 0 \implies \lambda = n\pi$.

In conclusion, the BVP

$$y'' + ky = 0, y(0) = 0, y(1) = 0$$

has non-trivial solutions only if $k = n^2\pi^2$ in which cases the solutions are

$$y = C \sin n\pi x$$

We refer to the values $n^2\pi^2$ as the **eigenvalues** of the BVP, and the functions $\sin n\pi x$ are the **eigenfunctions**.

Example 3.20. Find the eigenvalues and eigenfunctions of the BVP

$$y'' + 2y' + ky = 0, y(0) = 0, y(1) = 0$$

Solution. The characteristic equation is

$$m^2 + 2m + k = 0 \implies (m+1)^2 + (k-1) = 0 \implies m = -1 \pm \sqrt{k-1}$$

Now we divide this problem into cases:

- If $k < 1$ we have exponential solutions and we will find $y = 0$
- If $k = 1$, the solutions are

$$\begin{aligned} y = c_1 e^{-x} + c_2 e^{-x} &\implies y(0) = 0, y(1) = 0 \implies c_1 = 0, c_2 = 0 \\ &\implies y = 0 \text{ is the only solution} \end{aligned}$$

- If $k > 1$, then

$$y = e^{-x}\left[c_1 \cos \sqrt{k-1}x + c_2 \sin \sqrt{k-1}x\right]$$

and

$$\begin{aligned} y(0)=0, y(1)=0 \quad &\Longrightarrow \quad c_1=0, c_2\sin\sqrt{k-1}=0 \\ &\Longrightarrow \quad c_2=0 \text{ OR } \sqrt{k-1}=n\pi \implies k=n^2\pi+1 \end{aligned}$$

The eigenvalues are $k=n^2\pi^2+1$ and the eigenfunctions are $e^{-x}\sin(n\pi x)$.

4 Vector Differential Equations

In some applications, we encounter "coupled" equations, in which each functions are related to the derivatives of the other.

Example 4.1. Let x and y be the concentrations of a chemical (salt, maybe) in the two tanks of volume 100L each. Suppose that Tank 1 has x grams of chemical, 20L/min of pure water following in, 10L/min of Tank 2 flowing in, and 30L/min flowing into Tank 2. Also suppose that Tank 2 has 20L/min draining out to some unspecified source. We have

$$\begin{aligned} \frac{dx}{dt} &= -30\left(\frac{x}{100}\right)+10\left(\frac{y}{100}\right) \\ \frac{dy}{dt} &= 30\left(\frac{x}{100}\right)-30\left(\frac{y}{100}\right) \end{aligned}$$

or alternatively,

$$\begin{aligned} x'(t) &= -\frac{3}{10}x+\frac{1}{10}y \\ y'(t) &= \frac{3}{10}x-\frac{3}{10}y \end{aligned}$$

This is a **system** of linear equations. More generally, we might have

$$\begin{aligned} x' &= a_1x+b_1y+c_1z+f_1(t) \\ y' &= a_2x+b_2y+c_2z+f_2(t) \\ z' &= a_3x+b_3y+c_3z+f_3(t) \end{aligned}$$

where the equivalent matrix form is $\vec{x}'(t)=A\vec{x}+\vec{f}$ or

$$\vec{x}'(t)=\underbrace{\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\vec{x}(t)}+\underbrace{\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}}_{\vec{f}}$$

Our work will be shaped by one theorem which is as follows.

Theorem 4.1. *Any higher order ODE can be converted into a 1st-order vector DE. The same is true for higher order systems.*

Proof. (2nd-order Case) Consider the system

$$\begin{aligned} x''+p_1x'+q_1x &= f_1(x) \\ x''+p_2x'+q_2x &= f_2(x) \end{aligned}$$

If we let $\vec{x}=\begin{bmatrix} x & x' & y & y' \end{bmatrix}^T$, then $\vec{x}'=\begin{bmatrix} x' & x'' & y' & y'' \end{bmatrix}^T$ and

$$\vec{x}'=\begin{pmatrix} x' \\ -p_1x'-q_1x+f_1(t) \\ y' \\ -p_2y'-q_2y+f_2(t) \end{pmatrix}=\begin{pmatrix} 0 & 1 & 0 & 0 \\ -q_1 & -p_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -q_2 & -p_2 \end{pmatrix}\vec{x}+\begin{pmatrix} 0 \\ f_1(t) \\ 0 \\ f_2(t) \end{pmatrix}$$

□

Theorem 4.2. *(Converse of the Theorem above) Every n-th order vector DE can be expressed as an n-th order linear ODE.*

Proof. ($n = 2$ Case) Consider the system

$$\vec{x}' = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix} f(t) \\ g(t) \end{bmatrix}}_{\vec{f}}$$

Let V_i denote the i^{th} row or entry in a matrix or vector respectively. We can solve for y in $\vec{x}_1' = A_1\vec{x} + \vec{f}_1$ to get

$$y = \frac{x'}{b} - \frac{ax}{b} - \frac{f(t)}{b} \implies y' = \frac{x''}{b} - \frac{ax'}{b} - \frac{f'(t)}{b}$$

Plug this into $\vec{x}_2' = A_2\vec{x} + \vec{f}_2$ and we get

$$x'' - (a-d)x' + (ad - bc)x = f' - df + bg$$

$\square$

So the theory of linear ODEs extends easily to vector DEs and in particular, we have:

- (*Existence and Uniqueness*) The IVP $\vec{x}' = A\vec{x} + \vec{f}(t), \vec{x}(0) = \vec{x}_0$ has a unique solution on an interval I provided that the components of A and $\vec{f}$ are continuous on I.
- The *Principle of Superposition* holds.
- The general solution to a homogeneous problem will contain n linearly independent solutions.
- The general solution to an inhomogeneous problem can be expressed as $\vec{x} = \vec{x}_h + \vec{x}_p$.
- Our theorems on linear independence also carry over with this modification:

Definition 4.1. The Wronskian of the vector functions $\vec{f}_1, \vec{f}_2, ..., \vec{f}_n$ is the determinant of the matrix

$$\begin{bmatrix} \vec{f}_1 & \vec{f}_2 & \cdots & \vec{f}_n \end{bmatrix}$$

Example 4.2. Are the vector functions

$$\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}, \begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix}$$

independent? Calculating the Wronskian gives us

$$W(t) = \sin t \cos 2t - \cos t \sin 2t$$

which is non-zero on $(0, \pi/2) \implies$ linearly independent on $(0, \pi/2)$. Since $W(t)$ is *sometimes* zero on larger intervals, then functions cannot be solutions to a vector DE on any of those intervals.

4.1 Homogeneous Vector DEs

Consider the DE $\vec{x}' = A\vec{x}$. Recall that the DE $y' = ay$ has general solution $y = Ce^{ax}$. Perhaps $\vec{x} = A\vec{x}$ also has exponential solutions? We'll guess that $x = C_1e^{mt}$ and $y = C_2e^{mt}$. That is, we try $\vec{x} = \vec{C}e^{mt}$. Then, $\vec{x}' = m\vec{C}e^{mt}$ so

$$\vec{x}' = A\vec{x} \implies m\vec{C}e^{mt} = A\vec{c}e^{mt} \implies A\vec{c} = m\vec{C}$$

That is, $\vec{C}e^{mt}$ is a solution as long as m is an eigenvalue of A and $\vec{C}$ is corresponding eigenvector.

<u>Quick review of linear algebra:</u>

- $A\vec{v} = \lambda\vec{x} \implies (A - \lambda I)\vec{x} = 0$ and this will have non-zero solutions iff $(A - \lambda I)$ is **not** invertible.
- We thus need $\det(A - \lambda I) = 0$. We call this the **characteristic equation** of the vector DE.

- Once we have a value for λ, we can find associated eigenvectors by solving $(A - \lambda I)\vec{v} = \vec{0}$
- An $n \times n$ matrix will have n eigenvalues if we include multiplicity
- If an eigenvalue has multiplicity m, it will have anywhere from 1 to m linearly independent eigenvectors

Example 4.3. (Case I: Two Distinct Real Eigenvalues) Solve the system

$$\begin{aligned} x' &= 2x + 3y \\ y' &= 2x + y \end{aligned}$$

Solution. As a vector DE, this is

$$\vec{x}' = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \vec{x}$$

and using the eigenvalue algorithm gives us $\lambda = 4, -1$. For $\lambda = 4$, we will eventually get the system

$$-2v_1 + 3v_2 = 0 \implies v_2 = \frac{2}{3}v_1$$

Choosing $v_1 = 3$ gives us $v_2 = 2$ and so we may use $\vec{v} = \begin{bmatrix} 3 & 2 \end{bmatrix}^T$. Therefore *one* solution is

$$\vec{x}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4t}$$

For $\lambda = 1$, we eventually get

$$v_1 + v_2 = 0 \implies v_1 = -v_2$$

and setting $v_1 = 1$ gives us $\vec{v} = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$ and a second solution is

$$\vec{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

So the general solution is

$$\vec{x} = C_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

Now suppose our problem had $\vec{x}(0) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. Then

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = C_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \implies \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and we will get the solution $C_1 = 2/5, C_2 = -1/5$. Hence

$$\vec{x} = \frac{1}{5} \begin{bmatrix} 6e^{4t} - e^{-t} \\ 4e^{4t} + e^{-t} \end{bmatrix}$$

