ACTSC 445 (Winter 2013 - 1135)

Asset-Liability Management

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These notes are currently a work in progress, and as such may be incomplete or contain errors.

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Abstract

The purpose of these notes is to provide a secondary reference for students enrolled ACTSC 445. The official prerequisite to this course is STAT 330 and STAT 333 (or STAT 334), ACTSC 371, and ACTSC 231. However, this author believes that any student who has taken STAT 330 and ACTSC 231 will be more than prepared. While this course may seem easy at the beginning, do not be fooled. Near the immunization chapter, the content becomes fairly difficult and the exam drastically increase in difficulty.

Everything being said, though, this course is very well put together and counts towards the CIA accreditation program for exam MFE, so hence this author recommends that the reader spend a good portion of their total study and homework hours dedicated to this course.

1 Fixed Income Securities

We first define a few conventions.

Canadian convention:

$$F = P(1 + r_c \frac{n}{365})$$

US convention:

$$P = F(1 - r_D \frac{n}{360})$$

Throughout this course, we will be assuming the Canadian convention.

1.1 Forward and Spot Rates

We let $P(0, k)$ denote the price of a \$1 zero coupon bond maturing in k periods where $P(0, k) = (1 + s_k)^{-k}$ and s_k is the annualized *spot rate* for k periods (annualized rate over $[0, k]$).

We let $f_k = f_{k,k+1}$ denote the *forward rate* over $(k, k+1)$ which is the rate agreed upon today to borrow over $(k, k+1)$. For example, $f_0 = f_{0,1} = s_1$ and $f_1 = f_{1,2}$ is such that $(1 + s_2)^2 = (1 + f_0)(1 + f_1)$.

Finally, $r_k = r_{k.k+1}$ is known as the *short rate* or actual rate over the period $(k, k+1)$. The forward rate is the implied short rate that is derived from the spot rates.

For zero coupon bonds, the spot rate is just the *yield to maturity*. For coupon paying bond with face value F, price P, coupon rate c, and yield to maturity (ytm) y, to solve for a spot rate we need to solve the equation

$$P = \sum_{i=1}^{n-1} (Fc)(1 + s_i)^{-i} + (Fc + F)(1 + s_n)$$

if we are already given the other $n - 1$ spot rates. For a forward rate f_k, we solve the equation

$$(1 + s_k)^k (1 + f_k) = (1 + s_{k+1})^{k+1}$$

The yield of an investment is a constant rate that is the annual effective return of an investment. The yield is also the rate of reinvestment of a security at time 0.

At the end of the period of an investment, the realized rate return is the yield gained between the future values of the price calculated with the short rates and the price calculated with the forward rates. For example, an annuity paying k for 3 periods would have realized return

$$y^* = \frac{[k + k(1 + r_2) + k(1 + r_1)(1 + r_2)] - [k + k(1 + f_2) + k(1 + f_1)(1 + f_2)]}{[k + k(1 + f_2) + k(1 + f_1)(1 + f_2)]}$$

1.2 Bond Pricing

Given a bond with n coupons remaining, c and y are effective rates over the period between coupon payments, the price is given by

$$Price = \sum_{i=1}^{n} cF(1 + y)^{-i} + C(1 + y)^{-n}$$

<u>Pricing Between 2 Coupon Payment Dates</u>

Let D be the number of days between coupon payments and suppose d days have passed since the last coupon payment. At this time, the (dirty) price is

$$Dirty\ Price = P_{Dirty} = \left[\sum_{i=1}^{n} cF(1 + y)^{-i} + C(1 + y)^{-n} \right] \times (1 + y)^{\frac{d}{D}}$$

and the clean price is defined as

$$P_{Clean} = P_{Dirty} - cF\left(\frac{d}{D}\right) \quad \text{(Semi-theoretical Approach)}$$

Here, the *dirty price* is the actual/full price while the *clean/flat price* is the quoted price.

As a side note, the exact clean price is

$$P_{Clean} = P_{Dirty} - cF\left(\frac{(1+y)^{\frac{d}{D}} - 1}{y}\right)$$

Example 1.1. Consider a bond with price as at August 1, 2005. It's a 2-year 4% bond with a face value of \$100 and semi-annual coupons. The yield to maturity is 5%. The price at issue must be

$$P = 100(\frac{4\%}{2})\sum_{i=1}^{4}(1+\frac{5\%}{2})^{-i} + 100\left(1+\frac{5\%}{2}\right)^{-4} = \$98.12$$

We now calculate the full (dirty) and quoted (clean) price of the bond as of October 2, 2005 using the semi-theoretical method. Note that the next coupon date is February 1, 2006 $\implies D = 184$ days and from August 1, 2005 to October 2, 2005 we have $d = 62$ days. Thus the full price is

$$98.12\left(1+\frac{5\%}{2}\right)^{\frac{62}{184}} = \$98.94$$

and the quoted price is

$$98.94 - \frac{4\%}{2}(100)\left(\frac{62}{184}\right) = \$98.26$$

1.3 FIS Risks

Risks Associated with a FIS

(See course notes for detailed explanations)

These are: Interest Rate Risk, Reinvestment Risk, Credit Risk, Timing/Call Risk, Inflation Risk, Exchange-rate Risk, Volatility Risk

Interest Rate Risk

We first discuss how changes in the yield y changes a bond's price $P(y)$.

- For $\triangle y$ small, $|P(y+\triangle y) - P(y)| \approx |P(y-\triangle y) - P(y)|$
- For $\triangle y$ large, $|P(y+\triangle y) - P(y)| < |P(y-\triangle y) - P(y)|$
- The higher the original y the lower the interest rate risk (due to convexity)
- The higher the coupon rate, the lower the interest rate risk (more principal is paid off earlier)
- Note that the derivative of the price of a bond with respect to y is negative while the second derivative is positive (convex). Also take note of the following ordering:
- Premium bond < Par bond < Discount bond < Zero-coupon bond

1.4 Embedded Options

Call provision

Issuer (borrower) has the option to repay the redemption value at some time before maturity. Usually exercised when interest rates are low by the issuer (issuer calls and buys a bond with a lower coupon rate due to lower interest rates) and creates risks for the investor. Thus, the option lowers the price when compared to an otherwise equal vanilla bond.

Put provision

Investor (lender) has the option to sell back the bond (at par value) at some time before maturity. Usually exercised when interest rates are high by the investor (investor sells back the bond - take back the face value - and buys another one at a higher interest rate) and creates risks for the issuer. Thus, the option raises the price when compared to an otherwise equal vanilla bond.

Would a callable bond be more or less sensitive than an otherwise non-callable bond?

Answer. We first note that

$$P(\text{Callable Bond}) = P(\text{Non-callable bond}) - P(\text{Call option})$$

and since both bonds tend to move in the same direction with the callable bond asymptotically approaching the vanilla bond as interest rates rise along with the canceling effect of the two terms on the right, it is less sensitive.

What if we had a putable bond?

Answer. By the same reasoning

$$P(\text{Putable Bond}) = P(\text{Non-putable bond}) + P(\text{Put option})$$

and so it is relatively the same but slightly more sensitive close to the origin and less sensitive far away from it.

2 Duration and Convexity

In this unit, we explore several variants of duration an convexity.

2.1 Macaulay and Modified Duration

Assume that we are given a sequence of cash flows $\{A_{t_i}\}$ for times $\{t_i\}$ where $\{A_t, t > 0\}$ does not depend on y, the yield rate. The PV of these cash flows is

$$A(y) = \sum_{t>0} A_t(1+y)^{-t}$$

Suppose that y^* is the current ytm, The price is then

$$A^* = A^*(y) = \sum_{t>0} A_t(1+y^*)^{-t}$$

Consider an instantaneous change in the yield rate. Then $y^* \to y^* + \triangle y$ (parallel shift). The actual price change is

$$A(y^* + \triangle y) - A(y^*)$$

where the true price is $A(y^* + \triangle y)$. The approximate price change, using the second order Taylor expansion is

$$\begin{aligned} A(y^* + \triangle y) - A(y^*) &\approx A(y^*) + A'(y^*)\triangle y + A''(y^*)\frac{(\triangle y)^2}{2} - A(y^*) \\ &\approx A'(y^*)\triangle y + A''(y^*)\frac{(\triangle y)^2}{2} \end{aligned}$$

and for the first order expansion, we omit the seond term on the right. We define the *modified duration* for a FIS with cash flows $\{A_t, t > 0\}$ and current ytm y^* as

$$D_m = -\frac{A'(y^*)}{A(y^*)} = -\frac{1}{A(y^*)}\frac{d}{dy}A(y)\Big|_{y=y^*}$$

and so the first order approximation is

$$A(y^* + \triangle y) - A(y^*) \approx -D_m \cdot A(y^*) \cdot \triangle y \implies \frac{A(y^* + \triangle y) - A(y^*)}{A(y^*)} \approx -D_m \cdot \triangle y$$

which gives us an approximation for the % change in the price per unit change of yield rate y. For the FIS, $D_m = \sum_{t>0} t\frac{A_t(1+y^*)^{-t-1}}{A^*}$ where $A^* = A(y^*) = \sum_{t>0} A_t(1+y^*)^{-t}$. So

$$D_m = \sum_{t>0} t \cdot \frac{A_t(1+y^*)^{-t-1}}{\sum_{u>0} A_u(1+y^*)^{-u}} = \sum_{t>0} t \cdot \frac{A_t(1+y^*)^{-t-1}}{A^*}$$

and we can think of D_m as a "weighted" average of the times at which the cash flows are actually received. However, these "weights" are not truly weights in the sense that they do not sum to 1.We define the *Macaulay duration* D a as

$$D = \sum_{t>0} t \cdot \frac{A_t(1+y^*)^{-t}}{A^*} = (1+y^*)D_m$$

and note that the Macaulay duration of a zero-coupon bond (ZCB) maturing in T years is $D = T$. The Macaulay duration is a true/proper weight.

