ACTSC 446 (Winter 2014 - 1141)

Mathematical Models in Finance

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These notes are currently a work in progress, and as such may be incomplete or contain errors.

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Abstract

The purpose of these notes is to provide the reader with a secondary reference to the material covered in ACTSC 446. The formal prerequisite to this course is ACTSC 371, ACTSC 231 and STAT 333 or STAT 334. Readers should have a good background in linear algebra, basic statistics, and calculus before enrolling in this course.

1 Review

(Read the course notes on this section for more detail. They should be completely comprehensive.)

In the binomial model, two future cash flows weighted by probabilities p and $1-p$ are discounted by some rate. This can be computed using models like the CAPM and APT. In general, the interest rate is in the form

$$\text{Risk-free rate } + \text{ Risk Premium}$$

where the former is the base rate and the later is more specific to the underlying asset. Consider now a call option with time to maturity of 1-yr and a strike of \$30. Find the call's price if it is worth \$10 with $p_1 = 0.6$ and 0 with $p_2 = 0.4$. Unfortunately, we do not have any easy methods to find the discount rate / desired return on the call option.

Note that the above method involves finding the expected cash flow first and then discounting it using an appropriate rate of return, one that is adjusted for risk!

1.1 Risk-Neutral Pricing

The idea with this method is that we use the risk neutral rates for discounting and probabilities that are adjusted for risk. These adjusted probabilities are called risk neutral probabilities.

To start and motivate the definition, assume that arbitrage is not possible. Let $S_0 = 30, S_u = 40, S_d = 25$ and $C_u = 10, C_d = 0$. The idea here is to:

1. Create a portfolio that mimics the payoffs of the call at time 1. No arbitrage means the price of the call is equal to the price of the portfolio.
2. Assume that we can invest/borrow at the risk free rate r
3. Let α be the # of shares of stock purchased at $t = 0$ and β be the amount invested at the risk free rate at $t = 0$
4. In our portfolio P we have $P_{u(d)} = \alpha S_{u(d)} + \beta(1+r) = C_{u(d)}$ where we try to find α and β such that this holds
5. Solving, we can get:

$$\begin{aligned} \alpha &= \frac{C_u - C_d}{S_u - S_d} \\ \beta &= \frac{1}{1+r}\left[C_u - \frac{C_u - C_d}{S_u - S_d} \cdot S_u\right] \end{aligned}$$

6. Hence, the call price is

$$\alpha S_0 + \beta = ... = \frac{1}{1+r}\left[\overbrace{\frac{(1+r)S_0 - S_d}{S_u - S_d}}^{p_1} \cdot C_u + \overbrace{\frac{S_u - (1+r)S_0}{S_u - S_d}}^{p_2} \cdot C_d\right]$$

7. If $S_d < (1+r)S_0 < S_u$, then $0 < p_1 < 1$. We call p_1 and p_2 risk-neutral probabilities and our discount rate is our risk-free rate of return r. So we have the price as an EPV under this risk neutral probability measure. Note that the initial condition is also equivalent to

$$\frac{S_d - S_0}{S_0} < r < \frac{S_u - S_0}{S_0}$$

1.2 Derivatives

With respect to (wrt) options (stocks), a **long position** is when you buy (shares of and have ownership of) the option (stock) and a **short position** is when you sell or write (borrow, sell it on the market, and promise to return, some time in the future) the option (stock from usually a broker). For stocks, if any **dividends** are paid during this time, the original owner must be paid these dividends.

Long positions are usually taken when you expect prices to rise in the future (bullish) and vice versa for short positions (bearish).

A **call** (**put**) gives the owner the right but not the obligation to buy (sell) an underlying asset before a pre-specified time (maturity or expiration date) for a a price (strike or exercise price) set today.

A **European** option can only be exercised at the expiration date. An **American** option can be exercised any time before expiration. A **Bermudan** option can be exercised at only certain specified dates. With European options, let S_T be the spot price at expiration and K be the strike and exercise price. The payoff of a long call is

$$\max(S_T - K, 0) = (S_T - K)^+$$

and respectively for a put, the payoff is

$$\max(K - S_T, 0) = (K - S_T)^+$$

The profit for these options is therefore

$$\text{Payoff - FV of Premium}$$

The written version of these options has the negation of the above equations as the payoffs while the profit is

$$\text{Payoff + FV of Premium}$$

So what are they used for?

1. Speculating on volatility
   (a) Straddle: Buy a put and a call with the same strike price
2. As a means of insurance
   (a) Floor: provides insurance against a fall in price; long a stock and long a put
   (b) Cap: provides insurance against a rise in price; short a stock and long a call

1.3 Forwards and Futures

A **forward contract** is an obligation to buy or sell some underlying asset at some point in the future. The buyer will pay the seller at the time of delivery of the asset. A **futures contract** can be thought of an exchange-traded forward contract. Some differences from forward contracts is the margin account is marked-to-market. Futures contracts do have maintenance and initial margins.

The payoff for a forward contract is

$$\text{Payoff} = \begin{cases} \text{Spot Price } (S_T)\text{- Forward Price } (K) & \text{long position} \\ \text{Forward Price - Spot Price} & \text{short position} \end{cases}$$

and since the initial premium is zero, the profit on a forward contract equals its payoff. For the rest of this section, we will simply assume that forwards and futures are equivalent. That is, the price of a futures contract is the same as that of a forward contract.

Example 1.1. Consider the purchase or sale of a stock. There are 4 possible ways to do this:

1. Outright purchase: pay S_0 for the stock today ad receive it at time 0
2. Fully leveraged purchase: borrow S_0 at a risk-free rate r and the purchase stock today; pay off the loan with interest at some time T, an amount equal to $S_0 e^{rT}$ (assuming that r is continuously compounded)
3. Prepaid forward contract: pay for the stock today and receive it at time T

4. Forward contract: pay for the stock and receive it at time T at a price set today

We will be interested in finding the price of (3) and (4). We will use the notation of r as the continuously compounded risk free rate, T is the expiration date, $F_{0,T}$ is the price of a $T-$year forward contract, $F_{0,T}^P$ is the price of a prepaid $T-$year forward contract, and α is the continuously compounded risk adjusted interest rate where the stock pays no dividends.

Prepaid Forward

- Pricing method (1) : Pays no dividends so it doesn't matter when the stock is delivered to the buyer (could be at time T, at any time $(0,T)$ or even at time 0. It doesn't matter) and hence the fair price $F_{0,T}^P = S_0$, the stock price at time 0.
- Pricing method (2) : Using expected present value, $F_{0,T}^P = e^{-\alpha T}E_0(S_T)$. Now what is $E_0(S_T)$? Think of α as the yield of the stock and get
$$E_0(S_T) = S_0 e^{\alpha T} \implies F_{0,T}^P = e^{-\alpha T}[S_0 e^{\alpha T}] = S_0$$
- Pricing method (3) : We price using arbitrage (risk free profit) [May be on the midterm]. To do this, we construct a portfolio by buying a share of stock at S_0 and selling a prepaid forward at $F_{0,T}^P$:

Cash Flows		
	Time 0	Time T
(i) Buy Stock	$-S_0$	S_T
(ii) Sell Prepaid Forward	$F_{0,T}^P$	$-S_T$
Net	$F_{0,T}^P - S_0$	0

and hence no arbitrage implies that $F_{0,T}^P - S_0 = 0$ so $F_{0,T}^P = S_0$.

Now let's suppose that dividends are involved. In this case,
$$F_{0,T}^P = S_0 - PV(\text{All dividends paid over } (0,T))$$
If the stock pays discrete dividends of D_{t_j} at times t_j for $j = 1, 2, ..., n$ where $t_j < T$. Then,
$$F_{0,T}^P = S_0 - \sum_{j=1}^{n} PV_{0,t_j}(D_{t_j})$$
If the stock pays dividends at an annualized continuously compounded dividend yield of δ. Then,
$$F_{0,T}^P = S_0 e^{-\delta T}$$
Note in the latter case, if the smallest increment is 1 day, then the number of shares of stock at time T is
$$\left(1 + \frac{\delta}{365}\right)^{365T} \approx e^{\delta T}$$
Also in the latter case, we can apply the following arbitrage argument. We create a portfolio of $e^{-\delta T}$ of stock (and continuously reinvest dividends back into the stock) and sell a prepaid forward at $F_{0,T}^P$:

Cash Flows		
	Time 0	Time T
(i) Buy Stock	$-S_0 e^{-\delta T}$	S_T
(ii) Sell Prepaid Forward	$F_{0,T}^P$	$-S_T$
Net	$F_{0,T}^P - S_0 e^{\delta T}$	0

and hence no arbitrage implies that $F_{0,T}^P - S_0e^{-\delta T} = 0$ so $F_{0,T}^P = S_0e^{-\delta T}$. In Example 1 of the notes ($100 stock with quarterly $1 dividends), the price is 96.2409 for the prepaid. In Example 2 ($100 stock with continuous dividends at 5%), the price is 95.1229 for the prepaid.

Forward Contract

Basic intuition tells us that $F_{0,T} = FV(F_{0,T}^P) = F_{0,T}^P e^{rT}$ and hence the cases are:

1. No dividends: $F_{0,T} = S_0e^{rT}$
2. Discrete dividends: $F_{0,T} = S_0e^{rT} - \sum_{i=1}^n e^{r(T-t_i)}D_{t_i}$
3. Continuous dividends: $F_{0,T} = S_0e^{-\delta T}e^{rT} = S_0e^{(r-\delta)T}$

We can also examine the cash flows of two forward contracts, with $T_1 < T_2$ and $F_{0,T_2} > F_{0,T_1}e^{r(T_2-T_1)}$, to show if there is an arbitrage opportunity:

Cash Flows			
	Time 0	Time T_1	Time T_2
(i) Short T_2-yr Forward	0	0	$F_{0,T_2} - S_{T_2}$
(ii) Long T_1-yr Forward	0	$S_{T_1} - F_{0,T_1}$	0
(iii) Buy 1 share of stock at time T_1	0	$-S_{T_1}$	S_{T_2}
(iv) Borrow $F_{0,T}$ @ r at time T_1	0	F_{0,T_1}	$F_{0,T_1}e^{(T_2-T_1)}$
Net	0	0	$F_{0,T_2} - F_{0,T_1}e^{r(T_2-T_1)} > 0$

Sometimes, you may be given the **forward premium** which is defined as the ratio of the current forward price to the current stock price (i.e. $F_{0,T}/S_0$). If you are given the forward premium and the forward price, you can figure out the current stock price. Sometimes, you may be given the annualized forward premium which is

$$\frac{1}{T}\ln\left(\frac{F_{0,T}}{S_0}\right)$$

To create a **synthetic forward**, you borrow S_0 and buy share of stock at $T = 0$. The payoff is $S_T - S_0e^{rT}$ which is exactly the payoff of a long forward. The key here is to realize that

Forward=Stock − Zero Coupon Bond

Example 1.2. How would you create a long synthetic forward for a stock paying continuous dividends? Show that the payoff of the synthetic forward is the same as that of a long forward. Recall that the payoff of such a forward is $S_T - S_0e^{(r-\delta)T}$. Hence, the following cash flows can illustrate the solution:

Cash Flows		
	Time 0	Time T
(i) Buy $e^{-\delta T}$ shares of stock	$-S_0e^{-\delta T}$	S_T
(ii) Borrow $S_0e^{-\delta T}$ at the risk-free rate	$S_0e^{-\delta T}$	$S_0e^{(r-\delta)T}$
Net	0	$S_T - S_0e^{(r-\delta)T}$

Currency Contracts

Let x_0 be the USD/CAD exchange rate and r_c is the CAD-denominated risk-free interest rate. Suppose we want to purchase $1 CAD T years from now using CAD. Then the price of the $T-$year prepaid forward is $F_{0,T}^P = x_0e^{-r_cT}$. If r_u is the USD risk-free rate, then the price of a forward is $F_{0,T} = x_0e^{(r_u-r_c)T}$.

1.4 Put-Call Parity

Assume that the continuously compounded risk-free rate is r. Recall the put-call parity for European options with the same strike price and time to expiration T:

$$\begin{aligned} \text{Call Price} - \text{Put Price} &= PV_{0,T}(\text{Forward Price} - \text{Strike Price}) \\ C(K,T) - P(K,T) &= PV_{0,T}(F_{0,T} - K) = F_{0,T}^P - Ke^{-rT} \\ &= S_0 - PV_{0,T}(\text{Dividends} + K) \end{aligned}$$

Using this formula, we can make a **synthetic stock** (long call, short put, lend PV of strike and dividends), a **synthetic call** (long stock, long put, borrow PV of strike and dividends), a **synthetic put** (short stock, long call, lend PV of strike and dividends), and a **synthetic T-bill** (long stock, short call, long put); also called a "**Conversion**".