Example 4.4. (Case II: Complex Eigenvalues) Solve the system

$$\vec{x}' = \begin{bmatrix} -2 & 1 \\ -3 & -4 \end{bmatrix} \vec{x}$$

Solution. Under the eigenvalue algorithm, we will get the characteristic equation

$$(\lambda + 3)^2 + 2 = 0 \implies \lambda = -3 \pm \sqrt{2}i$$

For $\lambda = -3 + \sqrt{2}i$, we solve $(A - \lambda I)\vec{v} = 0$ to get

$$\begin{bmatrix} 1 - \sqrt{2}i & 1 \\ -3 & -1 - \sqrt{2}i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Setting $v_1 = 1$ gives

$$\vec{v} = \begin{bmatrix} 1 \\ -1+\sqrt{2}i \end{bmatrix}$$

Therefore, one solution is

$$\begin{aligned} \vec{x}_1 = \begin{bmatrix} 1 \\ -1+\sqrt{2}i \end{bmatrix} e^{(-3+\sqrt{2}i)t} &= e^{-3t}\left(\cos\sqrt{2}t + i\sin\sqrt{2}t\right)\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}\right] \\ &= e^{-3t}\left[\begin{pmatrix} \cos\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t - \sin\sqrt{2}t \end{pmatrix} + i\begin{pmatrix} \sin\sqrt{2}t \\ \sqrt{2}\cos\sqrt{2}t - \sin\sqrt{2}t \end{pmatrix}\right] \end{aligned}$$

and since the other eigenvalue only differs by the sign in i and the operations above are linear, then the second solution is

$$\begin{aligned} \vec{x}_2 = \begin{bmatrix} 1 \\ -1-\sqrt{2}i \end{bmatrix} e^{(-3-\sqrt{2}i)t} &= e^{-3t}\left(\cos\sqrt{2}t + i\sin\sqrt{2}t\right)\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} - i\begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}\right] \\ &= e^{-3t}\left[\begin{pmatrix} \cos\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t - \sin\sqrt{2}t \end{pmatrix} - i\begin{pmatrix} \sin\sqrt{2}t \\ \sqrt{2}\cos\sqrt{2}t - \sin\sqrt{2}t \end{pmatrix}\right] \end{aligned}$$

The real and imaginary parts are linearly independent solutions to the vector DE, so the general solution is

$$\vec{x} = e^{-3t}\left[C_1\begin{pmatrix} \cos\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t - \sin\sqrt{2}t \end{pmatrix} + C_2\begin{pmatrix} \sin\sqrt{2}t \\ \sqrt{2}\cos\sqrt{2}t - \sin\sqrt{2}t \end{pmatrix}\right]$$

Example 4.5. (Case III: Repeated Eigenvalues) Solve the system

$$\vec{x}' = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}\vec{x}$$

Solution. Under the eigenvalue algorithm, we will get the characteristic equation

$$(\lambda+1)^2 = 0 \implies \lambda = -1$$

The associated eigenvector will then be one that satisfies $v_1 = 2v_2$ and so we may use

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Therefore, one solution is

$$\vec{x}_1 = e^{-t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We need a second one. Can we multiply by t? In general, we have $\vec{x}_1 = \vec{v}e^{\lambda t}$ and we'll try

$$\begin{aligned} \vec{x} = \vec{v}te^{\lambda t} \quad &\implies \quad \vec{x}' = \vec{v}e^{\lambda t} + \lambda\vec{v}te^{\lambda t}, x' = A\vec{x} \\ &\implies \quad \vec{v}e^{\lambda t} + \lambda\vec{v}te^{\lambda t} = \underbrace{A\vec{v}}_{\lambda\vec{v}}te^{\lambda t} \\ &\implies \quad \vec{v}e^{\lambda t} = 0 \implies \vec{v} = 0 \end{aligned}$$

which is impossible since $\vec{v} \neq 0$ by properties of eigenvectors. So multiplication by t fails. What's wrong? In our example, if this had worked, we would have had

$$\vec{x} = C_1e^{-t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2te^{-t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

That is, $x = 2C_1e^{-t} + 2C_2te^{-t}, y = C_1e^{-t} + C_2te^{-t}$. Perhaps x and y should not be restricted to be multiples of one another? Maybe we could try the form

$$\vec{w} = \vec{v}te^{\lambda t} + \vec{w}e^{\lambda t}$$

where $\vec{w}$ is to be determined. Now

$$\begin{aligned} x' = Ax \quad &\implies \quad \vec{v}e^{\lambda t} + \lambda\vec{v}te^{\lambda t} + \lambda\vec{w}e^{\lambda t} = \underbrace{A\vec{v}}_{\lambda\vec{v}}te^{\lambda t} + A\vec{w}e^{\lambda t} \\ &\implies \quad (A - \lambda I)\vec{w} = \vec{v} \end{aligned}$$

where $\vec{w}$ is called a generalized eigenvector of A corresponding to the eigenvalue λ. Solving for $\vec{w}$ gives us

$$\vec{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

The general solution is therefore

$$\begin{aligned} \vec{x} &= C_1\vec{x}_1 + C_2\vec{x}_2 \\ &= C_1e^{-t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2\left[te^{-t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^{-t}\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right] \\ &= C_1e^{-t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2te^{-t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2e^{-t}\begin{bmatrix} -1 \\ 0 \end{bmatrix} \end{aligned}$$

Remark 4.1. (3 by 3 case)

- If λ has multiplicity 2 and 1 eigenvector, then one solution is $\vec{u}e^{\lambda t}$ and the second is $\vec{u}te^{\lambda t} + \vec{v}e^{\lambda t}$ where $(A - \lambda I)\vec{v} = \vec{u}$ which is exactly the same in the 2 by 2 case.
- If λ has multiplicity 3 and 1 eigenvector $\vec{u}$ then the solutions are $\vec{u}e^{\lambda t}$, $\vec{u}te^{\lambda t} + \vec{v}e^{\lambda t}$, $\frac{\vec{u}t^2}{2}e^{\lambda t} + \vec{v}te^{\lambda t} + \vec{w}e^{\lambda t}$ where

$$(A - \lambda I)\vec{v} = \vec{u}, (A - \lambda I)\vec{w} = \vec{v}, (A - \lambda I)^3\vec{u} = 0$$

- If λ has multiplicity 3 and 2 eigenvectors $\vec{u}_1$ and $\vec{u}_2$ then two solutions are $\vec{u}_1e^{\lambda t}$ and $\vec{u}_2e^{\lambda t}$. A third solution is $\vec{v}te^{\lambda t} + \vec{w}e^{\lambda t}$ where $\vec{v}$ is some vector in the eigenspace of λ and $\vec{w}$ is a generalized eigenvector where $(A - \lambda I)\vec{w} = \vec{v}$. Note that there will be only one choice of $\vec{v}$ which makes this solvable.

Example 4.6. (1) Solve

$$\vec{x}' = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 2 & 1 & 2 \end{bmatrix}\vec{x}$$

The eigenvalues are $\lambda = 1, -1, 4$. Corresponding eigenvectors are

$$\begin{bmatrix} -1 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

and the general solution is

$$\vec{x} = c_1\begin{bmatrix} -1 \\ -6 \\ 4 \end{bmatrix}e^t + c_2\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}e^{-t} + c_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}e^{4t}$$

(2) Solve $\vec{x}' = A\vec{x}$ where

$$\vec{x} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

The eigenvalues can be shown to be $\lambda = 1, 1 \pm i$ with the following eigenvectors. For $\lambda = 1$, we can use the vector $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$. For $\lambda = 1 + i$, we have

$$\begin{bmatrix} -i & 2 & -1 \\ 0 & -i & 1 \\ 0 & -1 & -i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We then get that $iv_2 + v_3 = 0, -v_2 - iv_3 = 0$. If we set $v_3 = 1$ then $v_2 = -i$ and $v_1 = -2 + i$. We will thus use

$\begin{pmatrix} -2+i & -i & 1 \end{pmatrix}^T$ and this gives

$$\begin{aligned} \vec{x}_2 = e^{(1+i)t} \begin{bmatrix} -2+i \\ -i \\ 1 \end{bmatrix} &= e^t(\cos t + i \sin t) \left(\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right) \\ &= e^t \left(\begin{bmatrix} -2\cos t - \sin t \\ \sin t \\ \cos t \end{bmatrix} + i \begin{bmatrix} \cos t - 2\sin t \\ -\cos t \\ \sin t \end{bmatrix} \right) \end{aligned}$$

We can now conclude that

$$\vec{x} = c_1 e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} -2\cos t - \sin t \\ \sin t \\ \cos t \end{bmatrix} + c_3 e^t \begin{bmatrix} \cos t - 2\sin t \\ -\cos t \\ \sin t \end{bmatrix}$$

(3) Consider

$$\vec{x}' = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ -2 & -2 & -1 \end{bmatrix} \vec{x}$$

This has $\lambda = 1, 1, 1$ and to get eigenvectors, we examine the system

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies u_1 + u_2 + u_3 = 0$$

where u_2 and u_3 are free. We may use

$$u_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

This gives us $e^t \vec{u}_1$ and $e^t \vec{u}_2$. For the 3rd, we need $(A - \lambda I)\vec{w} = \vec{v}$ where $\vec{v}$ is an eigenvector. Specifically,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

where the right side is unique. With this, we have to 2 free variables so we may set $w_2 = w_3 = 0$ and use $\vec{w} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$. The general solution to the DE is

$$\vec{x} = c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-t} + c_3 \left(\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t \right)$$

4.2 Inhomogeneous Vector DEs

The *method of undetermined coefficients* is useful for simple problems:

Example 4.7. Solve $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \vec{x} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Solution. Find $\vec{x}_h$ which will be

$$\vec{x}_h = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\vec{x}_p$ we guess

$$\vec{x} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \implies \vec{x}' = 0$$

so

$$\vec{x}' = A\vec{x} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \implies \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$\implies a_1 + a_2 = -2, 2a_2 = -1$$
$$\implies a_2 = -1/2, a_1 = -3/2$$

and the general solution is

$$\vec{x} = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$$

For the *method of variation of parameters*, the idea for ODEs is that given $\vec{x}_h = c_1\vec{x}_1 + c_2\vec{v}_2$, we assume that the solution can be written as $\vec{x} = u_1\vec{x}_1 + u_2\vec{x}_2$ for some functions u_1 and u_2. We differentiate to get

$$\vec{x}'(t) = u_1'\vec{x}_1 + u_1\vec{x}_1' + u_2'\vec{x}_2 + u_2\vec{x}_2'$$

Plug these into the DE $\vec{x}' = A\vec{x} + \vec{f}$ and we get

$$u_1'\vec{x}_1 + u_1\vec{x}_1' + u_2'\vec{x}_2 + u_2\vec{x}_2' = A(u_1\vec{x}_1 + u_2\vec{x}_2) + \vec{f}$$

Now, $A\vec{x}_1 = \vec{x}_1'$ and $A\vec{x}_2 = \vec{x}_2'$ so

$$u_1'\vec{x}_1 + u_2'\vec{x}_2 = \vec{f}$$

In component form, $u_1'x_{11} + u_2'x_{12} = f_1$ and $u_1'x_{12} + u_2'x_{22} = f_2$. We will always be able to solve this system for u_1 and u_2 since $\vec{x}_1$ and $\vec{x}_2$ are linearly independent. We'll obtain $u_1' = G_1(t), u_2' = G_2(t)$ and

$$\vec{x} = \left[\int G_1(t)dt\right]\vec{x}_1 + \left[\int G_2(t)dt\right]x_2$$

where *if we keep the constants of integration, this will be the general solution.*

Example 4.8. Solve $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\vec{x} + \begin{bmatrix} te^{-2t} \\ 3e^{-2t} \end{bmatrix}$.