If we rewrite $A(y)$ in terms of a continuously compounded rate say $\delta = \ln(1+y)$. Then

$$A(\delta) = \sum_{t\geq 0} A_t e^{-\delta t} \implies D_{A(\delta^*)} = \frac{-A'(\delta^*)}{A^*}$$

A bond with Macaulay duration $D = T$ has the same sensitivity of a ZCB maturing in T years.

Example 2.1. Consider an investment that pays \$50 at times 1 and 2 (in years). If the interest is 5% compounded semi-annually, find the Macaulay duration.

(Method 1) The annual effective rate is

$$y = \left(1 + \frac{0.05}{2}\right)^2 - 1$$

and

$$\begin{aligned} A(y) &= 50(1+y)^{-1} + 50(1+y)^{-2} \\ A'(y) &= -50(1)(1+y)^{-2} - 50(2)(1+y)^{-3} \end{aligned}$$

with the Macaulay duration equal to

$$D = -(1+y)\frac{A'(y)}{A(y)} = 1.49\ years$$

(Method 2) We use the semi-annual rate. Thus,

$$\begin{aligned} A(y^{(2)}) &= 50\left(1 + \frac{y^{(2)}}{2}\right)^{-2} + 50\left(1 + \frac{y^{(2)}}{2}\right)^{-4} \\ A'(y^{(2)}) &= -50(2)\left(1 + \frac{y^{(2)}}{2}\right)^{-3} - 50(4)\left(1 + \frac{y^{(2)}}{2}\right)^{-5} \end{aligned}$$

and again

$$D = -(1+y)\frac{A'(y)}{A(y)} = 1.49\ years$$

A third method is to calculate the continuously compounded rate $\delta = \ln\left[\left(1 + \frac{y^{(2)}}{2}\right)^2\right]$ and calculate the Macaulay duration directly by

$$D = \frac{-A'(\delta)}{A(\delta)} = 1.49\ years$$

We define the convexity of a FIS as

$$C = \frac{A''(y^*)}{A(y^*)} = \frac{1}{A(y^*)}\frac{d^2}{dy^2}A(y)\Big|_{y=y^*} = \frac{\sum_{t\geq 0} t(t+1)A_t(1+y^*)^{-t-2}}{A(y^*)}$$

and it improves the approximation in change in price

$$A(y^* + \triangle y) - A(y*) \approx -D_m \cdot A(y^*) \cdot \triangle y + C \cdot \frac{(\triangle y)^2}{2} \cdot A(y^*)$$

Example 2.2. Consider a 10 year 5% annual coupon bond with $F = 100$ and 4% ytm.

1) The bond price is

$$\sum_{i=1}^{10} 0.05(100)(1.04)^{-i} + 100(1.04)^{-10} = 5a_{\overline{10}|4\%} + 100(1.04)^{-10} = \$108.11$$

The duration is

$$D_m = \frac{-A'(4\%)}{A(4\%)} = \frac{5\sum_{t=1}^{10} t(1.04)^{-t-1} + 100(10)(1.04)^{-11}}{108.11} = 7.88$$

The convexity (by definition, as above) is 77.48

2) Compute the % change in price using modified duration, modified duration and convexity and recomputing the price. We present this using a summary table:

Method	-0.1%	+0.1%	-2%	+2%
Duration	0.788%	-0.788%	15.75%	-15.25%
Duration and Convexity	0.791%	-0.784%	17.301%	-14.202%
Actual Change	0.791%	-0.784%	17.424%	-14.31%

Example 2.3. Consider a portfolio of k securities with the price of 1 unit of asset i is A_i with duration (Macaulay or modified) or D_i, n_i the number of units of assets i. We assume (major assumption) that all assets have the same yield. Then we have the following measures:

(1) The value of a portfolio is $A(y) = \sum_{i=1}^{k} A_i n_i = A^*$

(2) The portfolio duration is $D_p = \sum_{i=1}^{k} w_i D_i, w_i = \frac{n_i A_i}{A^*}$

Example 2.4. Consider the following portfolio of two $F = 100$, 5 year bonds with 5% ytm.

	Bond 1	Bond 2
Coupon	10%	5%
Price	121.65	100
Macaulay Duration	4.25	4.55

The portfolio duration is

$$D_p = \frac{121.65}{221.65} \cdot 4.25 + \frac{100}{221.65} \cdot 4.55 = 4.38$$

and had the yields been different, the formula would only give an approximated duration.

2.2 Fisher-Weil (FW) Duration

This model assumes non-flat spot rate curves, with our spot rate being continuously compounded, and also assumes only parallel shifts. So let $s_t = t-$year continuously compounded spot rate. So the time zero value is

$$A(s_1, ..., s_n) = \sum_{t=1}^{n} A_t e^{-ts_t}$$

for any $\{s_1, ..., s_n\}$ continuously compounded spot rates and cash flows A_t. Suppose that $s_i \to s_i + \triangle s$. Then we can approximate A^* with

$$A^* = A(s_1 + \triangle s, ..., s_n + \triangle s) \approx A(s_1, ..., s_n) + \sum_{i=1}^{n} \frac{\partial A}{\partial s_i} \triangle s + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 A}{\partial s_i^2} (\triangle s)^2$$

and formally the *FW duration* is

$$D_{FW} = -\frac{1}{A^*}\sum_{i=1}^{n}\frac{\partial A}{\partial s_i}\Big|_{s_i=s_i^*} = \frac{1}{A^*}\sum_{t=1}^{n} tA_t e^{-t\cdot s_t^*}$$

and the *FW convexity* is

$$C_{FW} = \frac{1}{A^*}\sum_{i=1}^{n}\frac{\partial^2 A}{\partial s_i^2}\Big|_{s_i=s_i^*} = \frac{1}{A^*}\sum_{t=1}^{n} t^2 A_t e^{-t\cdot s_t^*}$$

and so the approximate % change in price is

$$-D_{FW}\triangle s + \frac{1}{2}C_{FW}(\triangle s)^2$$

2.3 Quasi-Modified Duration

The *quasi-modified duration* is a version of Fisher-Weil duration where we only have annual effective rates

$$D_Q = \frac{1}{A^*}\sum_{t>0} tA_t(1+s_t^*)^{-t-1}$$

Example 2.5. A security pays $50 at each of times $2, 4$, and 6. Suppose that $s_2 = 0.03, s_4 = 0.04$, and $s_6 = 0.07$. The price of this security is

$$\sum_{i=1}^{3} 50(1+s_{2i})^{-2i} = 123.1871$$

If $\triangle s = 0.01$, then the value of D_Q is

$$\begin{aligned} D_Q &= \frac{1}{A^*}\sum_{t>0} tA_t(1+s_t^*)^{-t-1} \\ &= \frac{50(2)(1.03)^{-3} + 50(4)(1.04)^{-5} + 50(6)(1.04)^{-7}}{A^*} \\ &= \frac{442.7245}{123.1871} = 3.5939 \end{aligned}$$

and we also have

$$\frac{\triangle A}{A^*} = -D_Q\triangle s = -0.035939$$

and the new price is

$$A^*[1 + (-0.035939)] = 118.7599$$

Aside. The true price in the above example is 118.87.

2.4 Partial Duration

This is a generalization of FW to non-parallel shifts in a non-flat yield curve. So suppose that $s_i^* \to s_i^* + \triangle s_i$. Then the *partial duration* with respect to s_t is

$$D_{m,t} = \begin{cases} \frac{1}{A^*} tA_t(1+s_t^*)^{-t-1} & \text{discrete case} \\ \frac{1}{A^*} tA_t e^{-ts_t^*} & \text{continuous case} \end{cases}$$

We can then create the following approximation

$$\frac{\triangle A}{A^*} \approx \sum_{k=1}^{n} -D_{m,k}\triangle s_k$$

Example 2.6. Let's redo the previous example using $\triangle s_2 = -0.01$, $\triangle s_4 = 0$ and $\triangle s_6 = 0.01$. We get

$$A^* = 123.1871$$

$$D_{m,2} = \frac{1}{A^*}2A_2(1+0.03)^{-3}, D_{m,6} = \frac{1}{A^*}6A_2(1+0.03)^{-7}$$
$$A_{New} = A^*(1 - D_{m,2}(-0.01) - D_{m,6}(0.01)) = ?$$

2.5 Effective Duration and Convexity

Using a central difference method, the *effective duration* and convexity are given by

$$D_m^e = \frac{A(y^* - \triangle y) - A(y^* + \triangle y)}{2A^* \triangle y}, C_m^e = \frac{A(y^* - \triangle y) - 2A^* + A(y^* + \triangle y)}{A^*(\triangle y)^2}$$

which are computed using finite difference methods. D_m^e is approximated using a central difference (approximation of the derivative at $y^* + \triangle y$ using $h = 2\triangle y$). C_m^e is approximated with a forward and backward difference (approximation of two derivatives A_1' and A_2' at $y^* + \triangle y$ and y^* respectively using $h = \triangle y$).