In **generalized put-call parity**, instead of having a strike price, we have a **strike asset** which can be a related asset of the underlying asset. If S_t is the underlying and Q_t is the strike asset, then

$$C(S_t, Q_t, T) - P(S_t, Q_t, T) = F_{0,T}^P(S_t) - F_{0,T}^P(Q_t)$$

Using this formula, we should infer:

American options should be more valuable than European options because they can be exercised more frequently. That is,

$$\begin{aligned} C_{Amer}(S,K,T) &\geq C_{Euro}(S,K,T) \\ P_{Amer}(S,K,T) &\geq P_{Euro}(S,K,T) \end{aligned}$$

Also note that:

1. Call option price
   (a) Cannot be negative $\implies$ Call Price ≥ 0
   (b) Parity equation $\implies$ Call price $\geq PV(F_{0,T}) - PV(K)$
   (c) Call Price $\leq S_0$ because payoff at time T is $\max(S_T - K, 0) \leq S_T \implies S_0 \geq C_{Amer}(S,K,T) \geq C_{Euro}(S,K,T) \geq \max(0, PV(F_{0,T}) - PV(K))$
2. Put option price
   (a) Cannot be negative $\implies$ Put Price ≥ 0
   (b) Parity equation $\implies$ Put price $\geq PV(K) - PV(F_{0,T})$
   (c) Put Price $\leq K$ because payoff at time T is $\max(K - S_T, 0) \leq K \implies K \geq P_{Amer}(S,K,T) \geq P_{Euro}(S,K,T) \geq \max(0, PV(K) - PV(F_{0,T}))$

Note that in early exercise, for the the American call option, at each point in time, we can:

1. Hold on to the option
2. Sell it at time t for $C_{Amer}(S,K,T-t)$
3. Exercise at time t for $S_t - K$

For a non-dividend paying stock, it is never optimal to exercise early.

Proof. We want to show that $C_{Amer}(S,K,T-t) \geq S_t - K$. Recall the parity equation

$$\begin{aligned} C_{Euro}(K,T) - P_{Euro}(K,T) &= S_t - Ke^{-r(T-t)} \\ &= S_t - K + \underbrace{K(1 - e^{-r(T-t)})}_{\geq 0} \\ &\geq S_t - K \end{aligned}$$

and hence $C_{Euro} \geq P_{Euro} + (S_t - K) \geq S_t - K$ which implies $C_{Amer} \geq C_{Euro} \geq S_t - K$. □

Next, for strike prices:

Given $K_1 < K_2 < K_3$, we have the following properties:

1. $C(K_1) \geq C(K_2)$
2. $P(K_1) \leq P(K_2)$
3. $C(K_1) - C(K_2) \leq K_2 - K_1$
4. $P(K_2) - P(K_1) \leq K_2 - K_1$

and these properties lead to what is called the convexity of prices (i.e. the absolute value of the slope of the option price with respect to (wrt) strike is ≤ 1):

- $\frac{C(K_1)-C(K_2)}{K_2-K_1} \geq \frac{C(K_2)-C(K_3)}{K_3-K_2}$ which says the call price curve is concave up
- $\frac{P(K_2)-P(K_1)}{K_2-K_1} \leq \frac{P(K_3)-P(K_2)}{K_3-K_2}$ which says the put price curve is concave up

Example 1.3. Suppose that call and put prices are given by:

Strike	50	55
Call Premium	16	10
Put Premium	7	14

Which no-arbitrage property is violated?

Answer: We see that $K_2 - K_1 = 5$ and $C(50) - C(55) = 6 > 5$ as well as $P(55) - P(50) = 7 > 5$ and so both properties are violated.

Which spread position would you use to affect arbitrage?

Answer: To make a risk-free profit,

1. Sell call with strike price of 50
2. Buy call with strike price of 55

Demonstrate that the spread position is an arbitrage.

Answer: See the table below

Cash Flows				
	Time 0	$S_T < 50$	$50 < S_T < 55$	$S_t > 55$
(i) Sell call with strike of 50	16	0	$-(S_T - 50)$	$-(S_T - 50)$
(ii) Buy call with strike of 55	-10	0	0	$S_T - 55$
Net	6	0	$50 - S_T \geq -5$	-5

We receive \$6 at time 0 and lose at most \$5 at time T so an arbitrage opportunity exists. So the profit is $6e^{rT} - 5 > 0$.

Example 1.4. Suppose that call and put prices are given by:

Strike	80	100	105
Call Premium	22	9	5
Put Premium	4	21	24.8

Which no-arbitrage property is violated?

Answer: Using the definition of convexity, we should have

$$\frac{K_3 - K_2}{K_3 - K_1} C(K_1) + \frac{K_2 - K_1}{K_3 - K_1} C(K_3) \geq C(K_2)$$

and since $\frac{K_3-K_2}{K_3-K_1} = \frac{5}{25} = 0.2$ with $0.2C(80) + 0.8C(105) = 8.4 < 9$ which is a violation of the convexity property. Similarly, $0.2P(80) + 0.8P(105) = 20.64 < P(100) = 21$ which is another violation.

Which spread position would you use to affect arbitrage? Demonstrate that the spread position is an arbitrage. (Both done below)

Answer: See the table below

Cash Flows					
	Time 0	$S_T < 80$	$80 < S_T < 100$	$100 < S_T < 105$	$S_T > 105$
(i) Buy 2 calls with strike 80	-44	0	$2(S_T - 80)$	$2(S_T - 80)$	$2(S_T - 80)$
(ii) Buy 8 calls with strike 105	-40	0	0	0	$8(S_T - 105)$
(iii) Sell 10 calls with strike 100	90	0		$10(S_T - 100)$	$10(S_T - 100)$
Net	6	0	$2(S_T - 80) \geq 0$	$8(105 - S_T) \geq 0$	0

1.5 Swaps

Suppose that $P(0,t)$ is the price of a ZCB with equation $[1 + r(0,t)]^{-t}$ where $r(0,t)$ is the **spot rate**. The **forward rate** is the rate locked in today to borrow / lend at some time in the future. We denote $r(t, t+k)$ as the forward rate set today for borrowing / lending over $(t, t+k)$.

We illustrate the concept of a swap with an example:

Example 1.5. The forward prices on a barrel of crude oil are \$40 and \$5 in years 1 and 2 respectively. the annual interest rates on ZCBs are 4% and 5% for years 1 and 2 respectively. What is the 2 year swap price on a barrel of crude oil?

Answer: Assume the swap price is level at R. The PV of the swap obligation is

$$\frac{40}{1.04} + \frac{45}{1.05^2} = \frac{R}{1.04} + \frac{R}{1.05^2} \implies R = 42.4271$$

2 Binomial Option Pricing

To price a call, we create a replicating portfolio. As an example, consider a call with maturity 1-year, strike of \$30 and two possible terminating values of \$3 and \$0. If the risk-free rate is 5%, we construct the portfolio as follows.

- Purchase $\triangle$ shares of stock at time 0
- Invest an amount of money B at the risk free rate at time 0

At time 1, we should have $3 = \triangle \cdot S_u + Be^r$ and $0 = \triangle S_d + Be^r$. This gives us $\triangle = \frac{3-0}{33-27}$ and $B = e^{-r}[C_u - \triangle \cdot S_u] = -12.84$. We call $\triangle$ the **delta** of the option. The price, based on a no-arbitrage argument, is (at time 0)

$$\begin{aligned} C_0 = \triangle \cdot S_0 + B &= \frac{C_u - C_d}{S_u - S_d} \cdot S_0 + e^{-r}\left[C_u - \triangle \cdot S_u\right] \\ &= \ldots \\ &= e^{-r}\left[\underbrace{\frac{e^r - d}{u - d}}_{p} \cdot C_u + \underbrace{\frac{u - e^r}{u - d}}_{q=1-p} \cdot C_d\right] \end{aligned}$$

where u is the up rate, d is the down rate, and p and $q = 1 - p$ are the called the **risk-neutral probabilities**. Remark that if $e^r < d$ then we can invest in the stock and lend the risk-free rate to make an arbitrage profit. Conversely, if $e^r > u$, then we long the risk-free rate and short the stock to make a profit.

Example 2.1. Assume that a stock pays dividends at a yield of δ and consider a call option that matures at time h. The payoff at time h is C_u if $S_h = uS_0$ and C_d otherwise. It can be shown that the no-arbitrage price of the call is $C_0 = \triangle \cdot S_0 + B$ where

$$\begin{aligned} \triangle &= \frac{C_u - C_d}{S_u - S_d} \\ B &= e^{-rh} \cdot \frac{uC_d - dC_u}{u - d} \end{aligned}$$

If we substitute $\triangle$ and B into C_0 and rearrange, then we can get

$$C_0 = e^{-r} \left[\underbrace{\frac{e^{(r-\delta)h} - d}{u - d}}_{p} \cdot C_u + \underbrace{\frac{u - e^{(r-\delta)h}}{u - d}}_{q=1-p} \cdot C_d \right]$$

as well as the risk-neutral probabilities p and q.

We implement this method in a lattice. In general,

$$\begin{aligned} F^p_{t,t+h} &= S_t e^{-\delta \cdot h} \\ F_{t,t+h} &= S_t e^{(r-\delta) \cdot h} \end{aligned}$$

and it turns out that $F_{t,t+h} = E_{\mathbb{Q}}[S_{t+h}]$ where $\mathbb{Q}$ is the risk-neutral probability measure. To see this, remark that

$$\begin{aligned} E\left[S_{t+h} | S_t\right] &= q \cdot u \cdot S_t + (1 - q) \cdot d \cdot S_t \\ &= \frac{e^{(r-\delta)h} - d}{u - d} \cdot u \cdot S_t + \frac{u - e^{(r-\delta)h}}{u - d} \cdot d \cdot S_t \\ &= \ldots \\ &= S_t e^{(r-\delta)h} = F_{t,t+h} \end{aligned}$$

We introduce uncertainty via a volatility coefficient σ where at time $t + h$ we have

$$\begin{cases} u \cdot S_t = F_{t,t+h} e^{\sigma\sqrt{h}} \\ d \cdot S_t = F_{t,t+h} e^{-\sigma\sqrt{h}} \end{cases}$$

(Example 13 from the notes has $\triangle = 1$ and $B_u = -28.5369$). In this procedure we calculate:

1. $\triangle_{u/d}$ from the forward C_u, S_u, C_d, S_d values
2. $B_{u/d}$ from the forward C_u, S_u, C_d, S_d values
3. $C_{u/d} = \max(\triangle \cdot S_{u/d} + B_{u/d}, (S - K)^+)$ for an American call and $P_{u/d} = \max(\triangle \cdot S_{u/d} + B_{u/d}, (K - S)^+)$ for an American put
4. Repeat until you reach the origin node

What we are doing here in the former term of the max function in the step 3 is trying to replicate the payoff of the option if it was held onto. For example, in a put, the payoff is $K - S_T$ and at time $T - 1$, the value is

$$Ke^{-r} - S_d = \underbrace{\triangle_d}_{=-1} S_d + \underbrace{B_d}_{=Ke^{-r}}$$

An extension to this model is to write it with *continuous dividends* so that

$$F_{t,t+h} = S_t e^{(r-\delta)h} \implies \begin{cases} u \cdot S_t = F_{t,t+h} e^{(r-\delta)h + \sigma\sqrt{h}} \\ d \cdot S_t = F_{t,t+h} e^{(r-\delta)h - \sigma\sqrt{h}} \end{cases}$$

with the new bond and delta values as

$$\triangle = e^{-\delta t} \cdot \frac{C_u - C_d}{S_u - S_d}, B = e^{-rh} \cdot \left[C_u - \triangle e^{\delta h} \cdot S_u\right]$$

and the option value is $\triangle \cdot S + B$.

Example 2.2. Given $u = 1.1$ and $d = 0.9$, with $h = 1$, we know that $u = e^{(r-\delta)h+\sigma\sqrt{h}}$ and $d = e^{(r-\delta)h-\sigma\sqrt{h}}$. We can solve for δ and σ being

$$\begin{cases} \sigma = \frac{1}{2}\left(\ln\left(\frac{u}{d}\right)\right) & = 0.1003 \\ \delta = r - \ln(ud)/2 & = 0.055 \end{cases}$$

We can then repeat the same four steps in the classical binomial lattice.