Solution. We know that

$$\vec{x}_h = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We assume that $\vec{x} = u_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + u_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and must solve

$$u_1' e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + u_2' e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} te^{-2t} \\ 3e^{-2t} \end{bmatrix} \implies \begin{cases} u_1'e^t + u_2'e^{2t} = te^{-2t} & (1) \\ u_2'e^{2t} = 3e^{-2t} & (2) \end{cases}$$

For equation (2), we can see that

$$u_2' = 3e^{-4t} \implies u_2 = -\frac{3}{4}e^{-4t} + c_2$$

and plugging this into (1) gives us

$$u_1'e^t + 3e^{-2t} = te^{-2t} \implies u_1' = (t-3)e^{-3t}$$
$$\implies u_1 = -\frac{1}{3}(t-3)e^{-3t} + \frac{1}{3}\int e^{-3t}dt$$
$$\implies u_1 = -\frac{1}{3}(t-3)e^{-3t} - \frac{1}{9}e^{-3t} + c_1$$
$$\implies u_1 = -\frac{t}{3}e^{-3t} + \frac{8}{9}e^{-3t} + c_1$$

using integration by parts with $u = t - 3$ and $dv = e^{-3t}$. Hence, the general solution is

$$\begin{aligned}\vec{x} &= \left(-\frac{t}{3}e^{-3t} + \frac{8}{9}e^{-3t} + c_1\right)e^t\begin{bmatrix}1\\0\end{bmatrix} + \left(-\frac{3}{4}e^{-4t} + c_2\right)e^{2t}\begin{bmatrix}1\\1\end{bmatrix}\\ &= c_1e^t\begin{bmatrix}1\\0\end{bmatrix} + c_2e^{2t}\begin{bmatrix}1\\1\end{bmatrix} + \left(-\frac{t}{3}e^{-2t} + \frac{8}{9}e^{-2t}\right)\begin{bmatrix}1\\0\end{bmatrix} + \left(-\frac{3}{4}e^{-2t}\right)\begin{bmatrix}1\\1\end{bmatrix}\end{aligned}$$

That is,

$$\begin{aligned}x(t) &= c_1e^t + c_2e^{2t} - \frac{1}{3}te^{-2t} + \frac{5}{36}e^{-2t}\\ y(t) &= c_2e^{-2t} - \frac{3}{4}e^{-2t}\end{aligned}$$

5 Partial Differential Equations (PDEs)

Partial differential equations (PDEs) involve functions of more than 1 variable.

Example 5.1. $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + x^2 \sin t$, for $u = u(x,t)$.

Example 5.2. Here are 3 famous PDEs:

1. The Heat Equation

   (a) Consider a metal bar of length L with uniform cross-sectional area and an insulated surface. Let $u(x,t)$ be the temperature at a distance x from one end at time t. It can be shown that $u(x,t)$ should obey the equation

   $$\frac{\partial u}{\partial t} = \gamma\frac{\partial^2 u}{\partial x^2}$$

   (b) Rough Explanation: The temperature will be increasing at any point where the temperature profile is concave up.

   (c) To complete the problem, we need side conditions:

      i. We'll need an "initial condition" $u(x,0) = f(x)$ which is technically a boundary condition.
      ii. We also need BCs at $x = 0$ and $x = L$ where if we imagine fixing the ends of the bar in ice, then we require $u(0,t) = u(L,t) = 0$.
      iii. If we insulated the ends , then we need $u_x(0,t) = u_x(L,t) = 0$.
      iv. Others are possible.

2. The Wave Equation

   (a) Now consider a string of length L, under tension. If the string is plucked, we expect it to vibrate. Let $u(x,t)$ be the vertical displacement of each point x at time t. Assuming that $|u| \ll L$, it can be shown that

   $$u_{tt} = \alpha^2 u_{xx}$$

   (b) Rough Explanation: The concavity of the string's profile determines the forces acting on each point, and hence determines the acceleration.

   (c) This time, for boundary conditions , we need 2 initial conditions:

      i. $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$
      ii. Also, we need BCs at $x = 0, L$. Usually, $u(0,t) = 0, u(L,t) = 0$ (or equations w.r.t u_x).

3. Laplace's Equation

   (a) This is $u_{xx} + u_{yy} = 0$. This is a kind of smoothness condition.

   (b) Application: Consider the 2nd-order Heat Equation $u_t = \gamma(u_{xx} + u_{yy})$. As $t \to \infty$, $u_t \to 0$ and so the "steady-state" temperature distribution will satisfy Laplace's equation.

   (c) Boundary conditions: Specified on the boundary and we usually want the object to be bounded (for easiness).

5.1 First Order Linear DEs

Some simple PDEs can be solved by "**partial integration**" (which is not integration at all - it's antidifferentiation).

Example 5.3. Consider the equation $u_y = -e^{-y}$ with the initial condition

$$u(x,0) = \frac{1}{1+x^2}$$

We can antidifferentiate to get

$$u(x,y) = e^{-y} + g(x)$$

Apply the IC to get

$$\frac{1}{1+x^2} = 1 + g(x) \implies g(x) = \frac{1}{1+x^2} - 1$$

and so

$$u(x,y) = e^{-y} + \frac{1}{1+x^2} - 1$$

We essentially have an ODE for each value of x. We say the lines $x = C$ are **characteristic curves** for the PDE.

Remark 5.1. Note that partial integration is not possible if both u_x and u_y are present. If the equation is **linear**, though, we can always introduce a change of variables which will eliminate a derivative where a linear 1st-order PDE has the form

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y)$$

Example 5.4. Consider the IVP

$$u_y = -2u_x, u(x,0) = e^{-x^2}$$

If we let $\xi = x - 2y, \eta = y$ then we may write

$$z = u(x,y) = \hat{u}(\xi,\eta)$$

Now,

$$\frac{\partial u}{\partial y} = \frac{\partial \hat{u}}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} \text{ and } \frac{\partial u}{\partial x} = \frac{\partial \hat{u}}{\partial \xi}\frac{\partial \xi}{\partial x}$$

$$\implies \frac{\partial u}{\partial y} = -2\hat{u}_\xi + \hat{u}_\eta \text{ and } \frac{\partial u}{\partial x} = \hat{u}_\xi$$

The PDE $u_y = -2u_x$ then becomes $-2\hat{u}_\xi + \hat{u}_\eta = -2\hat{u}_\xi$ and therefore $\hat{u}_\eta = 0$ and we can integrate (anti-differentiate) to get $\hat{u} = g(\xi)$ so $u(x,y) = g(x-2y)$. This chain rule here is the same one in Math247 involving the product of gradients and Jacobians. With our initial condition of $u(x,0) = e^{-x^2}$ we get $e^{x^2} = g(x)$ and

$$u(x,y) = e^{-(x-2y)^2}$$

Note that the lines $\xi = C$ (that is $x - 2y = C$) are the characteristics. We do have ODE-like behaviour but along these lines instead of along $x = C$.

Problem 5.1. How do we find the change of variables?

Lemma 5.1. *Consider the ODE $\frac{dy}{dx} = f(x,y)$. If its general solution can be written in implicit form as $\phi(x,y) = K$ then*

$$\frac{\phi_x}{\phi_y} = -\frac{dy}{dx} \implies \frac{\phi_x}{\phi_y} = -f(x,y)$$

Proof. Let $\phi(x,y) = K$ be the solution to $y' = f(x,y)$. Differentiate implicitly in x to get $\phi_x + \phi_y \frac{dy}{dx} = 0 \implies \phi_x/\phi_y = -dy/dx$. □

Example 5.5. For $\frac{dy}{dx} = 3xy^2$, we find $y = Ce^{x^3}$. Rewriting as $C = ye^{-x^3}$ we let $\phi(x,y) = ye^{-x^3}$. Then $\phi_y = e^{-x^3}$ and $\phi_x = -3x^2ye^{-x^3}$ so

$$\frac{\phi_x}{\phi_y} = -3x^2y$$

5.2 Change of Basis

Consider a 1st order linear PDE

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y)$$

where $a \neq 0$ and $b \neq 0$. We wish to introduce

$$\xi = \xi(x,y), \eta = \eta(x,y)$$

to replace $u(x,y)$ with $\hat{u}(\xi,\eta)$. By the chain rule,

$$\begin{aligned} u_x &= \hat{u}_\xi \xi_x + \hat{u}_\eta \eta_x \\ u_y &= \hat{u}_\xi \xi_y + \hat{u}_\eta \eta_y \end{aligned}$$

The PDE becomes

$$a(x,y)\left[\hat{u}_\xi \xi_x + \hat{u}_\eta \eta_x\right] + b(x,y)\left[\hat{u}_\xi \xi_y + \hat{u}_\eta \eta_y\right] + c(x,y)\hat{u} = f(x,y)$$

$$\implies \left[a\xi_x + b\xi_y\right]\hat{u}_\xi + \left[a\eta_x + b\eta_y\right]\hat{u}_\eta + c\hat{u} = f$$

The goal is to make one of the derivative terms vanish. Let's eliminate $\hat{u}_\eta$ by setting

$$a\eta_x + b\eta_y = 0$$

This means, that assuming that $\eta_y \neq 0$ we need

$$\frac{\eta_x}{\eta_y} = -\frac{b(x,y)}{a(x,y)}$$

So we need to pick η where $\eta(x,y) = K$ is the general solution to the ODE $\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$ from our Lemma above. What about ξ? Our only other constraint is that the transformation should be invertible. So we need the Jacobian to be non-zero:

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \xi_x\eta_y - \xi_y\eta_x \neq 0$$

If we simply let $\xi = x$ then this is satisfied.