2.6 Key Rate Duration

(s. 19-22) Shift the 1st key rate by say 1bp (0.01%). Recompute the spot rate curve (using linear interpolation). Recompute the price $\tilde{A}_1$ of the FIS using the new spot rate curve. The *key rate duration* is

$$\tilde{D}_{m,1} = \frac{-(\tilde{A}_1 - A^*)}{\triangle A^*}$$

and redo the previous steps for other ley rates to find $\tilde{D}_{m,2}, ..., \tilde{D}_{m,n}$. Linear interpolation is done by

$$s_t = \begin{cases} \tilde{s}_{t_1} & t < t_1 \\ \frac{t_{k+1}-t}{t_{k+1}-t_1}\tilde{s}_{t_k} + \frac{t-t_k}{t_{k+1}-t_1}\tilde{s}_{t_{k+1}} & t_{k-1} < t < t_{k+1} \\ \tilde{s}_{t_{\tilde{n}}} & t > t_{\tilde{n}} \end{cases}$$

Example 2.7. You are given 3 annual effective key rates of 2%, 3% and 4% for 1 year, 3 years, and 5 years respectively.

(1) Use linear interpolation to construct the spot rate curve form 0 to 6 years (use 1 year periods). That is, $s_1 = 2\%$, $s_2 = 2.5\%$, $s_3 = 3\%$, $s_4 = 3.5\%$, $s_5 = 4\%$ and $s_6 = s_5 = 4\%$ (s. 20).

(2) Using this curve, compute the value of a 6-year bond with 4% annual coupons. Assume a face value of F =\$100. The price is

$$P = \sum_{t=1}^{6} 4(1+s_t)^{-t} + 100(1+s_6)^{-6} = 100.3556$$

(3) Estimate the key rate durations $\tilde{D}_{m,1}, \tilde{D}_{m,3}$ and $\tilde{D}_{m,5}$ using a +1bp shift.

(i) Find $\tilde{D}_{m,1}$: Shift $\tilde{s}_1$ by 1bp and compute the new spot rate curve. We then have $s_1 = 2.01\%$,$s_2 = 2.505\%$, $s_3 = 3\%$, $s_4 = 3.5\%$, $s_5 = s_6 = 4\%$. So the new price using the new curve is $P = 100.3548$. Also,

$$\tilde{D}_{m,1} = \frac{-(100.3548 - 100.3556)}{(0.0001)(100.3556)} = 0.0753$$

(ii) Find $\tilde{D}_{m,3}$: Using the same methods as above, we get

$$\tilde{D}_{m,3} = \frac{-(100.3535 - 100.3556)}{(0.0001)(100.3556)} = 0.2103$$

(iii) Find $\tilde{D}_{m,5}$: Again, as above

$$\tilde{D}_{m,5} = \frac{-(100.3059 - 100.3556)}{(0.0001)(100.3556)} = 4.9485$$

(4) Find the approximate % change in price if $\triangle \tilde{s}_1 = 1\%$, $\triangle \tilde{s}_3 = -1\%$ and $\triangle \tilde{s}_5 = 1\%$. This can be done by observing that

$$\frac{\triangle A}{A^*} \approx -\tilde{D}_{m,1} \cdot \triangle \tilde{s}_1 - \tilde{D}_{m,3} \cdot \triangle \tilde{s}_3 - \tilde{D}_{m,5} \cdot \triangle \tilde{s}_5 = 4.813\%$$

3 Immunization

Immunization is a risk management technique that ensures for any small change in the interest rate, a portfolio of fixed income securities (FIS) will cover future liabilities.

Example 3.1. Suppose that we have \$1M in liabilities due in 5 years. We can invest in 3, 5, or 7 ZCBs with spot rates 6%. There is reinvestment risk in the 3 year bond which needs to be reinvested at time 3. There is market risk in the 7 year bond because you need to sell the bond at time 5 and you may not have enough capital to sell because of high spot rates.

Note that if half of the capital was invested in the 3 year and half in the 7 year bond, there will never be a loss at time 5 for any parallel shift in the flat yield curve.

3.1 Single Liability Immunization

Let $V_k(\hat{y})$ be the value of a portfolio at time k if the yield to maturity instantaneously (at $t = 0$) changes to $\hat{y}$.

For an immunizaed portfolio initially constructed at the current ytm y^*, with Macaulay duration D, for any new rate $\hat{y}$, we have

$$V_D(\hat{y}) \geq V_D(y^*)$$

Proof. See the unit 6 notes. Essentially, show that $V_D(y)$ is minimized when $y = y^*$.

s

The idea here is that for a single liability L_k due at some time k, we want to construct a portfolio such that $D = k$ where D is the portfolio duration. If

$$V_0(y^*) = L_k(1 + y^*)^{-k}$$

then

$$V_k(\hat{y}) \geq V_k(y^*) = V_0(y^*)(1 + y^*)^k = L_k$$

and the conditions for the immunization are

$$\sum_{t>0} A_t(1 + y^*)^{-t} = L(1 + y^*)^{-k}, \sum_{t>0} tA_t(1 + y^*)^{-t} = L_k(1 + y^*)^{-k}$$

where the first condition matches the present values of the liability and portfolio and the second condition is when duration between the two matches and the first holds.

Example 3.2. (s. 9) Consider a liability 5 years from now with a PV of \$3M (equiv. to a 5-year ZCB). You can invest in a 3-year zcb or a perpetuity, assuming a flat yield curve of 6%.

(1) What is the amount that should be invested in the two securities for the portfolio to be immunized? To solve this, let x_1 be the amount invested in the zcb and x_2 be the amount invested in the perpetuity. Note that the duration of a perpetuity with payments L is

$$D_m = -\left(\frac{-L}{y^2}\right)\left(\frac{y}{L}\right) = \frac{1}{y} \implies D = \frac{1+y}{y}$$

Next, observe that we get two equations:

$$\begin{aligned} \text{Match PV} &\implies x_1 + x_2 = 3M \\ \text{Match Duration} &\implies \frac{x_1}{x_1 + x_2}\left(\underbrace{3}_{\text{Mac. Dur. of ZCB}}\right) + \frac{x_2}{x_1 + x_2}\left(\frac{1.06}{0.06}\right) = 5 \\ &\implies 3x_1 + \frac{1.06}{0.06}x_2 = 15 \end{aligned}$$

and solving such a system gives us

$$x_1 = 2,590,909.09, x_2 = 409,090.91$$

(2) What is the fixed payment L for the perpetuity? This is simply by definition:

$$\frac{L}{y^*} = x_2 \implies L = x_2(y^*) = 24,545.45$$

(3) Suppose that spot rates increase immediately to 7%. Do we still have enough funds to pay the liability at time 5? The answer is yes. To see this, remark that in 5 years,

$$\begin{aligned} V_5(0.07) &= \underbrace{x_1(1.06)^3}_{\text{face value}}(1.07)^2 + \frac{L}{0.07} + Ls_{\overline{5}|0.07} \\ &= 4,024,752.42 \end{aligned}$$

and since the future value of the liability at 6% is

$$L_5 = (3M)(1.06)^5 = 4,014,676.75 < V_5(0.07)$$

we have enough to cover this change.

(4) Suppose that spot rates decrease immediately to 5%. Redoing the same steps as above,

$$V_5(0.05) = 4,028,648.24 > L_5$$

(5) Spot rates increased to 6.5% at time 1.

(i) How should the manager reconstruct the portfolio so that it is immunized over the remaining time?

First remark that the PV of the liability at time 1 is

$$3M(1.06)^5(1.065)^{-4} = 3,120,700.93$$

and so we want

$$\begin{cases} x_1 + x_2 = 3,120,700.93 & \text{(match PV at time 1)} \\ \frac{x_1}{x_1+x_2} \cdot 2 + \frac{x_2}{x_1+x_2} \cdot \frac{1.065}{0.065} = 4 & \text{(match duration)} \end{cases}$$

and this implies that

$$\begin{cases} x_1 = 2,686,806.68 \\ x_2 = 433,894.25 \end{cases}$$

(ii) Does the manager need any extra money to construct the portfolio?

The value of the portfolio at time 1 before construction is

$$2,590,909.09 \cdot \frac{1.06^3}{1.065^2} + L + \frac{L}{0.065} = 3,122,804.55 > x_1 + x_2 = 3,120,700.93$$

and so we will not need any additional funds.