Next, we can do this with *discrete dividends*. Assume over $(t, t+h)$ we receive dividends (with certainty) with a future value at time $t+h$ of D. We then have

$$F_{t,t+h} = F^p_{t,t+h} e^{rh} = \left[S_t - De^{-rh}\right] e^{rh} = S_t e^{rh} - D$$

and so

$$\begin{cases} u \cdot S_t = (S_t e^{rh} - D) e^{\sigma\sqrt{h}} \\ d \cdot S_t = (S_t e^{rh} - D) e^{-\sigma\sqrt{h}} \end{cases}, \triangle = \frac{C_u - C_d}{S_u - S_d}$$

and the new bond value is

$$\begin{aligned} B &= e^{-rh}\left[C_u - \triangle(S_u + D)\right] \\ &= e^{-rh}\left[\frac{S_u C_d - S_d C_u}{S_u - S_d}\right] - De^{-rh} \end{aligned}$$

Example 2.3. Assume a discrete dividend paying stock with $S_0 = 30$. Under the CRR model with time steps of length 1 year and $\sigma = 0.055$ (annual volatility), construct the binomial tree for the stock price over the next 2 years. If a dividend of \$1 is paid only at time 2 (with certainty), find the price of a European call option on the stock, with 3 yers to maturity and a strike price of \$30. The continuously compounded risk-free rate is $r = 5\%$.

- At time 0, $S_0^0 = 30$.
- At time 1, $S_1^0 = 29.8504$ and $S_1^1 = 33.3213$.
- At time 2, $S_2^0 = 28.755$, $S_2^1 = 32.0986$, $S_2^2 = 32.2088$, $S_2^3 = 35.9538$
- At time 3, $S_3^0 = 28.6116, S_3^1 = 31.9385, S_3^2 = 31.9385, S_3^3 = 35.6522, S_3^4 = 32.048, S_3^5 = 35.7745, S_3^6 = 35.7745$, and $S_3^7 = 39.9343$
- We can then calculate $C_{uu} = 6.4657, C_{ud} = 2.7205, C_{du} = 2.6105, C_{dd} = 0.3424$ and subsequently $C_u = 5.2714, C_d = 2.02$ and $C_0 = 3.4354$

2.1 Model Analysis

1. Risk-Neutral Probabilities

   (a) For the non-dividend paying stock, the R-N probabilities are

   $$q = \frac{e^r - d}{u - d}, 1 - q = \frac{u - e^r}{u - d}$$

   and

   $$\begin{aligned} H_\alpha &= e^{-(r-\delta)h}\left[qC[P]_{\alpha u} + (1-q)C[P]_{\alpha d}\right] \\ C[P]_\alpha &= \max(H_\alpha, E_\alpha) \end{aligned}$$

(b) In general, we want to price using the risk-neutral probabilities. It is easier than first finding the replicating portfolio and then pricing the security. How can we compute them easily? Under the risk-neutral measure, we must have

$$E_Q\left[S_{t+h}|S_t\right] = F_{t,t+h}$$

It can be shown that

$$q = \frac{F_{t,t+h}/S_t - d}{u-d} = \begin{cases} \frac{S_t e^{rh}/S_t - d}{u-d} = \frac{e^{rh}-d}{u-d} & \text{No dividends} \\ \frac{S_t e^{-\delta h} e^{rh}/S_t - d}{u-d} = \frac{e^{(r-\delta)h}-d}{u-d} & \text{Continuous dividends} \end{cases}$$

2. Delta

(a) For the non-dividend paying stock, we have $\triangle = \frac{C_u - C_d}{S_u - S_d}$

2.2 Lognormal Model

Definition 2.1. A random variable is said to be **lognormally distributed** with parameters μ and σ if it is of the form e^X where $X \sim N(\mu, \sigma^2)$.

Remark 2.1. The binomial model can be shown to approximate a log-normal distribution (continuous). To see this, we need to consider very time steps (makes the model more realistic). Consider time steps of size $1/n$ where n is very large and an interval of $n \cdot t$ steps. Let

$$S_t = S_0 u^{N_u} d^{N_d}$$

Suppose for simplicity that $u = e^{\sigma/\sqrt{n}}$ and $d = e^{-\sigma/\sqrt{n}}$ with $h = 1/n$. Note that $n \cdot t = N_u + N_d$. So

$$\begin{aligned} S_t &= S_0 e^{\sigma N_u/\sqrt{n}} e^{-\sigma N_d/\sqrt{n}} \\ &= S_0 e^{\sigma(N_u - N_d)/\sqrt{n}} \end{aligned}$$

If the R-N probabilities are $q = 1-q = 0.5$, then we want to re-write $1 \cdot N_u + (-1) \cdot N_d$. Let

$$X_i = \begin{cases} 1 & q = 0.5 \\ -1 & 1-q = 0.5 \end{cases}$$

Then,

$$1 \cdot N_u + (-1) \cdot N_d = \sum_{i=1}^{n \cdot t} X_i$$

Taking the limit of $S_t = S_0 e^{\sigma(N_u - N_d)/\sqrt{n}}$, we get

$$\lim_{n\to\infty} S_t = S_0 \lim_{n\to\infty} e^{\sigma\left(\sum_{i=1}^{nt} X_i\right)/\sqrt{n}}$$

Since $E[X_i] = 0, Var[X_i] = 1$, then using the **central limit theorem**,

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{nt} X_i - 0}{\sqrt{nt} \cdot 1} \xrightarrow[n\to\infty]{D} N(0,1)$$

and hence

$$\begin{aligned} \lim_{n\to\infty} S_t &= S_0 \lim_{n\to\infty} e^{\sigma\sqrt{t}\left(\sum_{i=1}^{nt} X_i\right)/\sqrt{nt}} \\ &= S_0 e^{\sigma\sqrt{t}(X)} \end{aligned}$$

and the exponent tends to $N(0, \sigma^2 t)$ where $X \sim N(0,1)$. So S_t is lognormally distributed with parameters 0 and $\sigma^2 t$. If we change $u = e^{(r-\delta)h + \sigma\sqrt{h}}$ and $u = e^{(r-\delta)h - \sigma\sqrt{h}}$ then $S_t = S_0 e^Y$ where $Y \sim N((r-\delta)t, \sigma^2 t)$. Equivalently, we can write

$$\ln\left(\frac{S_t}{S_0}\right) \sim N((r-\delta)t, \sigma^2 t)$$

where the left side is the continuously compounded return, $(r-\delta)t$ is the drift and $\sigma^2 t$ is the variance. If $t=1$, then σ^2 is the variance of the continuously compounded returns over 1 year. Now if σ^2_{yearly} is the variance of the continuously compounded returns, suppose we want to find $\sigma^2_{monthly}$. Since

$$r_{yearly} = \sum_{i=1}^{12} r_{monthly,i} \implies Var(r_{yearly}) = 12Var(r_{monthly,i})$$

Then, $\sigma_M = \sigma_Y/\sqrt{12}$. So in general, $\sigma\sqrt{h}$ represents one standard deviation of the continuously compounded returns.

3 Discrete-Time Securities Market

We start with some basic models.

3.1 Single Period Model

We now extend the one period binomial model for a single asset (from the previous chapter) to a discrete-time model for a market of securities. For now, we'll consider a single period model and then eventually extend it to a multiperiod model. Consider a market with N securities.

- The price of the j^{th} asset at time t is $S_j(t), t=0,1$. The price vector is thus

$$S(t) = \begin{bmatrix} S_1(t) & S_2(t) & ... & S_N(t) \end{bmatrix}_{1\times N}$$

- Suppose now at time 1, there are M possible states of the economy: $w_1, ..., w_M$. We use Ω to denote this state space:

$$\Omega = \{w_1, w_2, ..., w_M\}$$

- To make it more explicit that the time 1 price depends on one of the above states, we may write $S_j(1,w)$ instead of $S_j(1)$. We summarize the possible asset prices at time 1 in a matrix:

$$S(1,\Omega) = \begin{bmatrix} S_1(1,w_1) & S_2(1,w_1) & \cdots & S_N(1,w_1) \\ S_1(1,w_2) & S_2(1,w_2) & \cdots & S_N(1,w_2) \\ \vdots & \vdots & \ddots & \vdots \\ S_1(1,w_M) & S_2(1,w_M) & \cdots & S_N(1,w_M) \end{bmatrix}$$

  Note that the j^{th} column describes the possible prices of the j^{th} asset at time 1.

- Typically, the first asset is a bank account (risk-free bond) which earns interest at an annual effective rate of i. In this case, we have

$$S_1(0) = 1 \text{ and } S_1(1,w) = 1+i \text{ for all } w \in \Omega$$

- Our aim is to provide a framework for risk-neutral pricing. As we saw before, this was based on the assumption of no arbitrage opportunities.

- We need to hold a combination of securities at time 0 to construct a replication portfolio. Suppose the investor holds θ_j units of the j^{th} asset from 0 to 1. Our **trading strategy** is thus

$$\theta_{N\times 1} = \begin{bmatrix} \theta_1 & \theta_2 & ... & \theta_N \end{bmatrix}^T$$

  and so the value of the portfolio is $S(t)\theta$.

- (**Arbitrage**) If an arbitrage opportunity exists in the market, then there exists a trading strategy θ such that

$$S(0)\theta \leq 0 \text{ and } S_1(1,\Omega) > 0$$

If no arbitrage opportunities exist, then we say that the securities market model is arbitrage-free.

- We will now proceed to define what we call a state price vector. Its name is derived from the fact that the m^{th} element in this vector gives us the price (time-0 value) of \$1 received at time 1 if the state w_m occurs for $m = 1, ..., M$.

- A **state price vector** ψ is a strictly positive vector

$$\psi = \left[\begin{array}{cccc} \psi(w_1) & \psi(w_2) & ... & \psi(w_M) \end{array} \right]_{1\times M}$$

such that $S(0) = \psi S(1,\Omega)$.

- A security that pays \$1 at time 1 if the state w_m occurs and 0 otherwise is called the **Arrow-Debreu security for state** w_m.

- (**Completeness**) An arbitrage-free securities market is **complete** if and only if there is a unique state price vector. In

Theorem 3.1. *(Fundamental Theorem of Asset Pricing) The single period securities market model is arbitrage free if and only if there exists a state price vector.*

3.2 Multiperiod Model

We have n assets over a period $(0, T)$. We have a discrete model in the sense that each asset's price can only change at times $k = 0, 1, 2, ..., T$. We have a price vector

$$S(k) = \left[\begin{array}{cccc} S_1(k) & S_2(k) & ... & S_N(k) \end{array} \right]_{1\times N}$$

for each $k = 0, 1, ..., T$. In the multiperiod model, we have, for each asset, a sequence of random variables over time:

$$\left[\begin{array}{cccc} S_j(0) & S_j(1) & ... & S_j(T) \end{array} \right]$$

for $j = 1, 2, ..., N$. We call this our asset price **process**.

Example 3.1. (Sample path) Thins of a sample path as a "single" path (or realization) of the process over time. Possible paths for the 2 period binomial model are

$$w_1 = S_{uu}, w_2 = S_{ud}, w_3 = S_{du}, w_4 = S_{dd}$$

Assume that S_1 is the bank account with $S_1(0) = 1$ and $S_1(k+1) = S_1(k) \cdot (1 + i_k)$ for $k = 0, 1, 2, ..., T-1$. So

$$i_k = \frac{S_1(k+1)}{S_1(k)} - 1 \geq 0$$

and in the special case of $i_k = i$, we have $S_1(k) = (1+i)^k$. For $k \leq T$, P_k will denote the information available at time k (the asset prices from time 0 up to and including time k).

Example 3.2. Suppose that $\Omega = \{w_1, w_2, w_3, w_4, w_5\}$. Then,

$$P_0 = \Omega_0 = \{\Omega, \emptyset\}, P_1 = \Omega_1 = \{\emptyset, \Omega, \{w_1, w_2, w_3\}, \{w_4, w_5\}\}, P_2 = \Omega_2 = \mathbb{P}(\Omega)$$

With each P_k, we can answer questions such as "Did the stock price increase/decrease at time k?". Sometimes, we call $\{P_0, P_1, ..., P_T\}$ our information submodel.

Definition 3.1. We say that X is measurable with respect to P_k if $X(w)$ is constant within each partition of P_k.

Example 3.3. (2 Period Binomial Model) Suppose that $X(w_1) = X(w_2) = 3$ and $X(w_3) = X(w_4) = 4$. X is measurable with respect to (w.r.t) $P_1 = \{\{w_1, w_2\}, \{w_3, w_4\}\}$.

Remark 3.1. Suppose we have the information P_k at time k. Then $S_j(k)$ is measurable w.r.t. P_k. $S_j(k+1)$ is NOT measurable w.r.t. P_k. It is still random.

Definition 3.2. We say that X is adapted to the information submodel $\{P_0, ..., P_T\}$ if for each $k = 0, 1, ..., T$ we have that $X(k)$ is measurable with respect to P_k.

Example 3.4. With respect to our (general) multiperiod model, let $X(k)$ denote the maximum price of the j^{th} asset over the period $[0, k], k = 1, ..., T$. This process is adapted .