Example 5.6. Solve the IVP

$$xu_x + yu_y = 3u, u(x,1) = 1 - x^2, x \geq 0, y \geq 1$$

Solution. We start by solving the equation

$$\begin{aligned} \frac{dy}{dx} = \frac{b}{a} \implies \frac{dy}{dx} = \frac{y}{x} \quad &\implies \quad \int \frac{dy}{y} = \int \frac{dx}{x} \\ &\implies \quad C = \ln y - \ln x \end{aligned}$$

and $c_1 = \ln x - \ln y$ or $c_2 = x/y$. So we let $\xi = x$, $\eta = \ln x - \ln y$. Then $u_x = \hat{u}_\xi \xi_x + \hat{u}_\eta \eta_x = \hat{u}_\xi + \frac{1}{x}\hat{u}_\eta$ and $u_y = \frac{1}{y}\hat{u}_\eta$. The PDE then becomes $\xi\hat{u}_\xi = 3\hat{u}$ this is essentially equivalent to the ODE

$$x\frac{du}{dx} = 3u \implies u = C_1x^3$$

Therefore, $\hat{u} = f(\eta)\xi^3$ and so

$$u(x,y) = f(\ln x - \ln y)x^3 = h\left(\frac{x}{y}\right)x^3$$

Now $u(x,1) = 1 - x^2 \implies 1 - x^2 = x^3h(x)$ so $h(x) - \frac{1-x^2}{x^3}$ and thus

$$u(x,y) = x^3\left(\frac{1 - \left(\frac{x}{y}\right)^2}{\left(\frac{x}{y}\right)^3}\right) = ... = y^3 - x^2y$$

Note that we may also let $\xi = \phi(x,y)$ and $\eta = y$ and obtain an equation with no $\hat{u}_\xi$ term. This may be easier, or harder.

5.3 Second-Order Linear PDEs

A 2nd-order linear PDE is 2 variables has the form

$$a(x,y)u_{xx}+b(x,y)u_{xy}+c(x,y)u_{yy}+d(x,y)u_x+e(x,y)u_y+f(x,y)u=g(x,y)$$

We will try to simplify this using characteristics as we did for 1st order equations. That is we'll introduce

$$\xi=\xi(x,y) \text{ and } \eta=\eta(x,y)$$

and convert the DE to

$$A(\xi,\eta)\hat{u}_{\xi\xi}+B(\xi,\eta)\hat{u}_{\xi\eta}+C(\xi,\eta)\hat{u}_{\xi\xi}+D(\xi,\eta)\hat{u}_{\xi}+E(\xi,\eta)\hat{u}_{\eta}+F(\xi,\eta)\hat{u}=G(\xi,\eta)$$

The goal is to pick ξ and η so that $A=0\vee V=0\vee C=0$. What *are* these functions? This is a tedious calculation. We need to rewrite u_{xx},u_{xy}, etc. in terms of $\hat{u}_{\xi\xi},\hat{u}_{\xi\eta}$, etc.

Starting with u_{xx}, we know that $\hat{u}_x=\hat{u}_\xi\xi_x+\hat{u}_\eta\eta_x$ and hence

$$\begin{aligned}\hat{u}_{xx} &= \frac{\partial}{\partial x}(u_x)\\ &= \frac{\partial}{\partial x}(u_\xi)\xi_x+\hat{u}_\xi\frac{\partial}{\partial x}(\xi_x)+\frac{\partial}{\partial x}(\hat{u}_\eta)\eta_x+\hat{u}_\eta\frac{\partial}{\partial x}(\eta_x)\\ &= \left[\hat{u}_{\xi\xi}\xi_x+\hat{u}_{\xi\eta}\eta_x\right]\xi_x+\hat{u}_\xi\xi_{xx}+\left[\hat{u}_{\eta\xi}\xi_x+\hat{u}_{\eta\eta}\eta_x\right]\eta_x+\hat{u}_\eta\eta_{xx}\\ &= \hat{u}_{\xi\xi}\left(\xi_x\right)^2+2\hat{u}_{\xi\eta}\left(\xi_x\eta_x\right)+\hat{u}_{\eta\eta}\left(\eta_x\right)^2+\hat{u}_\xi\xi_{xx}+\hat{u}_\eta\eta_{xx}\end{aligned}$$

Repeating this for u_{xy} and u_{yy}, plugging in the results into the PDE, and rearranging gives

$$\begin{aligned}A(\xi,\eta) &= a(x,y)\left(\xi_x\right)^2+b(x,y)\xi_x\xi_y+c(x,y)\left(\xi_y\right)^2\\ B(\xi,\eta) &= 2a(x,y)\xi_x\eta_x+b(x,y)\left[\xi_x\eta_y+\eta_y\xi_x\right]+2c(x,y)\xi_y\eta_y\\ C(\xi,\eta) &= a(x,y)\left(\eta_x\right)^2+b(x,y)\eta_x\eta_y+c(x,y))\left(\eta_y\right)^2\end{aligned}$$

To eliminate the $\hat{u}_{\xi\xi}$ term, then we set $A=0$:

$$a\left(\xi_x\right)^2+b\xi_x\xi_y+c\left(\xi_y\right)^2=0$$

Assuming that $\xi_y\neq 0$, we can rewrite this as

$$a\left(\frac{\xi_x}{\xi_y}\right)^2+b\left(\frac{\xi_x}{\xi_y}\right)+c=0\implies\frac{\xi_x}{\xi_y}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

So we want to start by solving

$$\frac{dy}{dx}=\frac{b\pm\sqrt{b^2-4ac}}{2a}$$

We get 3 cases:

(1) Hyperbolic Equations: If $b^2-4ac>0$ then we may choose $\xi=\phi_1$ and $\eta=\phi_2$ where $\phi_1=K_1$ and $\phi_2=K_2$ are the general solutions to

$$\frac{dy}{dx}=\frac{b-\sqrt{b^2-4ac}}{2a} \text{ and } \frac{dy}{dx}=\frac{b+\sqrt{b^2-4ac}}{2a}$$

respectively. This will eliminate *both* $\hat{u}_{\xi\xi}$ and $\hat{u}_{\eta\eta}$ which leaves (after division by B)

$$\hat{u}_{\xi\eta}+\Phi(\xi,\eta,\hat{u},\hat{u}_\xi,\hat{u}_\eta)=0$$

This is the **canonical form of a hyperbolic equation**.

(2) Parabolic Equations: If $b^2-4ac=0$ then we can set $\xi=\phi(x,y)$ where $\phi=K$ is the general solution to $\frac{dy}{dx}=\frac{b}{2a}$ and this will eliminate $\hat{u}_{\xi\xi}$. We cannot also eliminate $\hat{u}_{\eta\eta}$ since ξ and η can't be the same, but the $\hat{u}_{\xi\eta}$ term DOES vanish!

If we choose $\xi = \phi$ as above so that $\frac{\xi_x}{\xi_y} = -\frac{b}{2a}$ then

$$\begin{aligned} B &= 2a\xi_x\eta_x + b\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2c\xi_y\eta_y \\ &= \underbrace{\left(2a\xi_x + b\xi_y\right)}_{=0}\eta_x + \underbrace{\left(b\xi_x + 2c\xi_y\right)}_{=\left(b\xi_x + \frac{b^2}{2a}\xi_y\right)=0}\eta_y \end{aligned}$$

This leaves us with

$$\hat{u}_{\eta\eta} + \Phi(\xi, \eta, \hat{u}, \hat{u}_\xi, \hat{u}_\eta) = 0$$

which call the **canonical form of a parabolic equation**.

(3) Elliptic Equations: If $b^2 - 4ac < 0$ then we cannot eliminate $\hat{u}_{\xi\xi}$ or $\hat{u}_{\eta\eta}$. It is still possible to reduce these equations to the form

$$\hat{u}_{\eta\eta} + \hat{u}_{\xi\xi} + \Phi(\xi, \eta, \hat{u}, \hat{u}_\xi, \hat{u}_\eta) = 0$$

Remark 5.2. The wave equation $u_{tt} = \alpha^2 u_{xx}$ is a hyperbolic, the heat equation $u_t = \gamma u_{xx}$ is parabolic, and Laplace's equation $u_{xx} + u_{yy} = 0$ is elliptic. The names simply reflect similarities in the forms of the equations:

(1) $u_{xy} = 1$ is hyperbolic, since the equation $xy = 1$ is a hyperbola

(2) $u_{xx} + 2u_{yy} = 1$ is elliptic since $x^2 + 2y^2 = 1$ is an ellipse

Note 2. We *may* be able to use ODE methods once the equation is put into canonical form.

Example 5.7. Solve $u_{xy} + u_y = xy$ with $u(x, 0) = 0$, $u(0, y) = \sin y$ and $x \geq 0, y \geq 0$. First, integrate (anti-differentiate) with respect to y to get

$$u_x + u = \frac{1}{2}xy^2 + f(x)$$

This is now equivalent to a 1st order ODE in x. The integrating factor is $I(x) = e^x$. Solving this gives us

$$\begin{aligned} \frac{d}{dx}(ue^x) = \frac{1}{2}\int xe^x y^2 dx + \int e^x f(x)dx + h(y) \quad &\Longrightarrow \quad u = e^{-x}\left[\frac{y^2}{2}\left(xe^x - e^x\right)\right] + e^{-x}\int e^x f(x)dx + e^{-x}h(y) \\ &\Longrightarrow \quad u(x,y) = \frac{1}{2}y^2(x-1) + g(x) + e^{-x}h(y) \end{aligned}$$

Now apply the boundary conditions:

$$\begin{aligned} (1)\ u(x,0) = 0 \quad &\Longrightarrow \quad 0 = g(x) + e^{-x}h(0) \\ (2)\ u(0,y) = \sin y \quad &\Longrightarrow \quad \sin y = -\frac{y^2}{2} + g(0) + h(y) \end{aligned}$$

From (1), we have $g(x) = -e^{-x}h(0)$ and from (2), we have $h(y) = \sin y + \frac{1}{2}y^2 - g(0)$. What's $g(0)$? We know $g(0) = -h(0)$. Now

$$\begin{aligned} u(x,y) &= \frac{y^2}{2}(x-1) - e^{-x}h(0) + e^{-x}\left[\sin y + \frac{y^2}{2} + h(0)\right] \\ &= \frac{y^2}{2}(x-1) - e^{-x}\sin y + \frac{1}{2}y^2 e^{-x} \end{aligned}$$