3.2 Multiple Liability Immunization

Let $\{A_t, t > 0\}, \{L_t, t > 0\}$ denote the asset and liability cash flows respectively. Define $A(y) = \sum_{t>0} A_t(1+y)^{-t}$ and $L(y) = \sum_{t>0} L_t(1+y)^{-t}$ be the PVs of the assets and liabilities respectively. We define the *present value of surplus* as

$$S(y) = A(y) - L(y)$$

We want to structure the asset portfolio so that for any small instantaneous change in the yield,

$$S(y^* + \triangle y) \geq S(y^*)$$

To do this, we use Reddington's Immunization Conditions:

(i) $S(y^*) = 0$ [match PV]

(ii) $S'(y^*) = 0$ [match duration]

(iii) $S''(y^*) > 0$ [dispersion / convexity condition]

Note that if (i) is true, then (ii) is equivalent to having $D_A = D_L$. To see this, Note that

$$S'(y^*) = 0 \iff A'(y^*) = L'(y^*) \iff \sum_{t>0} tA_t(1+y^*)^{-t-1} = \sum_{t>0} tL_t(1+y^*)^{-t-1}$$

and since $A(y^*) = L(y^*)$, divide by $A(y^*) = L(y^*)$ and multiply by $(1+y^*)$ to get

$$\frac{\sum_{t>0} tA_t(1+y^*)^{-t}}{A(y^*)} = \frac{\sum_{t>0} tL_t(1+y^*)^{-t}}{L(y^*)} \implies D_A = D_L$$

and usually it is easy to show that $\sum_{t>0} tA_t(1+y^*)^{-t} = \sum_{t>0} tL_t(1+y^*)^{-t}$ to satisfy condition (ii).

Now if both (i) and (ii) are satisfied, then (iii) is equivalent to $C_A \geq C_L$. To see this, note that

$$S''(y^*) \iff A''(y^*) \geq L^{''}(y^*) \iff \sum t(t+1)A_t(1+y^*)^{-t-2} \geq \sum t(t+1)L_t(1+y^*)^{-t-2}$$

and by (i), divide by $A(y^*) = L(y^*)$ to get the modified convexity of assets $\geq$ modified convexity of liabilities. Also, you multiply by $(1+y^*)^2$ throughout to get

$$\sum t(t+1)A_t(1+y^*)^{-t} \geq \sum t(t+1)L_t(1+y^*)^{-t}$$

and by condition (ii), subtract the equation found in (ii) from the above inequality to get

$$\sum t^2 A_t(1+y^*)^{-t} \geq \sum t^2 L_t(1+y^*)^{-t} \implies C_A \geq C_L$$

Example 3.3. Suppose that we have two liabilities, $L_5 = 10000$ and $L_8 = 20000$ with the option to invest in a 7-year ZCB. The current ytm is $y^* = 10\%$.

(1) How much should be invested in the 7 year zcb so that the PV is matched? We want A_7 such that

$$L_5(1+y^*)^{-5} + L_8(1+y^*)^{-8} = A_7(1+y^*)^{-7} \implies A_7 = 36700$$

(2) Is the portfolio is immunized? If it simple to check that the duration is matched. For convexity, we can check that

$$C_A = \frac{1}{A^*}\sum_{t>0} t^2 A_t(1+y^*)^{-t} = 49$$

and

$$C_L = \frac{1}{L^*}\sum_{t>0} t^2 L_t(1+y^*)^{-t} = 51$$

so the condition is not satisfied and the portfolio is not immunized.

Example 3.4. Suppose that we have two liabilities, $L_5 = 10000$ and $L_8 = 26620$ with the option to invest in a 3-year or 10-year ZCB. The current ytm is $y^* = 10\%$.

1) How much should be invested in the 3-year and 10-year ZCB so that the PV and duration are matched?

Match PV:

$$x_1 + x_2 = \underbrace{L_5(1.10)^{-5}}_{y_1} + \underbrace{L_8(1.10)^{-8}}_{y_2} = 18627.64$$

Match duration:

$$3x_1 + 10x_2 = 5y_1 + 8y_2$$

Solving for x_1 and x_2 gives us $x_1 = 7983.28$ and $x_2 = 10644.36$

2) Is the portfolio immunized? Check the convexity condition:

$$3^2x_1 + 10^2x_2 = 1136285.52 > 5^2y_1 + 8^2y_2 = 950006.62$$

Remark 3.1. Note that we assume

- A flat yield curve and a parallel shift
- Only immunizes for a small instantaneous shift
- At time 0, for $\triangle y$ small, $A(y) \geq L(y)$ so we can make a profit; thus we assumed that:

$$\begin{cases} (A) \text{ The model is arbitrage free} \\ (B) \text{ We need to continually rebalance the portfolio} \\ \text{so that PV and duration are matched} \end{cases}$$

3.3 Immunization Strategies

Bracketing Strategies

If we have liability cash flows $t_1^L < t_2^L < ... < t_n^L$ and asset cash flows at $t^- < t_1^L$ and $t_n^L < t^+$ then if the first two Redington conditions are satisfied, then the 3rd condition is satisfied. That is, if the portfolio is matched in PV and duration then $C_A > C_L$.

M^2 Strategy

We define the M^2 of an asset cash flow as

$$M_A^2 = \sum_{t>0} w_t^A (t - D_A)^2, w_t = \frac{A_t(1+y^*)^{-t}}{A(y^*)}$$

where D_A is the Macaulay duration of the cash flows. Similarly, we can define an M^2 for the liability cash flows.

Assuming that the first 2 Redington conditions old, the dispersion condition (convexity) is equivalent to checking if $M_A^2 \geq M_L^2$. Also,

(i) $M^2 = 0$ if there is a single cash flow

(ii) $M^2 \geq 0$ when all cash flows are non-negative

(iii) We can think of M_A^2 as the variance of a r.v. T where $P(T = t) = w_t$ so that $E(T) = \sum tw_t = D_A$

From the previous example, it is easy to check that $M_A^2 = 12 \geq M_L = 4$ and so we are immunized.

Implementation

There are often many portfolios satisfying the Redington conditions. To find a unique portfolio, we can do, for example,

$$\begin{aligned} \min \quad & M_A^2 \\ \text{subject to} \quad & M_A^2 \geq M_L^2 \\ & D_A = D_L \\ & PV(A) = PV(L) \end{aligned}$$

3.4 Generalized Redington Theory

Let the net cash flow at time t be denoted as $N_t = A_t - L_t$, $P(0,t) = e^{-t \cdot s_t}$, $n_t = N_t P(0,t)$ where the surplus is $\sum_{t>0} n_t$, FW Dollar Duration is $\sum_{t>0} tn_t$ and FW Dollar Convexity is $\sum_{t>0} t^2 n_t$ for continuously compounded spot rates $\{s_t\}$.

Let $\{\hat{s}_t\}$ denote the new spot rates and $\hat{P}(0,1) = e^{-t \cdot \hat{s}_t}$, $g(t) = \frac{\hat{P}(0,1)}{P(0,1)} - 1$, $\hat{S} = \sum N_t \cdot \hat{P}(0,t)$. Note that the change in surplus is

$$\hat{S} - S = \sum N_t \left[\hat{P}(0,t) - P(0,t) \right] = \sum n_t g(t)$$

We want to construct a portfolio with the property that $\hat{S} - S \geq 0$.

If $\sum n_t = 0$ and $\sum tn_t = 0$ and (1) the sequence $\{n_1, n_2, ..., n_k\}$ undergoes the sign change sequence $+, -, +$ (group of +, group of -, and then a group of +) then for any convex function ϕ,

$$\sum_{i=1}^{k} n_i \phi(t_i) \geq 0$$

In the alternative case (2) , where we have a sign change sequence $-, +, -$ then

$$\sum_{i=1}^{k} n_i \phi(t_i) \leq 0$$

and bringing everything together if $\sum n_t = 0$, $\sum tn_t = 0$, $\{n_{t_k}\}$ undergoes a $+, -, +$ sequence, and $g(t) = \frac{\hat{P}(0,t)}{P(0,t)} - 1$ is convex, then $\hat{S} - S \geq 0$.

Example 3.5. If there is a parallel shift in the spot rate curve, say $\hat{s}_t = s_t + c$. Then

$$g(t) = \frac{e^{-(s_t+c)t}}{e^{-s_t t}} - 1 = e^{-ct} - 1 \implies g'' > 0 \implies g \text{ is convex}$$

3.5 Dedication / Cash Flow Matching

We want to find the cheapest combination of assets such that the asset cash flows are at least as large as the liability cash flows ($A_t \geq L_t$ for all t). That is, we are trying to solve

$$\min \sum A_t P(0,t)$$

such that $A_t \geq L_t$ for all t.

4 Interest Rate Derivatives

(We skip over the section about call and put options since it's basic review; we will just leave the notation for reference)

$x^+ = \max(x, 0)$ where $(S_T - K)^+$ is for a call and $(K - S_T)^+$ is for a put.