Theorem 3.2. *Consider an arbitrage-free model with a risk-neutral probability measure Q. the time-0 value of an attainable European derivative with payoff X at time T is given by $E_Q(\frac{X}{S_1(T)})$.*

Proof. Note that

$$\frac{V^\theta(k)}{S_1(k)} = E_Q\left(\frac{V^\theta(t)}{S_1(t)}\Big|P_k\right)$$

and we set $t = T$ and $k = 0$ because of the definition of time-0 European option. The time 0 value of the derivative is

$$\underbrace{\frac{V^\theta(0)}{S_1(0)}}_{=1} = E_Q\left(\frac{V^\theta(T)}{S_1(T)}\Big|P_0\right) = E_Q\left(\frac{V^\theta(T)}{S_1(T)}\right) = E_Q\left(\frac{X}{S_1(T)}\right)$$

□

Definition 3.3. We say a cash flow in the form $c = \{c(k), k = 0, 1, ..., T\}$ is attainable if there exists a trading strategy such that

$$c^\theta(k, w) = S(k, w)\theta = c(k, w), \forall k, w$$

Theorem 3.3. *Consider an arbitrage-free model with a risk-neutral probability measure Q. The time-0 value of an attainable cash-flow stream c is given by*

$$E_Q\left(\sum_{k=0}^{T}\frac{c(k)}{S_1(k)}\right)$$

Definition 3.4. An arbitrage-free multiperiod model is said to be complete if every adapted cash-flow stream c is attainable (that is, every adapted cash-flow stream can be replicated by some trading strategy, not necessarily self financing).

Theorem 3.4. *A stochastic process $\psi = \{\psi(k), k = 0, 1, ..., T\}$ is said to be a state price process if the following hold*

- $\sum_{w\in\Omega}\psi(0, w) = 1$
- ψ is adapted and strictly positive
- For each $k = 0, 1, ..., T-1$, each $j = 1, 2, ..., N$ and each $H \in \mathcal{P}_k$

$$\sum_{w\in H}\psi(k, w)S_j(k, w) = \sum_{w\in H}\psi(k+1, w)S_j(k+1, w)$$

Definition 3.5. In the single period model we had $\psi = \frac{Q(w)}{1+i}$. We can find a unique parametrization in the multiperiod case. We have

$$\psi(k, w) = \frac{Q(H)}{|H|\cdot S_1(k, w)}$$

Note that if we set $k = T$, we can see that $\mathcal{P}_T$ are the events ,$\{w\}$ for $w \in \Omega$ with $|H| = 1$ and hence

$$Q(w) = \psi(T, w)\cdot S_1(T, w)$$

Remark 3.2. For a European derivative X and cash flow sequence c, we have

$$E_Q\left(\frac{X}{S_1(T)}\right) = \sum_{w\in\Omega}\psi(T, w)X(w)$$

$$E_Q\left(\sum_{k=0}^{T}\frac{c(k)}{S_1(k)}\right) = \sum_{w\in\Omega}\sum_{k=0}^{T}\psi(k, w)c(k, w)$$

Theorem 3.5. *Consider an arbitrage-free multiperiod model. The following are equivalent:*

(1) The model is complete

(2) The state-price process ψ is unique

(3) The risk-neutral probability measure Q is unique

4 Stochastic Calculus

(Most of the material here is *supplementary* to the course notes!)

Definition 4.1. A **standard Brownian motion** (BM) is a stochastic process $W = \{W_t, t \geq 0\}$ such that the following hold:

(1) $W_0 = 0$

(2) The process has stationary and normally independent and identically distributed increments where $W_{t_2} - W_{t_1} \sim N(0, t_2 - t_1)$.

(3) It has continuous sample paths

If $\mu \in \mathbb{R}$ and $\sigma > 0$. A linear transformation of a Brownian motion process

$$\tilde{W}_t = \mu t + \sigma W_t$$

is called a Brownian motion with drift μ and diffusion coefficient σ.

Example 4.1. Suppose that $\tilde{W}$ is a BM with zero drift and volatility coefficient σ. Then:

(1) $\left\{\frac{1}{\sigma}\tilde{W}_t, t \geq 0\right\}$ is a BM with zero drift and volatility coefficient 1.

(2) $\left\{\frac{1}{\lambda}\tilde{W}_{\lambda t}, t \geq 0\right\}$ is such that

$$\tilde{W}_{\lambda t_1} - \tilde{W}_{\lambda t_2} \overset{D}{=} \tilde{W}_{\lambda(t_1 - t_2)} \sim N(0, (t_1 - t_2)\lambda\sigma^2)$$

and hence it is a BM with zero drift and volatility coefficient $\frac{\sigma}{\sqrt{\lambda}}$.

(3) $\left\{\mu t + \tilde{W}_t, t \geq 0\right\}$ is a BM with drift μ and volatility coefficient σ.

Remark 4.1. We have the following properties about the BM process:

- The BM process is a Gaussian process where $(W_{t_1}, ..., W_{t_k})$ is MVN with $E[W_{t_i}] = 0$ and $Cov(W_{t_i}, W_{t_j}) = \min(t_i, t_j)$.
- The BM process is a Markov process.
- It will eventually hit every real number regardless of how large it is.
- No matter how large or how negative the BM is, the process will eventually go back to zero.
- The process has sample paths that are nowhere differentiable.

Example 4.2. Show that $Cov(W_{t_i}, W_{t_j}) = \min(t_i, t_j)$. WLOG, suppose that $t_i < t_j$. Then

$$\begin{aligned} Cov(W_{t_i}, W_{t_j}) &= Cov(W_{t_j} + W_{t_i} - W_{t_i}, W_{t_i}) \\ &= Cov(W_{t_j - t_i}, W_{t_i}) + Var(W_{t_i}) \\ &= t_i \end{aligned}$$

and in general $Cov(W_{t_i}, W_{t_j}) = \min(t_i, t_j)$.

4.1 Introduction to The Ito Integral

If $\delta(t)$ represents the number of shares held between $(t, t+dt)$, then for a BM $B(t)$ representing the stock prices, and infinitessimal time units dt, we have

$$\text{Profit}(T) = \int_0^T \delta(t)\, dB(t) = \lim_{\|P\|\to 0} \sum_{i=0}^{N-1} \delta(t_i)(B(t_{i+1}) - B(t_i))$$

called an **Ito integral**. If $B(t)$ were differentiable, then the above is just a Riemann-Stieltjes integral with

$$\text{Profit}(T) = \int_0^T \delta(t) B'(t) dt$$

but since $B(t)$ is nowhere differentiable and is of infinite bounded variation, we use the partition definition.

4.2 Quadratic Variation

The quadratic variation of a function g over an interval $[0, T]$ is given by

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} |g(t_{i+1}) - g(t_i)|^2$$

It turns our that if g is differentiable, then the quadratic variation of g over $[0, T]$ is 0. When we consider the quadratic variation of BM over $(0, T)$, the limit is defined in the sense of mean-squared convergence:

1. Define $QV[0,T] = \sum_{i=0}^{N-1} |B(t_{i+1}) - B(t_i)|^2$
2. We say that $QV(0,T)$ **converges in mean-squared** to L if

$$\lim_{N\to\infty} E([QV(0,T) - L]^2) = 0$$

It turns out that the quadratic variation of standard BM over an interval (a, b) is $b - a$.

Remark that if $E[QV(0,T)] = T$ and $Var[QV(0,T) - L] \to 0$

$$QV[0,T] = \underbrace{\sum_{i=1}^{N-1} |B(t_{i+1}) - B(t_i)|^2}_{\to T} \le \max_{0\le i\le N-1} \underbrace{|B(t_{i+1}) - B(t_i)|}_{\to 0} \times \underbrace{\sum_{i=1}^{N-1} |B(t_{i+1}) - B(t_i)|}_{\implies (\to\infty)}$$

This shows that:

- It is not possible that a process has both finite total variation and non-zero quadratic variation.
- If a process has finite total variation, then the quadratic variation must equal zero.

4.3 Conditional Expectation and Filtration

In the continuous time setting, instead of talking about $\mathcal{P}_k$ in $E[X|\mathcal{P}_k]$, we have $\{\mathcal{F}_t, t \ge 0\}$ also called a **filtration**. Each $\mathcal{F}_t$ is called a $\sigma-$**field** and the filtration models the information available over time.

If X is defined in terms of a Brownian motion process $B = \{B_t, t \ge 0\}$ then the filtration is usually chosen as the filtration generated by $\{B_t, t \ge 0\}$. This means that conditioning on $\mathcal{F}_s$ can be seen as conditioning with respect to $\{B_u, 0 \le u \le s\}$, the information over $[0, s]$.

The ideas of measurability and adaptability are also analogous to the those in discrete time. We say that X is **adapted** to $\{\mathcal{F}_t, t \ge 0\}$ if X_t is $\mathcal{F}_t$ is measurable for all t.

Suppose that X and Y are random variables and consider $\sigma-$fields $\mathcal{F}_s$ and $\mathcal{F}_t$ for $s \leq t$. We have the following properties of conditional expectation:

- $E[aX + bY|\mathcal{F}_t] = aE[X|\mathcal{F}_t] + bE[Y|\mathcal{F}_t]$
- $E(E(X|\mathcal{F}_t)) = E(X)$
- If X is $\mathcal{F}_t$ measurable, then $E[X|\mathcal{F}_t] = X$
- If Y is $\mathcal{F}_t$ measurable, then $E[XY|\mathcal{F}_t] = YE[X|\mathcal{F}_t]$
- $E(E(X|\mathcal{F}_s)|\mathcal{F}_t) = E(E(X|\mathcal{F}_t)|\mathcal{F}_s) = E(X|\mathcal{F}_s)$

4.4 Martingales

Suppose that the state space is Ω and that we have a probability measure, a filtration $\{\mathcal{F}_t, t \geq 0\}$ and an adapted stochastic process $M = \{M_t, t \geq 0\}$. If

- $E[|M_t|] < \infty$ for all t
- $E(M_t|\mathcal{F}_s) = M_s$ for all $s < t$

then M is a continuous **martingale** with respect to $\{\mathcal{F}_t, t \geq 0\}$. The above definition implies that $E[M_t] = E[M_0]$ for all t.

Example 4.3. A standard Brownian motion $W = \{W_t, t \geq 0\}$ is a continuous martingale with respect to its own filtration. For $s < t$, we have

$$\begin{aligned} E[W_t|\mathcal{F}_s] &= E[W_t - W_s|\mathcal{F}_s] + E[W_s|\mathcal{F}_s] \\ &= E[W_t - W_s] + W_s \\ &= E[W_t] - E[W_s] + W_s \\ &= W_s \end{aligned}$$

Example 4.4. Show that $\{W_t^2 - t, t \geq 0\}$ is a martingale with respect to the filtration generated by $\{W_t, t \geq 0\}$. To see this, we have

$$\begin{aligned} E[W^2 - t|\mathcal{F}_s] &= E[(W_t - W_s)^2|\mathcal{F}_s] + E[W_s^2|\mathcal{F}_s] + 2E[(W_t - W_s) \cdot W_s|\mathcal{F}_s] - t \\ &= Var(W_t - W_s) + W_s^2 + 2E[W_t - W_s|\mathcal{F}_s] - t \\ &= t - s + W_s^2 + 0 - t = W_s^2 - s \end{aligned}$$

Example 4.5. Let X be a random variable and define $M_t = E[X|\mathcal{F}_t]$ for $0 \leq t \leq T$. Show that $\{M_t, 0 \leq t \leq T\}$ is a martingale with respect to $\{\mathcal{F}_t, 0 \leq t \leq T\}$. By definition,

$$E[M_t|\mathcal{F}_s] = E\left[E\left[X|\mathcal{F}_t\right]|\mathcal{F}_s\right] = E[X|\mathcal{F}_s]$$

4.5 Ito Integral

Some properties of the Ito integral $I(T) = \int_0^T \delta(t)dB(t)$ include

- (Adaptedness) $I(T)$ is $\mathcal{F}_T$-measurable for all $T \geq 0$.
- (Linearity) If

  $$I(T) = \int_0^T \delta(t)dB(t) \text{ and } J(T) = \int_0^T \gamma(t)dB(t)$$

  then

  $$c_1 I(T) \pm c_2 J(T) = \int_0^T (c_1\delta(t) + c_2\gamma(t))dB(t)$$

- (Martingale) $I(T)$ is a martingale with respect to the filtration $\{\mathcal{F}_t, t \geq 0\}$ generated by B:

$$E\left[\int_0^T \delta(t)dB(t)\Big|\mathcal{F}_s\right] = \int_0^s \delta(t)B(t)$$

- (Ito Isometry) If $\delta(t)$ is deterministic, then

$$E[I^2(t)] = E\left[\int_0^T \delta^2(t)dt\right]$$

- (Normality) If δ is a deterministic function, then $I(T)$ is normally distributed.
- Remark that $I(T)$ is a zero mean continuous time martingale.