Example 5.8. (The Wave Equation) Consider $u_{tt} = \alpha^2 u_{xx}$ with $u(x, 0) = \gamma(x)$ and $u_t(x, 0) = 0$. In standard form, this is

$$\alpha^2 u_{xx} - u_{tt} = 0$$

To convert it to canonical form, we start by solving the equations

$$\frac{dt}{dx} = \frac{b^2 \pm \sqrt{b^2 - 4ac}}{2a} = \frac{\pm\sqrt{4\alpha^2}}{2\alpha^2} = \pm\frac{1}{\alpha}$$

That is, $\frac{dx}{dt} = \pm\alpha$ so $x = \pm\alpha t + C$ and we use $x \pm \alpha t$ as our new variables. So let $\xi = x + \alpha t$ and $\tau = x - \alpha t$. Then

$$u_t = \hat{u}_\xi \xi_t + \hat{u}_\tau \tau_t = \alpha\hat{u}_\xi - \alpha\hat{u}_\tau$$

We then get

$$\begin{aligned}
u_{tt} = \frac{\partial}{\partial t}(u_t) &= \frac{\partial}{\partial t}\left(\alpha \hat{u}_\xi - \alpha \hat{u}_\tau\right) \\
&= \alpha \frac{\partial}{\partial t}\left(\hat{u}_\xi\right) - \alpha \frac{\partial}{\partial t}\left(\hat{u}_\tau\right) \\
&= \alpha \left[\hat{u}_{\xi\xi}\xi_t + \hat{u}_{\xi\tau}\tau_t\right] - \alpha \left[\hat{u}_{\tau\xi}\xi_t + \hat{u}_{\tau\tau}\tau_t\right] \\
&= \alpha \left[\hat{u}_{\xi\xi}\left(\alpha\right) + \hat{u}_{\xi\tau}\left(-\alpha\right)\right] - \alpha \left[\hat{u}_{\tau\xi}\left(\alpha\right) + \hat{u}_{\tau\tau}\left(-\alpha\right)\right] \\
&= \alpha^2 \hat{u}_{\xi\xi} - 2\alpha^2 \hat{u}_{\xi\tau} + \alpha^2 \hat{u}_{\tau\tau}
\end{aligned}$$

Similarly, we obtain

$$u_{xx} = \hat{u}_{\xi\xi} + 2\hat{u}_{\xi\tau} + \hat{u}_{\tau\tau}$$

Thus, the original PDE becomes

$$\begin{aligned}
u_{tt} = \alpha^2 u_{xx} \quad &\Longrightarrow \quad \alpha^2 \hat{u}_{\xi\xi} - 2\alpha^2 \hat{u}_{\xi\tau} + \alpha^2 \hat{u}_{\tau\tau} = \alpha^2 \hat{u}_{\xi\xi} + 2\alpha^2 \hat{u}_{\xi\tau} + \alpha^2 \hat{u}_{\tau\tau} \\
&\Longrightarrow \quad 4\alpha^2 \hat{u}_{\xi\tau} = 0 \\
&\Longrightarrow \quad \hat{u}_{\xi\tau} = 0
\end{aligned}$$

This can be integrated to get the solution

$$\begin{aligned}
\hat{u}_\xi = f(\xi) \quad &\Longrightarrow \quad \hat{u} = \int f(\xi) d\xi + g(\tau) = F(\xi) + G(\tau) \\
&\Longrightarrow \quad u(x,t) = F(x + \alpha t) + G(x - \alpha t)
\end{aligned}$$

This is called **d'Alembert's solution**. If we apply our initial conditions,

$$\begin{aligned}
(1)\ u(x,0) = \gamma(x) \quad &\Longrightarrow \quad F(x) + G(x) = \gamma(x) \\
(2)\ u_t(x,0) \quad &\Longrightarrow \quad \alpha F'(x) - \alpha G'(x) = 0
\end{aligned}$$

Now both equations (1), (2), can be rearranged and modified to get

$$\begin{aligned}
\begin{cases} F'(x) + G'(x) = \gamma'(x) & (3) \\ F'(x) = G'(x) & (4) \end{cases} \quad &\Longrightarrow \quad F'(x) = G'(x) = \frac{1}{2}\gamma'(x) \\
&\Longrightarrow \quad F(x) = \frac{1}{2}\gamma(x) + c_1 = \frac{1}{2}\gamma(x) + c_2
\end{aligned}$$

where $c_1 + c_2 = 0$ from (3). Therefore our solutions is

$$u(x,t) = \frac{1}{2}\left[\gamma(x + \alpha t) + \gamma(x - \alpha t)\right]$$

The *interpretation* is done through an example. Suppose that

$$\gamma(x) = \begin{cases} 2 - x & x \geq 0 \\ 2 + x & x < 0 \end{cases}$$

At a later time, say $t = 2/\alpha$, we have

$$u(x,t) = \frac{1}{2}\left[\gamma(x+2) + \gamma(x-2)\right]$$

So copies of the triangle $\gamma(x)$ are traveling in opposite directions along the x-axis where α is a parameter for the speed of travel.

5.4 Separation of Variables

Here's the most commonly used approach: we *assume* that $u(x,t)$ can be expressed as $F(x)G(t)$.

Example 5.9. (Heat Equation) Solve $u_t = \gamma u_{xx}$ for $\gamma > 0, u(x,0) = 20\sin 3\pi x$, and $u(0,t) = 0, u(L,t) = 0$. we assume that $u(x,t) = F(x)G(t)$. Then,

$$\begin{aligned} u_t &= F(x)G'(t) \\ u_{xx} &= F''(x)G(t) \end{aligned}$$

Since $u_t = \gamma u_{xx}$, then

$$F(x)G'(t) = \gamma F''(x)G(t) \quad \implies \quad \frac{G'(t)}{\gamma G(t)} = \frac{F''(x)}{F(x)}$$

Because this equation holds for all x and all t, then both expressions must be equal to a constant c_0. We thus obtain two ODEs

$$\begin{cases} G'(t) &= c_0 \cdot \gamma G(t) \\ F''(x) &= c_0 \cdot F(x) \end{cases}$$

Starting with the second one, and adding side conditions,

$$\begin{aligned} u(0,t) = 0 &\implies F(0)G(t) = 0 \implies F(0) = 0 \\ u(L,t) = 0 &\implies F(L)G(t) = 0 \implies F(L) = 0 \end{aligned}$$

So we then have a boundary problem for F where

$$F'' - c_0 \cdot F = 0, F(0) = 0, F(L) = 0$$

It is easy to verify that there are no non-trivial solutions if $c_0 \geq 0$. So assume $c_0 < 0$ and let $c_0 = -\lambda^2$ where $F'' + \lambda^2 F = 0$. The general solution is

$$F(x) = c_1 \cos \lambda x + c_2 \sin \lambda x$$

Since $F(0) = 0 \implies c_1$ and $F(L) = 0 \implies c_2 \sin \lambda L = 0$ then

$$\lambda L = n\pi \implies \lambda = \frac{n\pi}{L}, n \in \mathbb{Z}$$

So non-trivial solutions exist only if $\lambda = n\pi/L$ and those solutions are

$$F_n(x) = c_n \sin\left(\frac{n\pi}{L}\right)$$

Now return to $G(t)$. We have

$$\begin{aligned} G'(t) &= c_0 \cdot \gamma G(t) \\ &= -\lambda^2 \gamma G(t) \\ &= -\frac{n^2\pi^2}{L^2}\gamma G(t) \end{aligned}$$

So

$$G_n(t) = A_n e^{-\frac{n^2\pi^2}{L^2}\gamma t}$$

and combining our results gives us

$$u_n(x,t) = F_n(x)G_n(t) = B_n e^{-\frac{n^2\pi}{L^2}\gamma t} \sin\left(\frac{n\pi}{L}x\right)$$

By the superposition principle, any linear combination of these is a solution. So

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\frac{n^2\pi}{L^2}\gamma t} \sin\left(\frac{n\pi}{L}x\right)$$

Now apply the IC $u(x,0) = 20\sin\left(\frac{3\pi x}{L}\right)$. This gives us $n = 3, B_3 = 20$ and all other $B_n's$ are 0. So

$$u(x,t) = 20e^{-\frac{9\pi^2}{L^2}}\sin\left(\frac{3\pi x}{L}\right)$$

5.5 The Fourier Transform

This is a technique designed for problems with infinite spatial domains. For example, the heat equation for an infinitely long bar $u_t = \gamma u_{xx}$ and $u(x,0) = f(x)$ where $x \in \mathbb{R}$.

Definition 5.1. The **Fourier Transform** of a function $f : \mathbb{R} \mapsto \mathbb{R}$ is

$$\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

provided that the integral converges. We'll all use the notation $\hat{f}(\omega)$ to denote $\mathcal{F}\{f(x)\}$. This converts a real-valued function of x into a new complex-valued function of ω. We'll need **Euler's Formula**:

$$e^{x+iy} = e^x(\cos y + i\sin y)$$

Example 5.10. Find $\mathcal{F}\{f(x)\}$ if

$$f(x) = \begin{cases} 0 & x < 0 \\ e^{-ax} & x \geq 0 \end{cases}$$

We get

$$\begin{aligned}
\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx &= \int_0^{\infty} e^{-(a+i\omega)x}dx \\
&= \lim_{t\to\infty}\int_0^t e^{-(a+iw)x}dx \\
&= \lim_{t\to\infty} \left.\frac{e^{-(a+i\omega)x}}{-(a+i\omega)}\right|_0^t \\
&= \frac{e^{-(a+i\omega)x}}{-(a+i\omega)} + \frac{1}{a+i\omega}
\end{aligned}$$

Now,

$$\left|\frac{-e^{-(a+i\omega)}}{a+i\omega}\right| = \frac{e^{-at}}{a+i\omega}|e^{-i\omega t}| = \frac{e^{-at}}{a+i\omega} \overset{t\to\infty}{\to} 0$$

since $e^{-at} \to 0$ as $t \to \infty$. So by the Squeeze Theorem,

$$\hat{f}(\omega) = \underbrace{\frac{e^{-(a+i\omega)x}}{-(a+i\omega)}}_{\to 0} + \frac{1}{a+i\omega} = \frac{1}{a+i\omega}$$

Remark 5.3. The Fourier transform (FT) is just one of many integral transforms

$$\hat{f}(\omega) = \int_{\alpha}^{\beta} f(x)K(\omega,x)dx$$

will always transform functions of x into functions of ω. Note that the FT does not always exist!