4.1 Interest Rate Caps and Floors

These derivatives are used together with floating rate loans to hedge against interest rate risk. Let L be the notional amount of the loan, T the time to maturity, i_t the floating interest rate and τ the reset period in years.

At times $t = 1, 2, ..., \frac{T}{\tau}$, the interest on the load is either settled:

- In arrears $\implies$ payment of Li_{t-1} at time t, or
- In advance $\implies$ payment of Li_t at time t

Example 4.1. Suppose that $i_1 = 7\%, i_2 = 8\%, i_3 = 9\%$ and we buy a floating rate bond with face value \$1000. For *caps,* we (the borrower) can purchase an interest rate cap at 5%. This would pay

$$\max(i_t - 5\%) \cdot 1000$$

So we receive payments of 20, 30, and 40 at $t = 1, 2, 3$ respectively. The net cash flows out (payments) would be \$50 for $t = 1, 2, 3$.

Let's define caps and floors explicitly now.

Caps are used to protect the borrower of a loan from increases in the interest rate. It is formed by a series of "caplets". At time t, the payoff from a caplet is

- $L(i_{t-1} - K)^+$ if settled in arrears
- $L(i_t - K)^+$ if settled in advance

Floors are used to protect the lender of a loan from decreases in the interest rate. It is formed by a series of "floorlets". At time t, the payoff from a floorlet is

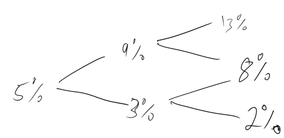
- $L(K - i_{t-1})^+$ if settled in arrears
- $L(K - i_t)^+$ if settled in advance

In general, the price of a cap/floor at time 0 is

$$E\left[\sum_{t=1}^{T/\tau} (\text{Payoff @ } t) \cdot \left(\frac{1}{1+i_0}\right)\left(\frac{1}{1+i_1}\right)\cdots\left(\frac{1}{1+i_{t-1}}\right)\right]$$

and we use backwards recursion to find the price of the cap. That is, we use the same method as the general binomial model. See Example 4 in the slides for an application of this.

Example 4.2. Price a 2-year cap settled in advance with $L = 100, K = 7\%$ and $\tau = 1$ year. With $q(t,n) = 0.5$ for all t and n using the following structure.



It can be shown that $V(1,0) = 0.4854$, $V(1,1) = 3.2110$ and $V(0,0) = 2.7126$.

Next, create a replicating portfolio to match the cash flows of the cap in the previous example. You can invest in a 3-yr ZCB and in the money market (eaching interest at the short rate.

Let $x_{t,n}$ and $y_{t,n}$ be the amounts invested in the ZCB and money market respectively. At time 0, we match the cash flows (c-f's) of the cap at time 1. That is,

$$x_{0,0}V_Z(1,1) + y_{0,0}(1.05) = 3.2110 + 2 \; [(0,0) \mapsto (1,1)]$$

$$x_{0,0}V_Z(1,0) + y_{0,0}(1.05) = 0.4854 + 0 \; [(0,0) \mapsto (1,0)]$$

where the \$2 comes from the cash flow that comes from the cap and the decimal valued parts on the right side are included because we assume that the value comes form selling off the cap (at its selling value). Solving simultaneously gives

$$\begin{cases} x_{0,0} = -49.9007 \\ y_{0,0} = 44.415 \end{cases}$$

Aside. The cost of the portfolio at time 0 is

$$x_{0,0}V_Z(0,0) + y_{0,0} = 2.7145$$

Rebalancing at time 1 at node $(1,1)$:

$$x_{1,1}V_Z(2,2) + y_{1,1}(1.09) = 6 \; [(1,1) \mapsto (2,2)]$$

$$x_{1,1}V_Z(2,1) + y_{1,1}(1.09) = 1 \; [(1,1) \mapsto (2,1)]$$

we get

$$\begin{cases} x_{1,1} = -122.2494 \\ y_{1,1} = 104.7621 \end{cases}$$

Similarly for node $(1,0)$ we get:

$$x_{1,0}V_Z(2,1) + y_{1,1}(1.03) = 1 \; [(1,0) \mapsto (2,1)]$$

$$x_{1,0}V_Z(2,0) + y_{1,0}(1.03) = 0 \; [(1,0) \mapsto (2,0)]$$

and

$$\begin{cases} x_{1,1} = -18.3486 \\ y_{1,1} = 17.4650 \end{cases}$$

4.2 Callable and Putable Bonds

For callable bonds, we have two ways to price:

(1) Price the bond by its components:

Callable Bond = Option Free Bond - Call Option

Let $B(t,n)$ be the value of the option free bond at (t,n), and $E(t,n) = \max(B(t,n) - F, 0)$ the payoff if the option is exercised at (t,n). Now as the holder of the call option, you would like to maximize its value. At node (t,n), there are two options, exercise at (t,n) or hold (do not exercise).

Let $H(t,n)$ be the continuation/holding value at (t,n), and $V(t,n) = \max(E(t,n), H(t,n))$, the option's value at (t,n) or the cash flow.

The algorithm works by using backward recursion to compute $V(t,n)$ starting with $B(T,n) = F, E(T,n) = 0, H(T,n) = 0$ and $V(T,n) = 0$. Specifically, for $t = T-1, T-2, ..., 1$ we step with

- Compute $B(t,n)$ and $E(t,n)$
- $H(t,n) = \frac{1}{1+i(t,n)}\left[q(t,n) \cdot V(t+1,n+1) + (1-q(t,n)) \cdot V(t+1,n)\right]$
- Compute $V(t,n) = \max(E(t,n), H(t,n))$

$V(0,0) = H(0,0)$ gives the price of the callable bond at time $t = 0$.

(2) Price the bond directly by considering at each node, the PV of coupons and principal (and you need to hold for the future) and the cost of calling the bond (usually equals F).

5 Interest Rate Models

The objective in this section is to learn to model interest (short rates) and price fixed income securities under these models.

5.1 Discrete Interest Rate Models

Example 5.1. Binomial Model

Let T be the number of time periods, i_t the short rate (as a random variable (r.v.)) at time t where $t = 0, 1, ..., T-1$ and $i(t,n)$ be the n^{th} possible value that i_t can take for $n = 0, 1, ..., N_t$ (indexing is from the bottom up). The model goes as follows:

1. Start with $i_0 = i(0,0)$.
2. Given $i(t,n)$ at time t, i_{t+1} can take only two possible values

(a) $i(t+1,n+1)$ with probability $q(t,n)$ (rate increases)

(b) $i(t+1,n)$ with probability $1-q(t,n)$ (rate decreases)

We will usually assume a recombining tree, i.e. whether we move up and then down, or move down and then up, we end up with the same interest rate.

the price of a T year ZCB is given by

$$P(0,T) = E\left[\Pi_{k=0}^{T-1}\left(\frac{1}{1+i_k}\right)\right]$$

where $\Pi_{k=0}^{T-1}\left(\frac{1}{1+i_k}\right)$ is the discount factor in terms of the short rates. As with other securities, we wilol find the price using backwards recursion. To be more specific about ZCB pricing, we have the following steps

1. Let $V(t,n)$ be the value of the bond at (t,n)
2. Initialization: $V(T,n) = F$, $n = 0,1,...,T$
3. Recursive Loop: For $t = T-1, T-2, ..., 0$ compute

$$V(t,n) = \frac{q(t,n)\cdot V(t+1,n+1) + [1-q(t,n)]\cdot V(t+1,n)}{1+i(t,n)}$$

4. Output $V(0,0) =$ price at time 0.

Example 5.2. Price a 2-year \$100 ZCB given $q(t,n) = 0.6$, $i(0,0) = 5\%$, $i(1,1) = 9\%$, $i(1,0) = 3\%$. The payment structure is that at time 2 we will always be paid \$100. We get $V(1,1) = 100/(1.09) = 91.74$, $V(1,0) = 100/1.03 = 97.09$ and $V(0,0) = \frac{(0.6)(91.74)+(97.09)(0.4)}{1.05} = 89.41$.

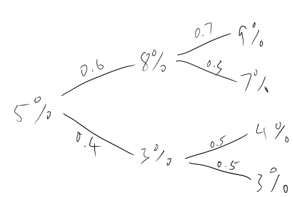
For pricing coupon bonds, we have similar steps with the notation $C(t,n)$ =Coupon at node (t,n). The steps are

1. Let $V(t,n)$ be the value of the bond at (t,n), $C(t,n)$ be the value of the coupon at node (t,n)
2. Initialization: $V(T,n) = F$, $n = 0,1,...,T$
3. Recursive Loop: For $t = T-1, T-2, ..., 0$ compute

$$V(t,n) = \frac{q(t,n)\cdot[V(t+1,n+1)+C(t+1,n+1)] + [1-q(t,n)]\cdot[V(t+1,n)+C(t+1,n)]}{1+i(t,n)}$$

4. Output $V(0,0) =$ price at time 0.

Example 5.3. Consider a 3-year \$100 bond with 5% annual coupons. Compute the price of this security using the following interest rate structure.



We can calculate the values to be $V(2,0) = 102.94, V(2,1) = 100.96, V(2,2) = 98.13, V(2,3) = 96.33, V(1,1) = 94.32, V(1,0) = 103.83$ and finally $V(0,0) = 98.21$.