Proof. (Ito Isometry) We have

$$E\left[\left(\int_0^T \delta(t)dB(t)\right)^2\right] = \lim_{N\to\infty} E\left[\left(\sum_{i=0}^{N-1} \delta(t_i)\underbrace{[B(t_{i+1}) - B(t_i)]}_{M_i}\right)^2\right] = \lim_{N\to\infty} E\left[\sum_{i=0}^{N-1} \delta^2(t_i)M_i^2 + \sum_{i\neq j} \delta(t_i)\delta(t_j)M_iM_j\right]$$

In the first term,

$$\begin{aligned}
\lim_{N\to\infty} E\left[\sum_{i=0}^{N-1} \delta^2(t_i)M_i^2\right] &= \lim_{N\to\infty} E\left[E\left[\sum_{i=0}^{N-1} \delta^2(t_i)M_i^2\Big|\mathcal{F}(t_i)\right]\right] \\
&= \lim_{N\to\infty} E\left[\sum_{i=0}^{N-1} \delta^2(t_i)E[M_i^2]\right] \\
&= \lim_{N\to\infty} E\left[\sum_{i=0}^{N-1} \delta^2(t_i)(t_{i+1} - t_i)\right] \\
&= \int_0^T \delta^2(t)dt
\end{aligned}$$

In the second term, we have by similar methods,

$$\begin{aligned}
\lim_{N\to\infty} E\left[\sum_{i\neq j} \delta(t_i)\delta(t_j)M_iM_j\right] &= \lim_{N\to\infty} 2E\left[E\left[\sum_{i<j} \delta(t_i)\delta(t_j)M_iM_j\Big|\mathcal{F}(t_i)\right]\right] \\
&= \lim_{N\to\infty} 2E\left[E\left[\sum_{i<j} \delta(t_i)\delta(t_j)(t_i)\underbrace{E[M_j]}_{=0}\Big|\mathcal{F}(t_i)\right]\right] \\
&= 0
\end{aligned}$$

□

Definition 4.2. (Formal definition of the Ito Integral) Suppose that δ is a process such that $\delta(t)$ is $\mathcal{F}_t-$measurable for all $t, 0 \leq t \leq T$ (it is *adapted*) and

$$E\left[\int_0^T \delta^2(t)\ dt\right] < \infty$$

which says that it is square integrable. Then the Ito integral over $[0, T]$ of δ with respect to B is written as

$$\int_0^T \delta(t)dB(t)$$

and is defined as the limit, in terms of mean-squared convergence, of

$$\sum_{i=0}^{N-1} \delta(t_i)(B(t_{i+1}) - B(t_i))$$

as $N \to \infty$.

Example 4.6. Suppose that $W = \{W(t), t \geq 0\}$ is a standard BM. Find the distribution of

$$X_T = \int_0^T t dW_t$$

using the definition of the Ito integral. Since $\delta(t) = t$ is deterministic, then X_T is normal. Now $E[X_T] = 0$ and

$$Var(X_T) = E[X_T^2] - E^2[X_T] = E[X_T^2] = E\left[\int_0^T t^2 dt\right] = \frac{T^3}{3}$$

so $X_T \sim N(0, T^3/3)$.

4.6 Ito's Lemma

Lemma 4.1. *(Ito's Lemma) If a Ito process has the form*

$$X_t = X_0 + \int_0^t \delta_1(s, W_s) ds + \int_0^t \delta_2(s, W_s) dW_s$$

then the equivalent differential form is

$$dX_t = \delta_1(t, W_t) + \delta_2(t, W_t) dW_t$$

Equivalently, if $f(t,x) \in C^2$ has the same dynamics as X_t and $Y_t = f(t, X_t)$ then

$$dY_t = f_t dt + f_{X_t} dX_t + \frac{1}{2} f_{X_t X_t} (dX_t)^2$$

with the rules (1) $dt \cdot dt = dt \cdot dW_t = dW_t \cdot dt = 0$, (2) $dW_t \cdot dW_t = dt$.

Example 4.7. Compute $\int_0^T W_t dW_t$ using Ito's formula on $f(t,x) = \frac{1}{2}x^2$. By definition, we have

$$dY_t = 0dt + W_t dW_t + (dW_t)^2 = d\left(\frac{1}{2}W_t^2\right) = W_t dW_t + \frac{1}{2}dt$$

If we integrate this over $[0, T]$, we get

$$\frac{1}{2}W_T^2 - \frac{1}{2}W_0^2 = \int_0^T W_t dW_t + \frac{1}{2}\int_0^T dt$$

and since $W_0 = 0$, we get

$$\int_0^T W_t dW_t = \frac{1}{2}\left(W_T^2 - T\right)$$

Example 4.8. Consider the GBM $dX_t = \mu X_t dt + \sigma X_t dW_t$. For a small period of time $(t, t+h)$,

$$\frac{X_{t+h} - X_t}{X_t} = \mu \cdot h + \sigma(W_{t+h} - W_t) = R_t$$

On average, over a small period of time, the return is μh and since $W_{t+h} - W_t \sim N(0, h)$, then

$$R_t \sim N(\mu h, \sigma^2 h)$$

approximately. Now solving explicitly, using Ito's Lemma on $f(x,t)=\ln x$ gives us

$$d(\ln X_t)=dY_t=\frac{dX_t}{X_t}+\frac{1}{2}\left(-\frac{1}{X_t^2}\right)(dX_t^2)=\mu dt+\sigma dW_t-\frac{1}{2X_t^2}\sigma^2X_t^2\left(dW_t^2\right)=\left(\mu-\frac{\sigma^2}{2}\right)dt+\sigma dW_t$$

and integrating over $(0,t)$ gives us

$$\ln(X_t)-\ln(X_0)=\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma\underbrace{\int_0^t dW_t}_{W_t}\implies X_t=X_0\exp\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma W_t$$

Example 4.9. In general, we may not always have a closed form and we may need numerical techniques to compute the integrals. Consider a special case of the **Ornstein-Uhlenbeck process**

$$dY_t=-\alpha Y_t dt+\sigma dW_t$$

Using Ito's formula with $f(t,x)=e^{\alpha t}x$, we get that

$$d(e^{\alpha t}Y_t)=\alpha e^{\alpha t}Y_t dt+e^{\alpha t}dY_t=\sigma e^{\alpha t}dW_t$$

and integrating from $(0,t)$ gives us

$$e^{\alpha t}Y_t-Y_0=\sigma\int_0^t e^{\alpha s}dW_s\implies Y_t=Y_0e^{-\alpha t}+\sigma e^{-\alpha t}\int_0^t e^{\alpha s}dW_s$$

(Additional question) What is the distribution of Y_t? Since we have a deterministic $\delta(t)$ in our Ito integral, Y_t is normal with

$$E[Y_t]=Y_0e^{-\alpha t}$$

$$Var[Y_t]=\sigma^2e^{-2\alpha t}\int_0^t e^{\alpha s}ds=\frac{\sigma^2}{2\alpha}\left[1-e^{-2\alpha t}\right]$$

by Ito isometry. Hence $Y_t\sim N(e^{-\alpha t}Y_0,\frac{\sigma^2}{2\alpha}\left[1-e^{-2\alpha t}\right])$

Example 4.10. The **Vasicek model** is a popular model used to describe the evolution of interest rates. The dynamics of the process are described by

$$dX_t=a(b-X_t)dt+\sigma dW_t$$

Using Ito's formula, with $f(t,x)=e^{\alpha t}x$, one can show that

$$X_t=X_0e^{-at}+b(1-e^{-at})+\sigma e^{-at}\int_0^t e^{as}dW_s$$

which is another special case of the Ornstein-Uhlenbeck stochastic process. Remark that since $X_{t+h}-X_t\approx a(b-X_t)$, then $X_t>b\implies$ the average change is positive while $X_t<b\implies$ the average change is negative. This is the mean reverting feature of the process where a adjusts the speed of the mean reversion. As an aside,

$$\begin{aligned}Cov(Y_t,Y_s)&=E[Y_t]E[Y_s]-E[Y_t]E[Y_s]\\&\cdots\\&=Cov\left(\int_0^s e^{-\alpha u}dW_u,\int_0^t e^{-\alpha v}dW_v\right)\cdot k,k\in\mathbb{R}\\&=E\left(\int_0^s e^{-\alpha u}dW_u\int_0^t e^{-\alpha v}dW_v\right)\end{aligned}$$

Example 4.11. The **Cox-Ingersoll-Ross** (CIR) model is defined by

$$dr(t)=a(b-r(t))dt+\sigma\sqrt{r(t)}dW_t$$

and has a mean reverting feature while keeping the interest rate positive. It can be shown that $r(t)$ has a non-central χ^2

distribution. What is the mean of $r(t)$? The integral form is

$$r(t) = r(0) + a\int_0^t (b - r(u))du + \sigma \int_0^t \sqrt{r(u)}dW_u$$

Taking expectations gives us

$$\begin{aligned} E[r(t)] = r(0) + a\int_0^t (b - E[r(u)]du \quad &\implies \quad \frac{dE[r(t)]}{dt} = a[b - E[r(t)]] \\ &\implies \quad \frac{d}{dt}E[r(t)] + aE[r(t)] = ab \\ &\implies \quad \frac{d}{dt}E\left[e^{at}E[r(t)]\right] = e^{at}ab \end{aligned}$$

and by integrating throughout,

$$e^{at}E[r(t)] - r(0) = b(e^{at} - 1) \implies E[r(t)] = r(0)e^{-at} + b(1 - e^{-at})$$

If we look at $E[r(t)]$ as $t \to \infty$, we can see that $E[r(t)] \to b$.

(We omit the "derivation" of Ito's Lemma and ask the reader to refer to the course notes)

4.7 Black-Scholes-Merton Model

In this model we have the following assumptions:

- We assume a state space Ω, a probability measure $\mathbb{P}$, a standard BM process $W = \{W_t, t \geq 0\}$ and a filtration $\mathcal{F} = \{\mathcal{F}_t, t \geq 0\}$ generated by W.
- The market consists of a stock (our risky asset), and a risk-free zero coupon bond (or a bank account earning the risk-free rate)
- We denote the time t value of the stock by S_t and assume the following dynamics for S_t:

  $$dS_t = \mu S_t dt + \sigma S_t dW_t$$

  where μ and σ are respectively the mean and volatility of the continuously compounded rate of the stock. Given S_0, we know that

  $$S_t = S_0 \exp\left(\left[\mu - \frac{\sigma^2}{2}\right]t + \sigma W_t\right)$$

- The time t value of the bank account is β_t and assuming a constant risk-free rate of return, the dynamics of β_t are $d\beta_t = r\beta_t dt$. This is an ODE with solution $\beta_t = \beta_0 e^{rt}$. We usually assume that $\beta_0 = 1$ so that $\beta_t = e^{rt}$, $t > 0$

Remark 4.2. (Black-Scholes PDE) For a replicating portfolio $V_t = a_t S_t + b_t \beta_t$, the dynamics of a self financing portfolio (see derivation in notes) are

$$dV_t = a_t dS_t + b_t d\beta_T = (\mu a_t S_t + r b_t \beta_t)dt + a_t \sigma S_t dW_t$$

If $V_t = C(t, S_t)$, where C is the option that we want to price, then

$$dV_t = \left[C_t + \mu S_t C_{S_t} + \frac{\sigma^2}{2}S_t^2 C_{S_t S_t}\right]dt + \sigma C_{S_t} S_t dW_t$$

Given the above equations, we match coefficients to get

$$a_t = C_{S_t}(t, S_t), b_t = \frac{1}{r\beta_t}\left[C_t + \frac{\sigma^2}{2}S_t^2 C_{S_t S_t}\right] \implies C_{S_t}S_t + \frac{1}{r\beta_t}\left[C_t + \frac{\sigma^2}{2}S_t^2 C_{S_t S_t}\right]\beta_t$$

Multiplying by r and rearranging, we have

$$C_t(t, S_t) + rC_{S_t}(r, S_t)S_t + \frac{\sigma^2}{2}S_t^2 C_{S_t S_t}(t, S_t) - rC(t, S_t) = 0$$

If $C(t, S_t)$ has payoff of $g(s)$ at time T, then we also require the boundary condition $C(T, S_t) = g(S_t)$. This is the well-known **Black-Scholes PDE** (B-S PDE) and $C(t, S_t)$ is arbitrage free price at time t of the security.

Example 4.12. In the case of a call option the price is given by

$$C(t, S_t) = S_t N(d_1) - Ke^{-r(T-t)} N(d_2)$$

and similarly for a put

$$P(t, S_t) = Ke^{-r(T-t)} N(-d_1) - S_t N(d_2)$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, d_2 = d_1 - \sigma\sqrt{T-t}, N(x) = \mathbb{P}(N(0,1) \leq x)$$

This is done via solving the B-S by performing a change of variables to solve a special version of the heat equation PDE. See AMATH 350 for details.

Example 4.13. In the case of a stock with dividends at a annualized and continuously compounded rate δ, the dynamics of the stock are

$$dS_t = (\mu - \delta) S_t dt + \sigma S_t dW_t$$

and the self financing condition becomes

$$dV_t = a_t dS_t + b_t d\beta_t + a_t(\delta S_t) dt$$

with the call price being

$$C(t, S_t) = S_t e^{-\delta(T-t)} N(d_1) - Ke^{-r(T-t)} N(d_2)$$

and d_1 and d_2 the same as the previous example.