Example 5.11. If $f(x) = 1$, then $\mathcal{F}\{f(x)\} = \mathcal{F}\{1\} = \int_{-\infty}^{\infty} e^{-i\omega x}dx$ which does not converge.

We'll normally consider only functions which satisfy the following conditions.

Theorem 5.1. *Let* $f : \mathbb{R} \mapsto \mathbb{R}$*. If*

(1) f is piecewise continuous on $\mathbb{R}$ AND

(2) f is absolutely integrable on $\mathbb{R}$

Then $\mathcal{F}\{f(x)\}$ exists.

Proof. Note that

$$\int_{-\infty}^{\infty} \left|f(x)e^{-i\omega x}\right| dx = \int_{-\infty}^{\infty} |f(x)| dx$$

So if f is absolutely integrable, then the 1st integral converges. Since f is piecewise continuous, the Triangle Inequality applies:

$$\left|\int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx\right| \leq \int_{-\infty}^{\infty} \left|f(x)e^{-i\omega x}\right| dx$$

Hence $\mathcal{F}\{f(x)\}$ converges as well. □

Note that in order for f to be absolutely integrable, we must have $f(x) \to 0$ as $x \to \pm\infty$. Therefore we have no reason to expect that functions such as $x, \sin x, e^x$, should have transforms.

Theorem 5.2. *(Linearity) Let $f, g : \mathbb{R} \mapsto \mathbb{R}$ where $\alpha, \beta \in \mathbb{R}$. If $\hat{f}$ and $\hat{g}$ exist, then*

$$\mathcal{F}\{\alpha f + \beta g\} = \alpha \hat{f} + \beta \hat{g}$$

Proof. We have

$$\begin{aligned}
\mathcal{F}\{\alpha f(x) + \beta g(x)\} &= \int_{-\infty}^{\infty} [\alpha f + \beta g] \, e^{-i\omega x} dx \\
&= \alpha \int_{-\infty}^{\infty} f e^{-i\omega x} dx + \beta \int_{-\infty}^{\infty} g e^{-i\omega x} dx \\
&= \alpha \hat{f} + \beta \hat{g}
\end{aligned}$$

□

Theorem 5.3. *(Transformation of Derivatives) Let f be differentiable on $\mathbb{R}$ and let $\mathcal{F}\{f(x)\} = \hat{f}(\omega)$. Then $\mathcal{F}\{f'(x)\}$ exists and in fact $\mathcal{F}\{f'(x)\} = i\omega \hat{f}(\omega)$.*

Proof. We have

$$\mathcal{F}\{f'(x)\} = \int_{-\infty}^{\infty} f'(x)e^{-i\omega x} dx = f(x)e^{-i\omega x}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} i\omega f(x)e^{-i\omega x} dx$$

If $\hat{f}(x)$ exists, then $f(x)e^{-i\omega x}\Big|_{-\infty}^{\infty} = 0$ so

$$\mathcal{F}\{f'(x)\} = \int_{-\infty}^{\infty} f'(x)e^{-i\omega x} dx = \int_{-\infty}^{\infty} i\omega f(x)e^{-i\omega x} dx = i\omega \hat{f}(\omega)$$

□

Corollary 5.1. *We can generalize this result, as well:*

$$\mathcal{F}\{f^{(n)}(x)\} = (i\omega)^n \hat{f}(\omega)$$

and also

$$\mathcal{F}\left\{\int_a^x f(x) dt\right\} = \frac{1}{i\omega} \hat{f}(\omega)$$

In theory, these results should allow us to transform ODEs to algebraic equations (they won't).

Example 5.12. Consider

$$y'' + ay' + by = f(x)$$

Applying the FT to both sides gives us

$$\begin{aligned}\mathcal{F}\{y'' + ay' + by\} = \mathcal{F}\{f(x)\} \quad &\implies \quad (i\omega)^2\hat{y} + a(i\omega)\hat{y} + b\hat{y} = \hat{f}(\omega) \\ &\implies \quad \hat{y} = -\frac{\hat{f}(\omega)}{\omega^2 - ia\omega - b} \\ &\implies \quad y(x) = \mathcal{F}\left\{\frac{-f(\omega)}{\omega^2 - ia\omega - b}\right\}\end{aligned}$$

The only problem is that $\hat{y}(\omega)$ won't usually exist. However, the same procedure can also convert some PDEs to ODEs.

Theorem 5.4. *(The Shifting Property) Let $f(x)$ be continuous and absolutely integrable on $\mathbb{R}$ with $\mathcal{F}\{f(x)\} = \hat{f}(x)$. Then $\mathcal{F}\{f(x-a)\} = \hat{f}(x)e^{-i\omega a}$*

Proof. We have

$$\mathcal{F}\{f(x-a)\} = \int_{-\infty}^{\infty} f(x-a)e^{-i\omega x}dx = \int_{-\infty}^{\infty} f(t)e^{-i\omega(t+a)}dt = e^{-i\omega a}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \hat{f}(x)e^{-i\omega a}$$

□

Corollary 5.2. *It is clear then, that $\mathcal{F}^{-1}\{e^{-i\omega a}\hat{f}(\omega)\} = f(x-a)$.*

Example 5.13. Consider the "unidirectional wave equation"

$$u_t + \alpha u_x = 0$$

The strategy is to eliminate the x derivative using a transform. To do so, we define

$$\mathcal{F}\{u(x,t)\} = \hat{u}(\omega, t) = \int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}dx$$

with this,

$$\mathcal{F}\{\alpha u_x\} = \alpha\mathcal{F}\{u_x\} = \alpha(i\omega)\hat{u}$$

Meanwhile

$$\mathcal{F}\{u_t\} = \int_{-\infty}^{\infty} \frac{\partial u}{\partial x}e^{-i\omega x}dx = \frac{\partial}{\partial x}\int_{-\infty}^{\infty} ue^{-i\omega x}dx = \hat{u}_t$$

Thus the original PDE becomes

$$\hat{u}_t + i\alpha\omega\hat{u} = 0$$

If we view ω as fixed, this in an ODE for $\hat{u}(t)$ where the general solution is

$$\hat{u} = \hat{G}(\omega)e^{-i\alpha\omega t}$$

Finally,

$$u(x,t) = \mathcal{F}^{-1}\{\hat{u}(\omega, t)\} = \mathcal{F}^{-1}\left\{\hat{G}(\omega)e^{-i\alpha\omega t}\right\} = G(x - \alpha t)$$

This is the same solution as we found by our previous/old method. Note that it is valid even when $\hat{G}$ does not exist.

Definition 5.2. The **convolution** of two functions f and g is

$$f * g(x) = \int_{-\infty}^{\infty} f(x-\tau)g(\tau)d\tau$$

This will exist if f and g are causal functions (0 for $x < 0$).

Example 5.14. Consider

$$f(x) = \begin{cases} 0 & x < 0 \\ x & x \geq 0 \end{cases}, g(x) = \begin{cases} 0 & x < 0 \\ e^{-x} & x \geq 0 \end{cases}$$

The convolution is

$$\begin{aligned} f * g(x) = \int_0^\infty f(x-\tau)e^{-\tau}d\tau &= \int_0^x (x-\tau)e^{-\tau}d\tau \\ &= -(x-\tau)e^{-\tau}\Big|_0^x - \int_0^x e^{-\tau}d\tau \\ &= 0 + x + e^{-\tau}\Big|_0^x \\ &= x + e^{-x} - 1 \end{aligned}$$

Like multiplication, convolution is commutative and distributes over addition. That is

- $f * g = g * f$
- $(\alpha f + \beta g) * h = \alpha(f * h) + \beta(g * h)$

Theorem 5.5. *(The Convolution Theorem) We have*

$$\mathcal{F}\{f * g\} = \hat{f}(\omega)\hat{g}(\omega) \implies \mathcal{F}^{-1}\{\hat{f}(\omega)\hat{g}(\omega)\} = f(x) * g(x)$$

Proof. By definition,

$$\begin{aligned} \mathcal{F}\{f * g\} &= \int_{-\infty}^{\infty} (f * g)(x)e^{-i\omega x}dx \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x-\tau)e^{-i\omega x}dx\right] g(\tau)d\tau \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\xi)e^{-i\omega(\xi+\tau)}d\xi\right] g(\tau)d\tau, \xi = x - \tau \implies d\xi = ddx \\ &= \int_{-\infty}^{\infty} f(\xi)e^{-i\omega\xi}d\xi \int_{-\infty}^{\infty} g(\tau)e^{-i\omega\tau}d\tau \\ &= \hat{f}(\omega)\hat{g}(\omega) \end{aligned}$$

□

Example 5.15. (Solution of the Heat Equation for an Infinitely Long Bar) Given that

$$u_t = \gamma u_{xx}, u(x,0) = f(x), u(x,t) \to 0 \text{ as } x \to \pm\infty$$

We'll use the FT method. Let $\hat{u}(\omega,t) = \mathcal{F}\{u(x,t)\}$. Then $u_t \mapsto \hat{u}_t$ and $u_{xx} \mapsto (i\omega)^2\hat{u} = -\omega^2\hat{u}$. Also, $u(x,0) = f(x) \mapsto \hat{u}(\omega,0) = \hat{f}(\omega)$. The BVP then becomes

$$\hat{u}_t = -\gamma\omega^2\hat{u}, \hat{u}(\omega,0) = \hat{f}(\omega)$$

The conditions $u(x,t) \to 0$ as $x \to \pm\infty$ have already been used in assuming that $\hat{u}$ exists. Solving this IVP gives us

$$\begin{aligned} \hat{u}(\omega,t) = H(\omega)e^{-\gamma\omega^2 t}, H(\omega) = \hat{f}(\omega) &\implies \hat{u}(\omega,t) = \hat{f}(\omega)e^{-\gamma\omega^2 t} \\ &\implies u(x,t) = \mathcal{F}^{-1}\{\hat{f}(\omega)e^{-\gamma\omega^2 t}\} = f(x) * \mathcal{F}^{-1}\{e^{-\gamma\omega^2 t}\} \end{aligned}$$