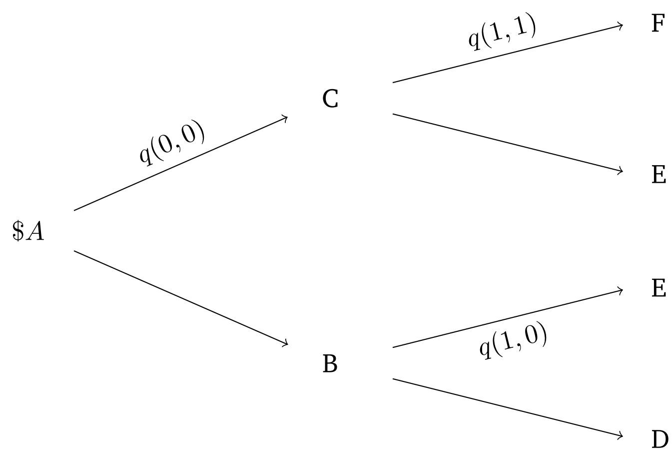
Definition 5.1. *Equilibrium models* generally produce ZCB prices that are not seen on the market while *no-arbitrage models* try to match the prices on the market.

Calibrate the interest rate binomial model so that it reproduces the actual term structure observed on the market. We use the prices for T zcb's to calibrate the tree. Let

1. $P(0, t)$ denote the model's price for a t-year zcb, and
2. $\hat{P}(0, t)$ denote the market price for a t-year zcb

There are $T(T+1)$ parameters in a recombining tree. We need to make simplifying assumptions.

Problem for a T-period model, we have $T(T+1)$ parameters but only T equations. For example, consider a two-period tree



Example 5.4. Ho-Lee Model

Assumes that $1 + i(t, n+1) = k[1 + i(t, n)]$ and $q(t, n) = q$. k and q are specified constants such that

$$\hat{P}(0, t) = P(0, t), t = 1, 2, \cdots, T$$

where the market price of the ZCB is equal to the theoretical/model price.

Example 5.5. Black-Dermon-Toy (BDT) Model

Assume that

$$i(t, n+1) = i(t, n)e^{2\sigma(t)}$$

and $q(t, n) = \frac{1}{2}$. (discrete approximation of the log-normal model).

Input: $\hat{P}(0, 1), \cdots, \hat{P}(0, T)$ and $\sigma(1), \cdots, \sigma(T-1)$.

Example: Calibrate a binomial tree from $t = 0$ to $t = 2$ according to the BDT model and based on the following input term structure:

t	s_t	$\sigma(t)$
1	6%	19%
2	7%	17.2%
3	8%	-

A forward procedure: use AD securities where the price of an Arrow-Debreu security $A(t, n)$ is the price of a security that pay \$1 at node (t, n) and \$0 elsewhere

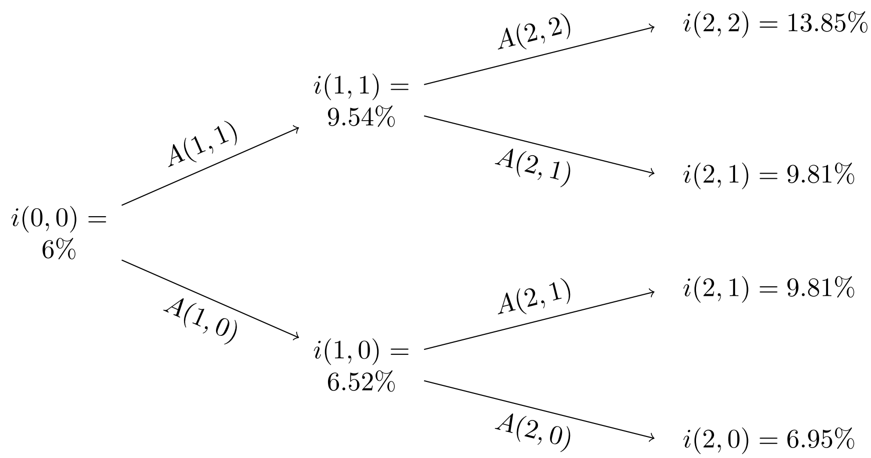
Time-0 calibration $i(0, 0) = 6\%, q(0, 0) = 0.5$.d

Find A-D price $A(1, 0) = \frac{0.5}{1+i(0,0)} = 0.4717$

$A(1, 1) = \frac{0.5}{1+i(0,0)} = 0.4717$

Time-1 calibration

Want to match $\hat{P}(0, 2) = P(0, 2)$. Now, $\hat{P}(0, 2) = \frac{1}{1.07^2}$. $P(0, 2) = \frac{A(1,0)}{1+i(1,1)} + \frac{A(1,0)}{1+i(1,0)}$ and $P(0, 3) = \frac{A(2,2)}{1+i(2,2)} + \frac{A(2,1)}{1+i(2,1)} + \frac{A(2,0)}{1+i(2,0)}$



In general we need to solve,

$$\hat{P}(0,t+1) = \sum_{k=0}^{t} \frac{A(t,k)}{1+i(t,k)}$$

Consider a function

$$f(x) = \sum_{k=0}^{t} \frac{A(t,k)}{1+xe^{2k\sigma(t)}} - \hat{P}(0,t+1)$$

so f is strictly decreasing in x,

$$f(0) = \hat{P}(0,t) - \hat{P}(0,t+1) > 0$$

and $\lim_{x\to\infty} f(x) = -\hat{P}(0,t+1) < 0$. This implies unique positive solution to the equation $f(x) = 0 \implies$ unique solution for $i(t,0)$.

5.2 Option Adjusted Spread

- Reasons for the spread:
- Compared to option-free bonds, bonds with embedded options come with repayment/reinvestment risk.
  - Using the calibrated model if we compute the price of such a bond, we will have the theoretical price, this may differ from the actual market price.
  - The OAS is a fixed/flat spread over the rates of the calibrated free that gives the theoretical price is equal to market price.
  - Prepayment/reinvestment risk for a callable bond can be defined as the risk that the principal with be repaid before maturity, and that the proceeds will have to be invested at a lower interest rate.
- OAS is the rate such that the binomial interest rate lattice shifted by the OAS equates the new theoretical price with the market price (uniform shift)
- The OAS of an option free bond is 0
- Here are the steps to compute V_+/V_-:
  1. Given the security's market price, find the OAS.
  2. Shift the spot-rate curve by a small quantity y.
  3. Compute a binomial interest-rate lattice based on the shifted curve obtained in Step 2.
  4. Shift the binomial interest-rate lattice obtained in Step 2 by the OAS.
  5. Compute V_+/V_- based on the lattice obtained in Step 4.

- The V_+/V_- values are used in the calculation of effective duration and convexity through the formulas:

$$D_m^e = \frac{V_- - V_+}{2V_0 \triangle y}, C_m^e = \frac{V_+ - 2V_0 + V_-}{V_0(\triangle y)^2}$$

5.3 Continuous Interest Rate Models

We briefly discuss the framework of continuous interest rate models and a few properties of several models.

Definition 5.2. A *brownian motion* $B = \{B_t, t > 0\}$ is a stochastic process such that

1) $B(0) = 0$.

2) For any $t_0 < t_1 < t_2$ the random variable $B(t_2) - B(t_1)$ and $B(t_1) - B(t_0)$ are independent of each other and so are any two non-overlapping intervals.

Furthermore, $B(t_2) - B(t_1)$ is equal in distance to $B(t_2 - t_1)$, and similarly for $B(t_1) - B(t_0)$ and $B(t_1 - t_0)$.

Finally, we have $B(\triangle) \sim N(0, \triangle)$ so that

$$B(t_B) - B(t_A) \overset{dist}{=} B(t_B - t_A) \sim N(0, t_B - t_A), t_B > t_A$$

Overall, we have $B(0) = 0$ and $B(t + \triangle) = B(t) + Z(t)$ where $\{Z(t), t \geq 0\}$ are i.i.d. $N(0, \triangle)$.

Example 5.6. Rendleman-Bartter (Lognormal) Model:

A special case is $dr(t) = \sigma \cdot r(t) \cdot dB(t)$. Intuitively,

$$r(t + \triangle) - r(t) = \sigma r(t) [B(t + \triangle) - B(t)] \implies \frac{r(t + \triangle) - r(t)}{r(t)} = \sigma [B(t + \triangle) - B(t)]$$

and this implies that the change in $r(t)$ is modelled by a Brownian Motion process where

$$\frac{r(t + \triangle) - r(t)}{r(t)} \sim N(0, \sigma^2 \triangle)$$

Solving the model gives

$$r(t) = r(0) \cdot e^{-\sigma^2 t/2 + \sigma B(t)}$$

where $B(t) \sim N(0, t)$.