(We omit the section of delta hedging which just says that we wish to hedge our risk by reducing the dynamics of the portfolio $V_t = -C(t, S_t) + \Delta_t S_t$ to be deterministic, via selecting $\Delta_t = C_{S_t}(t, S_t)$)

5 Risk Neutral Pricing

From previous sections, remark that if we had a risk-neutral probability measure $\mathbb{Q}$,then the time t price of a claim $X_T = g(S_T)$ at time T is given by

$$E^{\mathbb{Q}}\left[e^{-r(T-t)} X_T\right]$$

The question now is how do we obtain the $\mathbb{Q}$ measure? Suppose that $\mathbb{P}$ is the real world probability measure.

Definition 5.1. A risk-neutral probability measure is a probability measure $\mathbb{Q}$ on Ω such that

- $\mathbb{Q}$ is equivalent to $\mathbb{P}$ in the sense that for any $A \subseteq \Omega$, $\mathbb{Q}(A) = 0 \iff \mathbb{P}(A) = 0$.

- $\frac{S}{\beta}$ is a martingale under $\mathbb{Q}$

In this case, $\mathbb{Q}$ is also called an **equivalent martingale measure**.

Remark 5.1. (Important!) An Ito process is a martingale if and only if it has zero drift. See the notes for justification.

Example 5.1. Under what conditions is $\frac{S}{\beta}$ a martingale under $\mathbb{P}$?

[Direct Method] Given that

$$S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}, W_t \sim N(0, t)$$

We want to find conditions such that

$$E\left[e^{-rt} S_t | \mathcal{F}_u\right] = e^{-ru} S_u$$

Evaluating, we get

$$\begin{aligned} E\left[e^{-rt} S_t | \mathcal{F}_u\right] &= e^{-rt} E\left[S_t | \mathcal{F}_u\right] \\ &= S_0 e^{-rt} e^{\left(\mu - \frac{1}{2}\sigma^2\right)t} E\left[e^{\sigma W_t} | \mathcal{F}_u\right] \\ &= S_0 e^{-rt} e^{\left(\mu - \frac{1}{2}\sigma^2\right)t} E\left[e^{\sigma(W_t - W_u) + \sigma W_u} | \mathcal{F}_u\right] \end{aligned}$$

Now,

$$\begin{aligned} E\left[e^{\sigma(W_t-W_u)+\sigma W_u}|\mathcal{F}_u\right] &= e^{\sigma W_u}\cdot E\left[e^{\sigma(W_t-W_u)}|\mathcal{F}_u\right] \\ &= e^{\sigma W_u}\cdot E\left[e^{\sigma W_{t-u}}|\mathcal{F}_u\right] \\ &= e^{\sigma W_u}\cdot e^{\frac{1}{2}\sigma^2(t-u)} \end{aligned}$$

using moment generating functions. Substitute to get

$$\begin{aligned} E\left[e^{-rt}S_t|\mathcal{F}_u\right] &= e^{(\mu-r)(t-u)}\underbrace{S_0 e^{\left(\mu-\frac{1}{2}\sigma^2\right)u+\sigma W_u}}_{S_u}e^{-ru} \\ &= e^{(\mu-r)(t-u)}\left[S_u e^{-ru}\right] \end{aligned}$$

We thus need $\mu = r$ so that $\frac{S}{\beta}$ is a martingale under $\mathbb{P}$.

[Alternate Method] Find conditions such that $e^{-rt}S_t$ has zero drift. Use Ito's Lemma with $f(t,s) = e^{-rt}s$. Let $Y_t = e^{-rt}S_t$. We then have

$$\begin{aligned} dY_t &= -re^{-rt} + e^{-rt}\underbrace{dS_t}_{\mu S_t dt+\sigma S_t dW_t} \\ &= (\mu-r)e^{-rt}S_t dt + \sigma e^{-rt}S_t dW_t \\ &= \underbrace{(\mu-r)}_{=0}Y_t dt + \sigma Y_t dW_t \end{aligned}$$

This implies that $\mu = r$ which is the condition for a martingale under $\mathbb{P}$.

Example 5.2. Under $\mathbb{P}$,

$$dS_t = \mu S_t dt + \sigma S_t dW_t^{\mathbb{P}}$$

Choose a risk-neutral probability measure $\mathbb{Q}$ such that the stock's dynamic under $\mathbb{Q}$ is as follows

$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}}$$

where $W_t^{\mathbb{Q}}$ is a standard BM under $\mathbb{Q}$.

Aside. This process is not that different from how we chose $\mathbb{Q}$ for the (discrete) binomial model. We choose q such that

$$S_0 = \frac{1}{1+r}\left[q\cdot S_u + (1-q)S_d\right]$$

Example 5.3. (Ito's Formula) Show that $\frac{S}{\beta}$ is a martingale under $\mathbb{Q}$. We have,

$$\begin{aligned} d(e^{-rt}S_t) &= -re^{-rt}S_t dt + e^{-rt}\underbrace{dS_t}_{rS_t dt+\sigma S_t dW_t^{\mathbb{Q}}} \\ &= \sigma\left(e^{-rt}S_t\right)dW_t^{\mathbb{Q}} \end{aligned}$$

This process has zero drift and so $\frac{S}{\beta}$ is a martingale under $\mathbb{Q}$.

Example 5.4. If $C(t,s)$ is the time t price of a call option under the B-S model, show that $e^{-rt}C(t,s)$ is a martingale under $\mathbb{Q}$. Using $f(t,s) = e^{-rt}C(t,s)$, we have

$$\begin{aligned} f_t &= e^{-rt}\left[C_t(t,s) - rC(t,s)\right] \\ f_s &= e^{-rt}C_s(t,s) \\ f_{ss} &= e^{-rt}C_{ss}(t,s) \end{aligned}$$

and hence

$$\begin{aligned} d\left[e^{-rt}C(t,s)\right] &= e^{-rt}\left[C_t(t,S_t)-rC(t,S_t)\right]dt + e^{-rt}C_s(t,S_t)\underbrace{dS_t}_{=rS_t+\sigma S_t dW_t^{\mathbb{Q}}} \quad \frac{1}{2}e^{-rt}C_{ss}(t,S_t)\underbrace{(dS_t)^2}_{\sigma^2 S_t^2 dt} \\ &= e^{-rt}\left[-rC(t,S_t)+C_t(t,S_t)+rS_tC_s(t,S_t)+\frac{1}{2}\sigma^2S_t^2C_{ss}(t,S_t)\right]+\sigma S_t e^{-rt}C_s(t,S_t)dW_t^{\mathbb{Q}} \end{aligned}$$

The drift term equals zero by the B-S PDE and hence $e^{-rt}C(t,s)$ is a martingale under $\mathbb{Q}$.

5.1 Arbitrage

Definition 5.2. An arbitrage opportunity exists if there exists a self-financing portfolio such that:

(1) $V_0 \le 0$

(2) $\mathbb{P}(V_T \ge 0) = 1$ and $P(V_T > 0) > 0$

If no such arbitrage exists, then the market is said to be arbitrage free.

Theorem 5.1. *A model is arbitrage free if and only if there exists a risk-neutral probability measure. The above theorem implies that the Black-Scholes model is arbitrage free.*

Example 5.5. Find the price of a call option with payoff $\max(S_T-K,0)$ at time T. Assume the B-S model and price using the R-N method. You are given that if $X \sim N(a,b)$ then

$$E\left[e^X\chi_{\{e^X>c\}}\right] = \exp\left(a+\frac{b}{2}\right)\cdot\left[1-N\left(\frac{\ln c - a - b}{\sqrt{b}}\right)\right]$$

Solution. Under $\mathbb{Q}$,

$$S_T = S_0e^{(r-\frac{1}{2}\sigma^2)T+\sigma W_T} = e^X$$

where $X \sim N(\ln(S_0)+(r-\frac{1}{2}\sigma^2)T, \sigma^2T)$. So

$$\begin{aligned} C(0,S_0) &= E_{\mathbb{Q}}\left[e^{-rT}\max(S_T-K,0)\right] \\ &= e^{-rT}E_{\mathbb{Q}}\left[\max(S_T-K,0)\right] \\ &= e^{-rT}\left(E_{\mathbb{Q}}\left[S_T\cdot\chi_{\{S_T>K\}}\right]-K\cdot P(S_T>K)\right) \end{aligned}$$

Now

$$\begin{aligned} E_{\mathbb{Q}}\left[S_T\cdot\chi_{\{S_T>K\}}\right] &= \exp\left(\ln(S_0)+rT\right)\cdot\left[1-N\left(\frac{\ln K-\ln S_0-\left(r-\frac{1}{2}\sigma^2\right)-\sigma^2T}{\sigma^2\sqrt{T}}\right)\right] \\ &= S_0e^{rT}\left[1-N\left(\underbrace{\frac{\ln K-\ln S_0-\left(r-\frac{1}{2}\sigma^2\right)-\sigma^2T}{\sigma^2\sqrt{T}}}_{-d_1}\right)\right] \\ &= S_0e^{rT}\left[1-N(-d_1)\right] \\ &= S_0e^{rT}N(d_1) \end{aligned}$$

and

$$P(S_T>K) = P\left(N(0,1) > \underbrace{\frac{\ln(K)-\ln(S_0)-\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}}_{-d_2}\right) = 1-N(-d_2) = N(d_2)$$

Overall,

$$C(0,S_0) = e^{-rT}\left[S_0e^{rT}N(d_1)-K\cdot N(d_2)\right] = S_0N(d_1)-Ke^{-rT}N(d_2)$$

Example 5.6. Suppose that a European derivative with payoff X_T at maturity time T is **attainable** if there exists a self-financing and adapted trading strategy (a, b) such that

$$V_T = a_T S_T + b_T \beta_T = X_T$$

Suppose that (a, b) is self-financing. Show that under a risk neutral probability measure $\mathbb{Q}$, $\frac{V}{\beta}$ is a martingale.

Solution. V is self-financing implies that

$$dV_t = a_t dS_t + b_t d\beta_t = [a_t r S_t + b_t r \beta_t]\, dt + a_t \sigma S_t dW_t^{\mathbb{Q}}$$

[The above part is unnecessary] By Ito's Lemma,

$$\begin{aligned} d(e^{-rt}V_t) &= -re^{-rt}V_t dt + e^{-rt}dV_t \\ &= -re^{-rt}(a_t S_t + b_t \beta_t)dt + e^{-rt}(a_t dS_t + b_t d\beta_t) \\ &= -re^{-rt}a_t S_t dt + e^{-rt}a_t \underbrace{dS_t}_{rS_t dt + \sigma S_t dW_t^{\mathbb{Q}}} \\ &= a_t \sigma e^{-rt} S_t dW_t^{\mathbb{Q}} \end{aligned}$$

If X_T is attainable then there exists (a, b) such that $V_T = X_T$ so

$$E^{\mathbb{Q}}\left[\frac{X_T}{\beta_T}\right] = E^{\mathbb{Q}}\left[\frac{V_T}{\beta_T}\right] = \frac{V_0}{\beta_0} = V_0$$

since $\frac{X_T}{\beta_T}$ is a martingale and the price of X_T is arbitrage free.

Definition 5.3. An arbitrage-free market is said to be **complete** if every adapted cash flow stream can be replicated by some trading strategy (not necessarily self-financing).

Theorem 5.2. *An arbitrage-free model is complete if and only if there exists a unique risk-neutral probability measure $\mathbb{Q}$.*

5.2 Girsanov's Theorem

Definition 5.4. The **Radon-Nikodym derivative** of a probability measure $\mathbb{Q}$ with respect to $\mathbb{P}$ is a random variable $\frac{d\mathbb{Q}}{d\mathbb{P}}$ defined implicitly by

$$E^{\mathbb{Q}}(X) = E^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}X\right]$$

Theorem 5.3. *(Cameron-Martin-Girsanov Theorem) Let $W = \{W_t, 0 \le t\}$ be a $\mathbb{P}$ standard Brownian motion and let θ_t be a (bounded) adapted process such that*

$$E^{\mathbb{P}}\left[e^{\frac{1}{2}\int_0^T \theta_t^2 dt}\right] < \infty$$

Then there exists a measure $\mathbb{Q}$ such that

1) $\mathbb{Q}$ is equivalent to $\mathbb{P}$

2) $\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \theta_t dW_t - \frac{1}{2}\int_0^T \theta_t^2 dt\right)$

2) $Z_t = W_t + \int_0^t \theta_s ds$ is a Brownian motion for $0 \le t \le T$

Remark 5.2. To convert between the real world measure $\mathbb{P}$ and the risk-neutral measure $\mathbb{Q}$, note that

$$\begin{aligned} dS_t = \mu S_t dt + \sigma S_t dW_t^{\mathbb{P}} &= rS_t dt + (\mu - r)S_t dt + \sigma S_t dW_t^{\mathbb{P}} \\ &= rS_t dt + +\sigma S_t \left[\frac{\mu - r}{\sigma} dt + dW_t^{\mathbb{P}}\right] \\ &= rS_t dt + +\sigma S_t d\left[\frac{\mu - r}{\sigma} t + W_t^{\mathbb{P}}\right] \\ &= rS_t dt + +\sigma S_t d\left[\underbrace{\int_0^t \frac{\mu - r}{\sigma} ds}_{\theta} + W_t^{\mathbb{P}}\right] \end{aligned}$$

Girsanov tells us that $\exists$ a $\mathbb{Q}$ measure such that (from (3) of theorem), $W_t^{\mathbb{Q}} = W_t^{\mathbb{P}} + \int_0^t \frac{\mu-r}{\sigma} ds$ and $W_t^{\mathbb{Q}}$ is a $\mathbb{Q}$-standard B-M.