We still have to calculate

$$\begin{aligned} \mathcal{F}^{-1}\{e^{-\gamma\omega^2 t}\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\gamma\omega^2 t}e^{i\omega x}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(\gamma\omega^2 t - i\omega x)}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\gamma t\left(\left[\omega - \frac{ix\omega}{2\gamma t}\right]^2 + \frac{x^2}{4\gamma^2 t^2}\right)}d\omega \\ &= \frac{1}{2\pi}e^{-\frac{x^2}{4\gamma t}}\int_{-\infty}^{\infty} e^{-\gamma t\left[\omega - \frac{ix}{2\gamma t}\right]^2}d\omega \end{aligned}$$

Now let $v = \sqrt{\gamma t}\left(\omega - \frac{ix}{2\gamma t}\right) \implies dv = \sqrt{\gamma t}d\omega$. We get that

$$\mathcal{F}^{-1}\{e^{-\gamma\omega^2 t}\} = \frac{1}{2\pi}e^{-\frac{x^2}{4\gamma t}} \underbrace{\int_{-\infty}^{\infty} e^{-v^2}\frac{dv}{\sqrt{\gamma t}}}_{=\sqrt{\pi}/\sqrt{\gamma t}} = \frac{1}{2\sqrt{\pi\gamma t}}e^{-\frac{x^2}{4\gamma t}}$$

Call this $g(x) = \mathcal{F}^{-1}\{e^{-\gamma\omega^2 t}\}$. So

$$u(x,t) = f(x) * g(x) \quad = \quad \frac{1}{2\sqrt{\pi\gamma t}}\int_{-\infty}^{\infty} f(x-\tau)e^{-\frac{\tau^2}{4\gamma t}}d\tau$$

Let $y = -\frac{\tau}{2\sqrt{\gamma t}} \implies dy = -\frac{d\tau}{2\sqrt{\gamma t}}$ with $y \to \pm\infty \implies \tau \to \mp\infty$. So

$$\begin{aligned} u(x,y) &= \frac{1}{2\sqrt{\pi\gamma t}}\int_{\infty}^{-\infty} f(x+2y\sqrt{\gamma t})e^{-y^2}\left(-2\sqrt{\gamma t}dy\right) \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x+2y\sqrt{\gamma t})e^{-y^2}dy \end{aligned}$$

We can use this as a general solution for the heat equation for an infinite bar.

Example 5.16. Suppose that $f(x) = \chi_{[-1,1]}$. Then

$$f(x+2y\sqrt{\gamma t}) = \begin{cases} 1 & y \in \left(-\frac{1-x}{2\sqrt{\gamma t}}, \frac{1-x}{2\sqrt{\gamma t}}\right) \\ 0 & \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned} u(x,y) &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x+2y\sqrt{\gamma t})e^{-y^2}dy \\ &= \frac{1}{\sqrt{\pi}}\int_{-\frac{1-x}{2\sqrt{\gamma t}}}^{\frac{1-x}{2\sqrt{\gamma t}}} e^{-y^2}dy \\ &= \frac{1}{2}\left[\text{erf}\left(\frac{1-x}{2\sqrt{\gamma t}}\right) - \text{erf}\left(-\frac{1-x}{2\sqrt{\gamma t}}\right)\right] \end{aligned}$$

where $\text{erf}(x) = \frac{2}{\pi}\int_0^x e^{-t^2}dt$.

6 Stochastic Differential Equations (SDEs)

In this section, we only discuss one particular SDE, the **Black-Scholes Equation**.

6.1 The Black-Scholes Equation

Definition 6.1. A **European Call Option** is the right to purchase a commodity for an agreed upon price K at an agreed upon time T. We'll call K the **strike price** and T the **strike time**.

The value of an option depends on the price of the commodity S and on time t. More naturally, it depends on the time remaining $T - t$. We'll denote the value of our option as $F(S,t)$. We'll need several simplifying assumptions:

1. We'll assume that trading is continuous
2. We'll assume that assets are infinitesimally divisible
3. We'll ignore transaction costs

4. We'll permit short-selling
5. We'll assume that our assets pay no dividends

The stock price depends on time as well where $S = S(t)$ and we assume that the change in S contains both a deterministic and random component, and that its behaviour is Malthusian. That is, if we write

$$\begin{aligned} \Delta S &= S(t + \Delta t) - S(t) \\ &= f_{det}(S, t, \Delta t) + f_{rand}(S, t, \Delta t) \end{aligned}$$

The Malthusian assumption gives us $f_{det} = \mu S \Delta t$. For the random contribution, we'll assume that $f_{rand} = \sigma S \Delta W(t)$ where $\Delta W(t)$ is a random variable whose probability density function depends on time. We'll assume that it has mean 0 and variance Δt, where we call such a process a **Wiener process**. We then have

$$\Delta S \approx S(\mu \Delta t + \sigma \Delta W(t))$$

We call μ the **growth rate** and call σ the **volatility**. Letting $\Delta t \to 0$ the in this also goes to 0 and so it is customary to write

$$dS = \mu S\ dt + \sigma S\ dW(t)$$

which we will call assumption 6. One useful fact is that $[W(t)]^2$ has mean Δt and variance $2(\Delta t)^2$. To incorporate our assumptions about $S(t)$ into F, recall that a function of 2 variables $f(x, y)$ can be expanded in a Taylor series

$$f(x, y) = f(x_0, y_0) + \nabla f(x_0, y_0)(\mathbf{x} - x_0) + \frac{1}{2!}(\mathbf{x} - x_0)^T \nabla^2 f(x_0, y_0)(\mathbf{x} - x_0)$$

If $\Delta \mathbf{x} = (\mathbf{x} - x_0)$, and $\Delta f = f(x, y) - f(x_0, y_0)$, then this equation reduces to

$$\Delta f = [\nabla f(x_0, y_0)] [\Delta \mathbf{x}] + \frac{1}{2} [\Delta \mathbf{x}]^T \left[\nabla^2 f(x_0, y_0)\right] [\Delta \mathbf{x}]$$

So we can expand $F(S, t)$ about an arbitrary point (S_0, t_0) and if $\Delta S = \mu S \Delta t + \sigma S \Delta W(t)$ get

$$\begin{aligned} \Delta F &= F_S [\mu S \Delta t + \sigma S \Delta W(t)] + F_t \Delta t + \frac{1}{2} F_{SS} [\mu S \Delta t + \sigma S \Delta W(t)]^2 + F_{St} [\mu S \Delta t + \sigma S \Delta W(t)] \Delta t \\ &= F_S \mu S \Delta t + F_S \sigma S \Delta W(t) + F_t \Delta t + \frac{1}{2} F_{SS} \mu^2 S^2 \Delta t^2 + F_{SS} \mu \sigma S^2 \Delta t \Delta W(t) + \\ &\quad + \frac{1}{2} F_{SS} \sigma^2 S^2 [\Delta W(t)]^2 + F_{St} \mu S (\Delta t)^2 + F_{St} \sigma S \Delta W(t) \Delta t + \frac{1}{2} F_{tt} (\Delta t)^2 + ... \end{aligned}$$

Now if Δt is small, then we can neglect terms order $(\Delta t)^2$. Furthermore, since $\Delta W(t)$ has mean 0 and variance Δt, we can justifiably neglect terms of order $\Delta t \Delta W(t)$. For the $[\Delta W(t)]^2$ term, since it has mean Δt and a very small variance, $2(\Delta t)^2$ we'll approximate it as Δt. This leaves

$$\Delta F = F_S \mu S \Delta t + F_S \sigma S \Delta W(t) + F_t \Delta t + \frac{1}{2} F_{SS} \sigma^2 S^2 \Delta t$$

or, as a stochastic DE,

$$dF = \sigma S F_S dW(t) + \left(F_t + \mu S F_S + \frac{1}{2} \sigma^2 S^2 F_{SS}\right) dt$$

To eliminate the stochastic term, we make two more assumptions:

1. Assume that we can always invest at a constant interest rate r
2. Arbitrage is almost immediately eliminated, that is no arbitrage is possible
   (a) This also assumes that the free market will automatically and instantaneously price financial instruments fairly

This allows a clever trick: Consider a portfolio constructed by buying one option and selling ϵ units of stock. This portfolio has value $\Pi(t) = F - \epsilon S$. This change in value over time Δt is

$$\begin{aligned}
\Delta\Pi &= \Delta F - \epsilon \Delta S \\
&\approx \sigma S F_S dW(t) + \left(F_t + \mu S F_S + \frac{1}{2}\sigma^2 S^2 F_{SS}\right) dt - \epsilon\left[\mu S \Delta t + \sigma S \Delta W(t)\right] \\
&= \left[\sigma S F_S - \epsilon \sigma S\right] \Delta W(t) + \left[F_t + \mu S F_S + \frac{1}{2}\sigma^2 S^2 F_{SS} - \epsilon \mu S\right] \Delta t
\end{aligned}$$

Now if we set $\epsilon = F_S$ then the stochastic term disappears! We are then left with the deterministic expression

$$\begin{aligned}
\Delta\Pi = \left[F_t + \frac{1}{2}\sigma^2 S^2 F_{SS}\right]\Delta t \quad &\Longrightarrow \quad \frac{d\Pi}{dt} = \left[F_t + \frac{1}{2}\sigma^2 S^2 F_{SS}\right] \\
&\Longrightarrow \quad r\Pi = F_t + \frac{1}{2}\sigma^2 S^2 F_{SS}
\end{aligned}$$

since if arbitrage cannot exists, then the value of Π must match the value of an invest at interest rate r. Substituting in Π gives us

$$r(F - F_S S) = F_t + \frac{1}{2}\sigma^2 S^2 F_{SS} \implies \frac{1}{2}\sigma^2 S^2 F_{SS} + rSF_S + F_t - rF = 0$$

which is the Black-Scholes equation for a European call option.

What is the domain of $F(S,t)$? We must have $S \geq 0$ and we want to consider $t \in [0, T]$ (time for purchase until strike time). What do we know about the values of F on these boundaries? Consider time T:

- If $S(T) > K$, then the owner of the option will exercise it (purchasing the stock at strike K) for a profit of $S - K$.
- If $S(T) < K$ then the owner will let it expire and it will become worthless.

Therefore, $F(S,T) = \max(S - K, 0) = (S - K)^+$. This is our initial condition. Secondly, observe that since $dS = \mu S\ dt + \sigma S\ dW(t)$, if S is ever 0, ten it will remain 0. Therefore, the value of F will also be $F(0,t) = 0$. On the other hand, if the value of the option becomes very large, then we be certain that we will exercise the option. That is, $\lim_{S\to\infty} F(S,t) = S - K \approx S$. We simplified this in the sense that if S is very large, then K is negligible.

So our IVP is

$$\begin{aligned}
F_t &= -\frac{1}{2}\sigma^2 S^2 F_{SS} - rSF_S + rF \\
F(S,T) &= (S-K)^+ \\
F(0,t) &= 0 \\
F \to S \quad &\text{as} \quad S \to \infty
\end{aligned}$$

There are several ways to simplify this:

- We could normalize the coefficient of $S^2 F_{SS}$ by letting $\tau = \frac{1}{2}\sigma^2 t$. This makes

  $$F_t = \frac{\partial F}{\partial t} = \frac{\partial F}{\partial\left(\frac{2}{\sigma^2}\tau\right)} = \frac{\sigma^2}{2} \cdot \frac{\partial F}{\partial \tau} = \frac{1}{2}\sigma^2 F_\tau$$

  which gives

  $$F_\tau = -S^2 F_{SS} + ... + ...$$

- We could instead reverse time by letting $\tau = T - t$ which turns our final condition into a true initial condition.
- We do both at once and let $\tau = \frac{1}{2}\sigma^2(T - t)$ to get

  $$F_\tau = S^2 F_{SS} + ... + ...$$

- Now note that K appears only in the condition $F(S,T) = (S-K)^+$. We could divide by K to get

  $$\frac{F(S,T)}{K} = \left(\frac{S}{K} - 1\right)^+$$

Replacing F with $\frac{F}{K}$ and S with $\frac{S}{K}$, we'll eliminate K from the problem.

- Lastly, recall that the equation
$$ax^2y'' + bxy' + cy = 0$$
can be converted to a constant coefficient equation by letting $x = e^t$. We can use the same strategy here: let $S = e^x$.

- We can combine the two ideas above as well by setting
$$\frac{S}{K} = e^x$$

- We will be doing everything at once to get the following substitutions:
$$x = \ln\left(\frac{S}{K}\right), \tau = \frac{1}{2}\sigma^2(T-t), v = \frac{F}{K}$$
or reversely,
$$S = Ke^x, t = T - \frac{2}{\sigma^2}\tau, F = Kv$$
This gives
$$F_t = \frac{\partial F}{\partial t} = \frac{\partial (Kv)}{\partial \tau} \cdot \frac{d\tau}{dt} = \frac{-K\sigma^2}{2}v_\tau$$
$$F_S = \frac{\partial F}{\partial S} = \frac{\partial (Kv)}{\partial x} \cdot \frac{dx}{dS} = \frac{K}{S}v_x$$
$$F_{SS} = \frac{\partial}{\partial S}\left(\frac{K}{S}v_x\right) = -\frac{K}{S^2}v_x + \frac{K}{S}\frac{\partial}{\partial x}(v_x)\frac{dx}{dS} = -\frac{K}{S^2}v_x + \frac{K}{S^2}v_{xx} = \frac{K}{S}\left(v_{xx} - v_x\right)$$

So the PDE $F_t = -\frac{1}{2}\sigma^2S^2F_{SS} - rSF_S + rF$ becomes
$$-\frac{K\sigma^2}{2}v_\tau = -\frac{\sigma^2}{2}S^2\left(\frac{K}{S^2}\left(v_{xx} - v_x\right)\right) - rS\left(\frac{K}{S}v_x\right) + rKv \implies v_\tau = v_{xx} + \left(\frac{r - \sigma^2/2}{\sigma^2/2}\right)v_x - \frac{r}{\sigma^2/2}v$$
$$\implies v_\tau = v_{xx} + \left(\frac{2r}{\sigma^2} - 1\right)v_x - \frac{2r}{\sigma^2}v$$
$$\implies v_\tau = v_{xx} + (\delta - 1)v_x - \delta v, \delta = \frac{2r}{\sigma^2}$$

What happens to the boundary conditions? If $F = (S-K)^+$ when $t = T$ then $v = (e^x - 1)^+$ when $\tau = 0$. If $F = 0$ when $S = 0$, then $v \to 0$ as $x \to -\infty$. If $F \to S$ as $S \to \infty$, then $v \to e^x$ as $x \to \infty$. So we have
$$\begin{aligned} v_\tau &= v_{xx} + (\delta-1)v_x - \delta v \\ v(x,0) &= (e^x - 1)^+ \\ \lim_{x\to\infty} v &= e^x \\ \lim_{x\to-\infty} v &= 0 \end{aligned}$$

From a past assignment, this PDE can be converted into the heat equation $u_\tau = u_{xx}$ by letting $v = \exp\left[\frac{1}{2}(1-\delta)x - \frac{1}{4}\left(\delta+1\right)^2\tau\right]u$. that is, $v = e^{\alpha x - \beta^2\tau}$ where $\alpha = \frac{1}{2}(1-\delta)$ and $\beta = \frac{1}{2}(\delta+1)$. The initial conditions also become $u(x,0) = \left(e^{\beta x} - e^{-\alpha x}\right)^+$. The new boundary conditions are complicated and difficult to interpret. We'll ignore them, though, because our next is as follows.

We already know that a solution to the problem $u_\tau = \gamma u_{xx}, u(x,0) = f(x), x \in (-\infty, \infty)$ is
$$u(x,\tau) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-y^2}f(x + 2y\sqrt{\gamma\tau})dy$$

So set $\gamma = 1$ and plug in our $f(x)$ to get

$$u(x,\tau) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} \left(e^{\beta(x+2y\sqrt{\tau})} - e^{-\alpha(x+2y\sqrt{\tau})} \right)^+ dy$$

Now observe that

$$\begin{aligned} e^{\beta(x+2y\sqrt{\tau})} - e^{-\alpha(x+2y\sqrt{\tau})} > 0 &\iff \beta(x+2y\sqrt{\tau}) > -\alpha(x+2y\sqrt{\tau}) \\ &\iff (\alpha+\beta)(x+2y\sqrt{\tau}) \\ &\implies x + 2y\sqrt{\tau} > 0 \\ &\implies y > \frac{-x}{2\sqrt{\tau}} \end{aligned}$$

since $\alpha + \beta = 1$. Therefore,

$$u(x,\tau) = \frac{1}{\sqrt{\pi}} e^{\beta x} \int_{-\frac{x}{2\sqrt{\tau}}}^{\infty} e^{2\beta y\sqrt{\tau} - y^2} dy - \frac{1}{\sqrt{\pi}} e^{-\alpha x} \int_{-\frac{x}{2\sqrt{\tau}}}^{\infty} e^{-2\alpha y\sqrt{\tau} - y^2} dy$$

Computing the first integral gives us

$$\begin{aligned} I &= \frac{e^{\beta x + \beta^2 \tau}}{\sqrt{\pi}} \int_{-\frac{x}{2\sqrt{\tau}}}^{\infty} e^{-(y-\beta\sqrt{\tau})^2} dy \\ &= \frac{e^{\beta x + \beta^2 \tau}}{\sqrt{\pi}} \int_{\frac{x+2\beta\tau}{\sqrt{2\tau}}}^{-\infty} e^{-\frac{\omega^2}{2}} \left(-\frac{d\omega}{\sqrt{2}} \right) \\ &= e^{\beta x + \beta^2 \tau} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x+2\beta\tau}{\sqrt{2\tau}}} e^{-\frac{\omega^2}{2}} d\omega \\ &= e^{\beta x + \beta^2 \tau} \Phi\left(\frac{x+2\beta\tau}{\sqrt{2\tau}} \right) \end{aligned}$$

where $\omega = -\sqrt{2}(y - \beta\sqrt{\tau}) \implies d\omega = -\sqrt{2}dy$ and $y \to \infty \implies \omega \to -\infty$. Also, because if $y = -\frac{x}{2\sqrt{\tau}}$, then $\omega = \frac{x+2\beta\tau}{\sqrt{2\tau}}$. We can construct a similar solution for the second integral to get

$$u(x,\tau) = e^{\beta x + \beta^2 \tau} \Phi\left(\frac{x+2\beta\tau}{\sqrt{2\tau}} \right) - e^{-\alpha x + \alpha^2 \tau} \Phi\left(\frac{x-2\alpha\tau}{\sqrt{2\tau}} \right)$$

which is one form of the solution of the Black-Scholes PDE. The only step now is to replace the variables above with the original variables. This will not be done because it's WAY too tedious. The steps can be found in the course notes.

6.2 Verification of Solutions

Does $u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + 2y\sqrt{\gamma t}) e^{-y^2} dy$ satisfy the heat equation $u_t = \gamma u_{xx}, u(x,0) = f(x)$? The IC is easy to check:

$$u(x,0) = \frac{f(x)}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} dy = f(x)$$

Next, we check the PDE. First, we need to be able to differentiate u. If $f(x)$ is twice differentiable, then

$$\begin{aligned} u_t &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f'(x + 2y\sqrt{\gamma t}) \left(\frac{y\sqrt{\gamma}}{\sqrt{t}} \right) e^{-y^2} dy \\ &= \frac{\sqrt{\gamma}}{\sqrt{\pi t}} \int_{-\infty}^{\infty} f'(x + 2y\sqrt{\gamma t}) e^{-y^2} dy \end{aligned}$$

and

$$u_{xx} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f''(x + 2y\sqrt{\gamma t}) e^{-y^2} dy$$

Apply **integration by parts** by letting

$$u = e^{-y^2}, dv = f''(x + 2y\sqrt{\gamma t})dy \implies du = -2ye^{-y^2}dy, v = \frac{1}{2\sqrt{\gamma t}}f'(x + 2y\sqrt{\gamma t})$$

to get

$$u_{xx} = \frac{1}{\sqrt{\pi}}\left[\frac{e^{-y^2}}{2\sqrt{\gamma t}}f'(x + 2y\sqrt{\gamma t})\right]_{-\infty}^{\infty} + \underbrace{\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{y}{\sqrt{\gamma t}}f'(x + 2y\sqrt{\gamma t})e^{-y^2}dy}_{\frac{1}{\gamma}u_t}$$

Our solution works if the first term is zero. This is true as long as $\lim_{x\to\pm\infty} f(x) < O(e^{y^2})$.

Index