Example 5.7. Suppose that $W_t \sim N(0,t)$ under $\mathbb{P}$. Given that $\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\alpha W_t - \frac{1}{2}\alpha^2 t}$, find $E^{\mathbb{Q}}\left[e^{u \cdot Z_t}\right]$ where $Z_t = W_t + \int_0^t \alpha dt = W_t + \alpha t$.

Solution. By Girsanov, we should arrive at the conclusion that $Z_t \sim N(0,t)$. We have

$$\begin{aligned} E^{\mathbb{Q}}\left[e^{uZ_t}\right] &= E\left[e^{uZ_t} \frac{d\mathbb{Q}}{d\mathbb{P}}\right] \\ &= E\left[e^{u(W_t + \alpha t)} e^{-\alpha W_t - \frac{1}{2}\alpha^2 t}\right] \\ &= e^{u\alpha t - \frac{1}{2}\alpha^2 t} E\left[e^{(u-\alpha)W_t}\right] \\ &= e^{u\alpha t - \frac{1}{2}\alpha^2 t} e^{\frac{1}{2}t(u-\alpha)^2} \\ &= e^{\frac{1}{2}tu^2} \end{aligned}$$

using the MGF of a $N(\mu, \sigma^2)$ which is $\exp(\mu \cdot u + \frac{1}{2}\sigma^2 u^2)$. So Z is a standard BM under $\mathbb{Q}$.

Example 5.8. (SEE A3 for a better version) Consider the simple case of a 3 period binomial tree. The option will pay, at time 3, a payoff of

$$\max\left(\sqrt[4]{S_0 S_1 S_2 S_3} - K, 0\right)$$

To price this, we first find the distribution of

$$Y_n = \sqrt[n+1]{S_0 S_{\frac{T}{n}} S_{\frac{2T}{n}} \ldots S_{\frac{nT}{n}}}$$

under the B-S framework. In general,

$$\begin{aligned} S_t = S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t} &\implies \ln S_t = \ln S_0 + \left[\left(r - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right] \\ &\implies Y_n = S_0 \exp\left(\frac{1}{n+1}\left[\left(r - \frac{1}{2}\sigma^2\right) \underbrace{\sum_{k=1}^{n} \frac{k \cdot T}{n}}_{\frac{T(n+1)}{2}} + \sigma\left(\sum_{k=1}^{n} W_{\frac{kT}{n}}\right)\right]\right) \\ &\implies Y_n = S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)\frac{T}{2} + \sigma\left(\sum_{k=1}^{n} W_{\frac{kT}{n}}\right)/n+1\right) \\ &\implies \ln Y_n \sim N\left(\ln S_0 + \left(r - \frac{1}{2}\sigma^2\right)\frac{T}{2}, Var(\ln Y_n)\right) \end{aligned}$$

Now in general,

$$\begin{aligned} Var\left(\sum_{k=1}^{n} W_{\frac{kT}{n}}\right) &= nZ_1 + (n-1)Z_2 + ... + 2Z_{n-1} + Z_n, \{Z_t\} \sim \text{iid } N\left(0, \frac{T}{n}\right) \\ &= \frac{T}{n}\sum_{i=1}^{n} i^2 = \frac{T}{n} \cdot \frac{n(2n+1)(n+1)}{6} = \frac{T(2n+1)(n+1)}{6} \end{aligned}$$

Hence

$$\ln Y_n \sim N\left(\ln S_0 + \left(r - \frac{1}{2}\sigma^2\right)\frac{T}{2}, \frac{\sigma^2 T(2n+1)}{6(n+1)}\right)$$

Taking $n \to \infty$ and setting $Y = \lim_{n\to\infty} Y_n$, we get that

$$\ln Y_n \to \ln Y \sim N\left(\ln S_0 + \left(r - \frac{1}{2}\sigma^2\right)\frac{T}{2}, \frac{\sigma^2 T}{3}\right)$$

Next, we proceed to find the price of a security which pays $\max(Y-K,0)$ at time T (European Asian Call Option). We know the price of a security which pays $\max(S_T - K^*, 0)$ at time T. The idea here is to consider $\max(Y-K,0)$ and write it in such a way that we have it in the form of a call's payoff. In particular, write $\max(Y-K,0)$ as $\max(e^A S_T - K, 0)$ where A to be determined. Then

$$\max(e^A S_T - K, 0) = e^A \max(S_T - K^*, 0), K^* = Ke^{-A}$$

Let $\sigma_*^2 = \frac{\sigma^2}{3}$ which will be the variance of some stock S^*. So we should have

$$\ln S^* \sim N\left(\ln S_0 + \left(r - \frac{1}{2}\sigma^2\right)\frac{T}{2}, \sigma_*^2 T\right)$$

We can write Y in the form

$$Y = S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)\frac{T}{2} + \sigma_* W_T}$$

and we find A such that

$$Y = S_0 e^{A + \left(r - \frac{1}{2}\sigma^2\right)T + \sigma_* W_T} = e^A S_T^*$$

Combining the two equations above gives us

$$A + \left(r - \frac{1}{2}\sigma^2\right)T = \left(r - \frac{1}{2}\sigma^2\right)\frac{T}{2} \implies A = -\left(r - \frac{1}{2}\sigma^2\right)\frac{T}{2}$$

Then the payoff is

$$\max(Y-K,0) = \max(e^A S_T^* - K, 0) = e^A \max(S_T^* - Ke^{-A}, 0)$$

This is the payoff of e^A units of a call option on S^* with strike $K^* = Ke^{-A}$. Based on the B-S formula for the call price, the price of this option is

$$e^A \left[S_0 N(d_1) - K^* N(d_2)\right] = S_0 e^A N(d_1) - KN(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{S_0}{K^*}\right) + \left(r + \frac{1}{2}\sigma_*^2\right)T}{\sigma_*\sqrt{T}} \\ d_2 &= d_1 - \sigma_*\sqrt{T} \end{aligned}$$

Example 5.9. (Gap Option) Find the price of a security which pays at time T, $S_T - K_1$ where $S_T > K_2$ and 0 otherwise. We have that the payoff is

$$\begin{aligned}\text{Payoff} &= \begin{cases} S_T - K_1 & S_T > K_2 \\ 0 & S_T \le K_2 \end{cases} \\ &= \begin{cases} S_T - K_2 + (K_2 - K_1) & S_T > K_2 \\ 0 & S_T \le K_2 \end{cases} \\ &= \max(S_T - K_2, 0) + (K_2 - K_1)\chi_{\{S_T > K_2\}}\end{aligned}$$

So the price is

$$E^{\mathbb{Q}}\left[e^{-rT}\max(S_T - K_2, 0)\right] + E^{\mathbb{Q}}\left[e^{-rT}(K_2 - K_1)\chi_{\{S_T > K_2\}}\right] = e^{-rT}(K_2 - K_1)\cdot\mathbb{Q}(S_T > K) + \left[S_0 N(d_1) - K_2 e^{-rT} N(d_2)\right]$$

where

$$d_1 = \frac{\ln\left(\frac{S_0}{K_2}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, d_2 = d_1 - \sigma\sqrt{T}$$

Now

$$\mathbb{Q}(S_T > K_2) = P\left(N(0,1) > \frac{\ln\left(\frac{S_0}{K_2}\right) - \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) = 1 - N\left(\frac{\ln\left(\frac{S_0}{K_2}\right) - \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) = 1 - N(-d_2) = N(d_2)$$

Hence the price is

$$S_0 N(d_1) - K_2 e^{-rT} N(d_2) + (K_2 - K_1) e^{-rT} N(d_2) = S_0 N(d_1) - K_1 e^{-rT} N(d_2)$$

5.3 Estimating Volatility

There are several ways to do this:

- Method 1: Use Historical Volatility
  - Under the B-S Model, recall that $\epsilon_i = \ln\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)$ has variance $\sigma^2(t_i - t_{i-1})$
  - Sample $n+1$ stock prices in the past such that $t_i - t_{i-1} = \Delta t$, where we sample at regular intervals. Let our data be $\epsilon_1, \epsilon_2, ..., \epsilon_n$ where $Var(\epsilon_i) = \sigma^2 \Delta t$.
  - An estimator of σ is

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n}(\epsilon_i - \bar{\epsilon})^2}{n-1}}$$

- Method 2: Use Implied Volatility
  - Find a "benchmark" option on the same underlying asset in the market. Suppose the price is P. Under the B-S mode, suppose the "theoretical" price is $C(\sigma, t, S_t)$. Set $P = C(\sigma, t, S_t)$ and solve for σ. The estimate $\hat{\sigma}$ is called the **implied volatility**.
  - Plotting the strike against the implied volatility gives a concave up curve, also known as a **volatility smile**.

5.4 Greeks

The Greeks are simply partial derivatives of the option's value with respect to the model's parameters. In particular,

$$\text{Delta} = \frac{\partial V(t, S_t)}{\partial S_t}, \text{Delta} = \frac{\partial^2 V(t, S_t)}{\partial S_t^2}$$

where $V(t, S_t)$ is the price of some derivative on an underlying asset S.

Example 5.10. Given that

$$C(t,S_t)+Ke^{-r(T-t)}=P(t,S_t)+S_t$$

derive the delta of a put option. Taking the derivative with respect to S_t gives us

$$\frac{\partial C}{\partial S_t}+0=\frac{\partial P}{\partial S_t}+1 \implies \Delta_p=N(d_1)-1$$

and additionally, if we take second partials, we find that $\Gamma_c=\Gamma_p$.

Definition 5.5. When it comes to mis-specification of parameters, we also have Greeks to measure the sensitivity of our price to mis-speicification. The following are some other Greeks:

$$\begin{aligned}
\text{Rho} &= \rho &&= \frac{\partial V}{\partial r}=K(T-t)e^{-r(T-t)}N(d_2)\\
\text{Theta} &= \theta &&= \frac{\partial V}{\partial t}=\frac{-S_t\phi(d_1)\sigma}{2\sqrt{T-t}}-rKe^{-r(T-t)}N(d_2)\\
\text{Vega} &= \nu &&= \frac{\partial V}{\partial \sigma}=S_t\phi(d_1)\sqrt{T-t}
\end{aligned}$$

Example 5.11. For a PutOnCall,

$$\begin{aligned}
P(0) &= e^{-rT}E^{\mathbb{Q}}\left[E\left[\max\left(L-C(S_{T_1},T_1,T_2,K),0\right)\middle|\mathcal{F}_{T_1}\right]\right]\\
&= e^{-rT_1}E^{\mathbb{Q}}\left[E\left[\chi_{\{S_{T_1}<s^*\}}\left(C(s^*,T_1,T_2,K)-C(S_{T_1},T_1,T_2,K)\right)\middle|\mathcal{F}_{T_1}\right]\right]\\
&= e^{-rT_1}E^{\mathbb{Q}}\left[E\left[\chi_{\{S_{T_2}<S_{T_2^*}\}}\left((S_{T_2^*}-K)^+-(S_{T_2}-K)^+\right)\middle|\mathcal{F}_{T_1}\right]\right]\\
&= e^{-rT_1}E^{\mathbb{Q}}\left[\chi_{\{S_{T_2}<S_{T_2^*}\}}\left((S_{T_2^*}-K)^+-(S_{T_2}-K)^+\right)\right]\\
&= \ldots
\end{aligned}$$

The parity of these compound options is

$$\underbrace{\text{CallOnCall}}_{e^{-rT_1}\max(C-L,0)}+Le^{-rT_1}=\underbrace{\text{PutOnCall}}_{e^{-rT_1}\max(L-C,0)}+\underbrace{C(S_0,0,T_2,K)}_{e^{-rT_2}\max(S_T-K,0)}$$

$$\begin{aligned}
&\implies e^{-rT_1}\left[\max(C-L,0)-\max(L-C,0)\right]+Le^{-rT_1}=e^{-rT_2}\max(S_{T_2}-K,0)\\
&\implies e^{-rT_1}\left[C-L\right]+Le^{-rT_1}=e^{-rT_2}\max(S_{T_2}-K,0)\\
&\implies e^{-rT_1}C=e^{-rT_1}\max(S_{T_2}-K,0)
\end{aligned}$$

Example 5.12. The parity equation for knock-out option is

Knock-out +Knock-In = Ordinary Option

5.5 Examples

The following are problems from **Tutorial 9**:

1. a) We have

$$\text{Payoff}=\max(S_T,G)=G+\max(S_T-G,0)$$

and applying the B-S formula, on the latter term which is a call with strike G and maturity T, gives us

$$\begin{aligned}
\text{Price} &= E^{\mathbb{Q}}\left[\{G+\max(S_T-G,0)\}e^{-rT}\right]\\
&= Ge^{-rT}+E^{\mathbb{Q}}\left[e^{-rT}\max(S_T-G,0)\right]\\
&= Ge^{-rT}+\left[S_0N(d_1)-Ge^{-rT}N(d_2)\right]\\
&= S_0N(d_1)+Ge^{-rT}\left[1-N(d_2)\right]\\
&= S_0N(d_1)+Ge^{-rT}N(-d_2)
\end{aligned}$$

where $d_1 = \left[\ln(S_0/G) + (r + \frac{1}{2}\sigma^2)T\right] / \left[\sigma\sqrt{T}\right]$ and $d_2 = d_1 - \sigma\sqrt{T}$.

b) We have

$$\begin{aligned} \text{Payoff} &= 10 \cdot I_{\{S_T > 110\}} + 2 \cdot I_{\{S_T \leq 110\}} \\ &= 2 + 8 \cdot I_{\{S_T > 110\}} \end{aligned}$$

and the price is

$$\begin{aligned} \text{Price} &= E^{\mathbb{Q}}\left[e^{-rT}\{2 + 8 \cdot I_{\{S_T > 110\}}\}\right] \\ &= 2e^{-rT} + 8e^{-rT}\mathbb{Q}(S_T > 110) \\ &= 2e^{-rT} + 8e^{-rT}\left[1 - \mathbb{Q}(S_T \leq 110)\right] \\ &= 2e^{-rT} + 8e^{-rT}\left[1 - N\left(\frac{\ln\left(\frac{110}{S_0}\right) - \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right] \end{aligned}$$

since $\ln S_T \sim N\left(\ln S_0 + \left(r - \frac{1}{2}\sigma^2\right)T, \sigma^2 T\right)$.

2. *Aside.* Suppose that $dS_{1,t} = \alpha_1 S_{1,t}dt + \sigma_1 S_{1,t}dW_t$ and we introduce another stock S_2 such that $dS_{2,t} = \alpha_2 S_{2,t}dt + \sigma_2 S_{2,t}dW_t$. Show that in order to avoid arbitrage, we must have

$$\frac{\alpha_1 - r}{\sigma_1} = \frac{\alpha_2 - r}{\sigma_2}$$

To see this, note that

$$\begin{aligned} V_1 = S_2 - \frac{\sigma_2 S_2}{\sigma_1 S_1} \cdot S_1 = \left(1 - \frac{\sigma_2}{\sigma_1}\right) S_2 \quad &\Longrightarrow \quad dV = dS_2 - \frac{\sigma_2 S_2}{\sigma_1 S_1} \cdot dS_1 \\ &\Longrightarrow \quad dV = \alpha_2 S_2 dt + \sigma_2 S_2 dW_t - \left(\frac{\sigma_2 S_2}{\sigma_1 S_1}\right)\left[\alpha_1 S_1 dt + \sigma_1 S_1 dW_t\right] \\ &\Longrightarrow \quad (1)\ dV = S_2\left(\alpha_2 - \frac{\alpha_1\sigma_2}{\sigma_1}\right)dt \end{aligned}$$

where V is a riskless portfolio. For such a porfolio, we must have

$$(2)\ dV = rV\ dt = r\left[1 - \frac{\sigma_2}{\sigma_1}\right] S_2 dt$$

We must have (1) = (2) and so

$$r\left[1 - \frac{\sigma_2}{\sigma_1}\right] = \left[\alpha_2 - \frac{\alpha_1\sigma_2}{\sigma_1}\right] \Longrightarrow \frac{\alpha_2 - r}{\sigma_2} = \frac{\alpha_1 - r}{\sigma_1}$$

Question 2 is just an application of this concept.

3. a) Given $S_3 = S_1^a S_2^b$, we have

$$\frac{\partial S_3}{\partial t} = 0, \frac{\partial S_3}{\partial S_1} = aS_1^{a-1}S_2^b = \frac{aS_3}{S_1}, \frac{\partial S_3}{\partial S_2} = bS_1^a S_2^{b-1} = \frac{bS_3}{S_2}$$

$$\frac{\partial^2 S_3}{\partial S_1^2} = \frac{a(a-1)S_3}{S_1^2}, \frac{\partial^2 S_3}{\partial S_2^2} = \frac{b(b-1)S_3}{S_2^2}, \frac{\partial^2 S_3}{\partial S_2 S_1} = \frac{abS_3}{S_1 S_2}$$

Substitute to get

$$\begin{aligned} dS_3 &= aS_3\frac{dS_1}{S_1} + bS_3\frac{dS_2}{S_2} + \frac{1}{2}a(a-1)S_3\frac{(dS_1)^2}{S_1^2} + \frac{1}{2}b(b-1)S_3\frac{(dS_2)^2}{S_2^2} + \frac{1}{2}abS_3\frac{(dS_1)(dS_2)}{S_1 S_2} \\ &= S_3\left[a\alpha_1 + b\alpha_2 + \frac{1}{2}a(a-1)\sigma_1^2 + \frac{1}{2}b(b-1)\sigma_2^2\right]dt + a\sigma_1 dZ_1 + b\sigma_2 dZ_2 \end{aligned}$$

b) $S_1(0) = x_1, S_2(0) = x_2$ gives

$$S_3(0) = [S_1(0)]^a [S_2(0)]^b = x_1^a x_2^b$$

c) Method 1: Use Ito's Lemma on $Y_t = \ln S_{3,t}$ to get

$$dY_t = \left(a\alpha_1 + b\alpha_2 - \frac{\alpha}{2}\sigma_1^2 - \frac{b}{2}\sigma_2^2 \right) dt + a\sigma_1 dZ_1 + b\sigma_2 dZ_2$$

and hence

$$S_3 = S_3(0) e^{\left(a\alpha_1 + b\alpha_2 - \frac{\alpha}{2}\sigma_1^2 - \frac{b}{2}\sigma_2^2\right)dt + a\sigma_1 dZ_1 + b\sigma_2 dZ_2}$$

6 Fixed Income Securities

We cover interest rate models and bond pricing here.

6.1 Interest Rate Models

The following models are for a short rate r_t.

1. Lognormal Model:

$$dr_t = \mu r_t \, dt + \sigma r_t \, dW_t \implies r_t = r_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}, t \geq 0$$

   (a) Advantage: Positive interest rates.
   (b) Disadvantage: No mean reverting feature. Possible that interest rates can become large.

2. Vasicek Model:

$$dr_t = a(b - r_t)dt + \sigma \, dW_t$$

   (a) Advantage: Mean-reverting feature.
   (b) Disadvantage: Interest rates can become negative.

3. CIR (Cox-Ingersoll Ross) Model:

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t} \, dW_t$$

   (a) Advantage: Mean reversion feature. Positive rates (can show that r_t has a non-central χ^2 distribution).

6.2 Bond Pricing

Consider a T-year zero coupon bond with face value \$1. Our objective is find B_0, the time 0 price of the bond. If r_t is continuously compounded, we can compute

$$B_0 = E\left[e^{-\int_0^T r_t \, dt} \right]$$

Assume the Vasicek model for the short rate process $\{r_t, t \geq 0\}$ where $dr_t = a(b - r_t)dt + \sigma \, dW_t$. We then solve for r_t using Ito's Lemma using $Y_t = f(r_t, t) = e^{at} r_t$ to get

$$dY_t = abe^{at}dt + \sigma e^{at} dW_t$$

In integral form

$$Y_t - Y_0 = ab \underbrace{\int_0^t e^{au} du}_{b(e^{at}-1)} + \sigma \int_0^t e^{au} dW_u \implies r_t = r_0 e^{-at} + b(1 - e^{-at}) + \sigma e^{-at} \int_0^t e^{au} dW_u$$

Since the term inside the Ito integral is deterministic, then r_t is normally distributed. Now

$$\begin{aligned} E[r_t] &= r_0 e^{-at} + b(1-e^{-at}) + \sigma e^{-at} \underbrace{E\left[\int_0^t e^{au} dW_u\right]}_{=0} \\ &= r_0 e^{-at} + b(1-e^{-at}) \end{aligned}$$

and also

$$\begin{aligned} Var(r_t) &= \sigma^2 e^{-2at} E\left[\left(\int_0^t e^{au} dW_u\right)^2\right] \\ &= \sigma^2 e^{-2at} E\left[\int_0^t e^{2au} du\right] \\ &= \frac{\sigma^2}{2a}\left[1-e^{-2at}\right] \end{aligned}$$

since the first moment is zero and by the isometric property. Now look at $X_T = \int_0^T r_t \, dt$. Since r_t is normally distributed, X_T is also normally distributed. So

$$B_0 = E\left[e^{-\int_0^T r_t dt}\right] = E\left[e^{-X_T}\right] = e^{-E[X_T] + \frac{1}{2} Var(X_T)}$$

Now

$$\begin{aligned} E[X_T] = E\left[\int_0^T r_t dt\right] = \int_0^T E[r_t] dt &= \int_0^T \left[(r_0 - b)e^{-at} + b\right] dt \\ &= \frac{r_0 - b}{a}\left[1 - e^{-aT}\right] + b \cdot T \end{aligned}$$

and

$$Var(X_T) = Var\left(\int_0^T r_t dt\right) = Cov\left(\int_0^T r_t dt, \int_0^T r_u du\right) = \int_0^T \int_0^T Cov(r_t, r_u) \, dt \, du$$

Calculating $Cov(r_t, r_u)$ gives us

$$\begin{aligned} Cov(r_t, r_u) &= Cov\left(\sigma e^{-at} \int_0^t e^{-as} dW_s, \sigma e^{-au} \int_0^u e^{-as} dW_s\right) \\ &= \sigma^2 e^{-2a(u+t)} Cov\left(\int_0^t e^{-as} dW_s, \int_0^u e^{-as} dW_s\right) \end{aligned}$$

Suppose that $u < t$. Then

$$\int_0^t e^{-as} dW_s = \int_0^u e^{-as} dW_s + \int_u^t e^{-as} dW_s$$

Since $\int_0^u e^{-as} dW_s$ is independent of $\int_u^t e^{-as} dW_s$, by the independent increment assumption, then

$$\begin{aligned} Cov\left(\int_0^t e^{-as} dW_s, \int_0^u e^{-as} dW_s\right) &= Var\left(\int_0^u e^{-as} dW_s\right) \\ &= E\left(\left(\int_0^u e^{-as} dW_s\right)^2\right) \\ &= \int_0^u e^{-2as} ds = \frac{1 - e^{-2au}}{2a} \end{aligned}$$

In general, for any u, t,

$$Cov(r_t, r_u) = \sigma^2 e^{-a(t+u)} \left(\frac{1 - e^{-2a \cdot \min(u,t)}}{2a}\right)$$

Now

$$
\begin{aligned}
Var(X_T) = \int_0^T \int_0^T Cov(r_t, r_u) dt\, du &= \int_0^T \int_0^u Cov(r_t, r_u) dt\, du + \int_0^T \int_u^T Cov(r_t, r_u) dt\, du \\
&= \frac{\sigma^2}{2a} \int_0^T \int_0^u \left(e^{a(t-u)} - e^{-a(t+u)} \right) dt\, du + \frac{\sigma^2}{2a} \int_0^T \int_u^T \left(e^{-a(t-u)} - e^{-a(t+u)} \right) dt\, du \\
&= \frac{\sigma^2}{2a^3} \left[2aT - 3 + 4e^{-aT} - e^{-2aT} \right]
\end{aligned}
$$

Then, once more,

$$
B_0 = e^{-E(X_T) + \frac{1}{2} Var(X_T)}
$$

where $E(X_T)$ and $Var(X_T)$ are given above. In the more general form, the ZCB issued at time t and maturing at time T is

$$
B(t, T, r_t) = e^{-A(t,T) r_t + D(t,T)}
$$

where

$$
\begin{aligned}
A(t, T) &= \frac{1 - e^{-a(T-t)}}{a} \\
D(t, T) &= \left(b - \frac{\sigma^2}{2a^2} \right) [A(t, T) - (T - t)] - \frac{\sigma^2 A(t, T)}{4a}
\end{aligned}
$$

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