Example 5.7. Vasicek Model:

This model is described by $dr(t) = a(b - r(t))\, dt + \sigma dB(t)$ and intuitive speaking, on average,

$$r(t + \triangle) - r(t) = a[b - r(t)]\triangle$$

since $B(t + \triangle)$ and $B(t)$ are zero mean r.v.s

Cases:

i) If $b > r(t)$ then $\implies$ $a(b - r(t))\triangle$ is positive $\implies$ $r(t + \triangle) - r(t)$ is positive $\implies$ change in $r(t)$ is positive $\implies$ tends upwards to b

ii) If $b < r(t)$ then $\implies$ $a(b - r(t))\triangle$ is negative $\implies$ $r(t + \triangle) - r(t)$ is negative $\implies$ change in $r(t)$ is negative $\implies$ tends downwards to b

The above two cases are collectively known as "*the mean reversion feature of interest rates*". Solving this model gives

$$r(t) = r(0)e^{-at} + b(1 - e^{-at}) + \sigma e^{-at} \int_0^t e^{as}\, dB(s)$$

This particular model captures the mean reversion feature in the sense that $r(t)$ tends to fluctuate about b. The main drawback is that interest rates can become negative.

Example 5.8. Cox-Ingersoll-Ross (CIR) Model:

This model is described by $dr(t) = a(b - r(t))\ dt + \sigma\sqrt{r(t)}dB(t)$. With $a, b > 0$, this model captures the mean reversion feature while keeping interest rates positive.

5.4 Pricing Securities

Theoretical Price

For a security that pays V_T at time t, the price is given by

$$V_0 = E\left[e^{-\int_0^T r(t)\ dt}V_T\right]$$

for a specified risk neutral probability measure.

Example 5.9. Suppose that a security pays c_t at times $t = \triangle, 2\triangle, 3\triangle, ..., N\triangle$. Assume that the short rate process is given by

$$r(t) = r(0)e^{-\frac{1}{2}\sigma^2 t + \sigma B(t)}$$

If $V_t = e^{-\int_0^{t\triangle} r(s)\ ds}$ the time 0 price is

$$c_0 = E\left[\sum_{t=1}^{N} c_t V_t\right]$$

and for simple securities, we may be able to find a closed form for c_0. However, for more complex securities, we will need to use simulation.

Note 1. In the CIR model, if $r(t)$ is small and $b > 0$, the volatility term $\sigma\sqrt{r(t)}dB(t)$ will be small and with $b > r(t)$ the drift term $a(b - r(t))dt$ will be relatively large. Overall the drift term more than offsets the volatility term, and so, prevents the interest rate from becoming negative.

Monte Carlo Simulation

Example 5.10. Suppose that we want to estimate $E[X]$ and we have observed values $\{x_1, ..., x_n\}$. The estimate is given by $\frac{1}{n}\sum_{i=1}^{n} x_i$. In practice, we may need to simulate $x_1, x_2, .., x_n$ according to some model.

For example if we assume $X \sim N(0, 1)$ we simulate an x using, for example, `rnorm(1,0,1)` in R OR `norm.inv(rand(1),0,1)` in Excel.

How will this be used in pricing securities?

- Price is given by the expected present value
- With cash flows c_t at $t = 1, 2, ..., N$ the price is $c_0 = E\left[\sum_{i=1}^{N} c_t v_t\right]$ where $v_t = e^{-\int_0^t r(s)\ ds}$

Using Monte Carlo simulation,

1. We simulate a set of discount factors $\{v_1, v_2, ..., v_N\}$
2. Find the simulated price $\sum_{i=1}^{N} c_t v_t$
3. Repeat steps 1 and 2, n times where n is large; at the end of the process, we have n simulated prices $c_0^1, .., c_0^n$
4. The estimated price of the security is given by $\frac{1}{n}\sum_{i=1}^{n} c_0^i$

For simulation of the discount factors, write $e^{-(r_0 + r_1 + ... + r_{t-1})}$ and then simulate the sample path $\{r_1, ..., r_n\}$.

Suppose, for example, we use the log normal model. Here, $r_t = r_0 e^{-\sigma^2 t/2 + \sigma B(t)}$ and so

$$\frac{r_t}{r_{t-1}} = r_0 e^{-\sigma^2 t/2 + \sigma[B(t) - B(t-1)]} \implies r_t = r_{t-1}e^{-\sigma^2 t/2 + \sigma[B(t) - B(t-1)]}$$

where we do this so that r_t depends on the previous short rate r_{t-1} and so that we can now simulate $B(t) - B(t-1)$ independently of previous simulations. Now since $B(t) - B(t-1) \sim N(0,1)$ then

$$r_t = r_{t-1}e^{-\sigma^2 t/2 + \sigma Z}$$

where $Z \sim N(0,1)$.

6 Risk Measures

We will be focusing on VaR and CTE with a deep emphasis on VaR.

6.1 VaR and CTE

Definition 6.1. The value-at-risk is the maximum loss on a portfolio that can occur over a specified period with a given probability (level of confidence).

Example 6.1. Next month's VaR is $1000 at a 95% level of confidence level. You are 95% confident that the loss over the next month will not exceed $1000.

Definition 6.2. Mathematically, denote V_n be the portfolio value after n periods and L_n be the loss of the portfolio over n periods. Say $F_{L_n}(l)$ is the cdf of L_n and $S_{L_n}(l) = 1 - F_{L_n}(1)$ the survival distribution of L_n. If L_n is a continuous distribution then $VaR_{\alpha,n}$ is written as

$$P(L_n > VaR_{\alpha,n}) = 1 - F_{L_n}(VaR_{\alpha,n}) = 1 - \alpha$$

which is the the $100\alpha^{th}$ percentile of the distribution of $F_{L_n}(l)$.

Example 6.2. Suppose that $L_n \sim Exp(\lambda)$. Then $CTE_{\alpha,n} = \frac{1-\ln(1-\alpha)}{\lambda}$.

Definition 6.3. For more general loss distributions, (e.g. discrete, continuous, mixed), $VaR_{\alpha,n}$ is defined as

$$VaR_{\alpha,n} = \inf\{l \in \mathbb{R} | F_{L_n}(l) \geq \alpha\} = \inf\{l \in \mathbb{R} | P(L_n > l) \leq 1 - \alpha\}$$

that is, it is the small value of l such that $F_{L_n}(l) \geq \alpha$.

Example 6.3. We are given

l	0	100	1000	10000
$P(L = l)$	0.9	0.04	0.052	0.0008

which is equivalent to

l	0	100	1000	10000
$P(L \leq l)$	0.9	0.94	0.992	1

We thus have $VaR_{0.9,n} = 0, VaR_{0.95,n} = 1000, VaR_{0.99,n} = 1000, VaR_{0.995,n} = 10000$.

Remark 6.1. VaRcan be calculated on a tree. The l values are the far child nodes and the probabilities are the sum of the probabilities of all paths that reach a particular node. The cdf can be extrapolated from there.

Definition 6.4. Alternatively, VaR can be interpreted as the change in portfolio value $\triangle V = V_n - V_0 = -L_n$ since $VaR_{\alpha,n}$ is such that

$$P(L_n \geq VaR_{\alpha,n}) = 1 - \alpha \implies P(\triangle V \leq -VaR_{\alpha,n}) = 1 - \alpha$$

Also, if V^* is such that $P(V_n \leq V^*) = 1 - \alpha$ then $VaR_{\alpha,n} = V_0 - V^*$.

Definition 6.5. The *conditional tail expectation* is the average loss that can occur if loss exceeds $VaR_{\alpha,n}$. For a loss distribution L_n and confidence α this is

$$CTE_{\alpha,n} = E[L_n | L_n \geq VaR_{\alpha,n}]$$

Example 6.4. Using the previous example, the conditional tail expectation is

$$CTE_{0.95,n} = \frac{\sum_{\text{all } l \text{ w/ } L \geq VaR_{\alpha,n}} l \cdot Pr(L_n = l)}{\sum_{\text{all } l \text{ w/ } L \geq VaR_{\alpha,n}} Pr(L_n = l)} = \frac{1000(0.052) + 10000(0.008)}{0.052 + 0.008} = 2200$$

6.2 Modeling VaR

To compute VaR, we need to identify the risk factors and make assumptions on the distribution of these factors. Here are our assumptions:

1. The change in portfolio value is linearly related to the risk factors
2. Risk factors are normally distributed

In this section, we will examine three cases

- Case 1: One factor case
  - Assume only one risk factor R which represents the return on the portfolio over n periods
  - We then have $V_n = V_0(1+R) \implies \triangle V = V_n - V_0 = V_0 R$. If $R \sim N(\mu_R, \sigma_R^2)$ then $\triangle V \sim N(\mu_V = \mu V_0, \sigma_V^2 = \sigma_R^2 V_0^2)$. Thus, an expression for VaR is given by

    $$VaR_{\alpha,n} = V_0 \sigma_R z_\alpha - V_0 \mu_R$$

    where z_α is the α percentile of a $N(0,1)$ distribution. That is $P(N(0,1) \leq z_\alpha) = \alpha$.
  - To see this, $VaR_{\alpha,n}$ is such that

    $$P(\triangle V \leq -VaR_{\alpha,n}) = 1 - \alpha$$

    which implies that

    $$P\left(\underbrace{\frac{\triangle V - \mu_V}{\sigma_V}}_{N(0,1)} \leq \underbrace{-\frac{VaR_{\alpha,n} - \mu_V}{\sigma_V}}_{z_{1-\alpha}}\right) = 1 - \alpha \implies -\frac{VaR_{\alpha,n} - \mu_V}{\sigma_V} = z_{1-\alpha} = -z_\alpha$$

    by symmetry. The result follows from solving for $VaR_{\alpha,n}$.
  - Usually, R represents the daily return. However, if we use $R(n) = \sum_{t=1}^n R_t$ which a sum of daily returns with $R_t \overset{iid}{\sim} N(\mu_R, \sigma_R^2)$ we have that

    $$VaR_{\alpha,n} = V_0 \sigma_R \sqrt{n} z_\alpha - V_0 \mu_R$$

    over a period of n days. To see this, remark that

    $$\triangle V = V_0 \sum_{t=1}^n R_t \sim \left(\mu_V = n\mu_R V_0, n\sigma_R^2 V_0^2\right)$$

    and repeating the same procedure as above we get the result.
  - As a rule of thumb, if μ is unknown, set it to 0.
  - *Example.* Suppose that you have a portfolio of \$10M in RIM. The daily volatility of the stock return is 0.02. Find $VaR_{0.95,10 \text{ days}}$ of the portfolio if the return is normally distributed. It can be shown that

    $$VaR_{0.95,10} = V_0 \sqrt{n} \sigma_R z_{0.95} - nV_0 \mu_R = (10M)(\sqrt{10})(0.02) - 0 = 1,040,389$$

- Case 2: Two factor case
  - Now suppose that we have two risk factors, so that we can write

$$V_n - V_0 = \triangle V = V_0(w_1(1+R_1) + w_2(1+R_2)) - V_0$$

where the $w_i's$ are weights. For example, if the risk factors are the returns on two assets, then $w_i = n_i S_i / V_0$ where n_i is the number of shares of asset i and S_i is the price. We also will generally assume that

$$\left(\begin{array}{c} R_1 \\ R_2 \end{array} \right) \sim BVN\left(\left[\begin{array}{c} \mu_1 \\ \mu_2 \end{array} \right], \left[\begin{array}{cc} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{array} \right] \right)$$

If we are interested in the return R_V of the portfolio, the above implies that

$$R_V \sim N(\mu_V, \sigma_V^2), \mu_V = w_1\mu_1 + w_2\mu_2, \sigma_V^2 = w_1^2\sigma_1^2 + w_2\sigma_2^2 + 2\rho w_1 w_2 \sigma_1\sigma_2$$

which will imply that

$$VaR_{\alpha,n} = V_0\sigma_V z_\alpha - V_0\mu_V \approx V_0\sigma_V z_\alpha$$

if we assume that μ_1 and μ_2 are approximately 0.

Duration approach

- This is simply using the modified duration D_m of a portfolio as a linear approximation of a portfolio's change in value:

$$\triangle V = -V_0 D_m \triangle y$$

and if we assume $\triangle y$ is a parallel shift as well as normal with volatility σ_y then

$$VaR_{\alpha,n} = V_0 D_m \sigma_y z_\alpha$$

Cash-flow mapping

- This method breaks down a portfolio into it constituent cash flows and focuses on creating a "replicating" portfolio using zero coupon bonds that match the volatilities of the individual cash flows
- For example, consider the case of a single cash flow C_τ at time τ and suppose that we have information on the mean and volatilities of several zero coupon bonds $\{P(0,i)\}_{i=1}^N$ (modeled as a multivariate normal) with return means $\{\mu_i = 0\}_{i=1}^N$, return volatilities $\{\sigma_i\}_{i=1}^N$, and correlation structure $\{\rho_{ij}\}_{i\neq j}$
  - If $k < \tau < k+1$ then we use linear interpolation to get μ_τ and σ_τ and $P(0,\tau)$
  - Let $V_0 = C_\tau P(0,\tau)$
  - We then find the weights in $P(0,k)$ and $P(0,k+1)$ such that the volatility of the two ZCB portfolio is equal to the volatility of the interpolated volatility
  - That is, if α is the weight in $P(0,k)$, then we find α such that

$$\sigma_\tau^2 = \alpha^2\sigma_k^2 + (1-\alpha)^2\sigma_{k+1}^2 + 2\rho_{k,k+1}\alpha(1-\alpha)\sigma_k\sigma_{k+1}$$

  - We then invest αV_0 in $P(0,k)$ and $(1-\alpha)V_0$ in $P(0,k+1)$
  - We then have

$$\sigma_V^2 = V_0^2(w_k^2\sigma_k^2 + w_{k+1}^2\sigma_{k+1}^2 + 2\rho_{k,k+1}w_k w_{k+1}\sigma_k\sigma_{k+1})$$

and

$$VaR_{\alpha,n} = \sqrt{n}\sigma_V z_\alpha$$

- This concept can easily be extended to multiple cash flows by normalizing the weights w_i each time a new cash flow is added in the calculation

Delta-Normal Method

- For a portfolio with multiple factors, we have through a first order Taylor expansion,

$$dV \approx \sum_{i=1}^{m} \frac{\partial V}{\partial f_i} df_i = \sum_{i=1}^{m} \triangle_i df_i = \sum_{i=1}^{m} f_i \triangle_i \frac{df_i}{f_i} = \sum_{i=1}^{m} f_i \triangle_i R_i$$

where $\triangle_i = \partial V / \partial f$, assuming that $R_i \sim N(0, \sigma_i^2)$

- We can then compute

$$Var(dV) = \sigma_V^2 = \sum_{i=1}^{m} (f_i \triangle_i)^2 Var(R_i) + 2 \sum_{i \neq j} f_i f_j \triangle_i \triangle_j Cov(R_i, R_j)$$

and using $\mu_V = 0$, we can approximate VaR as

$$VaR_{\alpha,n} \approx \sigma_V z_\alpha$$

- For the special case of options,

$$dV = \triangle dS = S_0 \triangle \frac{dS}{S_0} = S_0 \triangle R_S$$

where $\triangle$ is the delta of the option. Thus we can use the approximation

$$\sqrt{Var(dV)} = S_0 |\triangle| \sigma_S = \sigma_V \implies VaR_{\alpha,1} = \sigma_V z_\alpha$$

Multinomial Quadratic Model

- Using a second order Taylor expansion, and under the case of a single risk factor $R_f \sim N(0, \sigma_f^2)$, we have

$$\begin{aligned} dV &= \triangle \times f \times R_f + \frac{1}{2} \Gamma \times f^2 \times R_f^2 \\ &= \frac{1}{2} \Gamma f^2 \left[\left(R_f + \frac{\triangle}{\Gamma f} \right)^2 - \left(\frac{\triangle}{\Gamma f} \right)^2 \right] \end{aligned}$$

- This implies that

$$\frac{dV}{\frac{1}{2} \Gamma f^2} + \left(\frac{\triangle}{\Gamma f} \right)^2 = \left(R_f + \frac{\triangle}{\Gamma f} \right)^2$$

and if we define

$$Y = \frac{dV}{\frac{1}{2} \Gamma f^2} + \left(\frac{\triangle}{\Gamma f} \right)^2 = \left(R_f + \frac{\triangle}{\Gamma f} \right)^2$$

it turns out that $Y \sim \chi_{1,q}^2$ which is a non-centralized χ_1^2 random variable with non centrality parameter

$$q = \left(\frac{\mu_f + \frac{\triangle}{\Gamma f}}{\sigma_f} \right)^2$$

- If y_α is the value such that $P(Y \leq y_\alpha) = 1 - \alpha$ then

$$P\left(dV \leq \frac{1}{2} \Gamma f^2 \sigma_f^2 y_\alpha - \frac{\triangle^2}{2\Gamma} \right) = 1 - \alpha$$

and hence

$$VaR_{\alpha,n} \approx \frac{\triangle^2}{2\Gamma} - \frac{1}{2} f^2 \sigma_f^2 y_\alpha$$

Delta-Gamma-Normal Method

- Working with a non-centralized χ^2 distribution is very difficult in practice, which is why an alternative is to assume that $dV \sim N(\hat{\mu}, \hat{\sigma})$ where we want to match the first two moments of the theoretical dV from the quadratic model above

- With the assumption that $R \sim N(0, \sigma_f^2)$, the first two moments are

$$\begin{aligned} E(dV) &= \frac{1}{2}\Gamma \times f^2 \times \sigma_f^2 \\ E\left[(dV)^2\right] &= \triangle^2 f^2 \sigma_f^2 + 2\left[E(dV)\right]^2 \end{aligned}$$

and hence $\hat{\mu}_V = E(dV)$, $\hat{\sigma}^2 = E\left[(dV)^2\right] - \hat{\mu}^2$

- We can then use the classical approximation of VaR, which is

$$VaR_{\alpha,n} = \hat{\sigma} z_\alpha - \hat{\mu}$$

<u>Cornish-Fisher Expansion</u>

- In practice, the returns on a portfolio tend to be heavily skewed; if a portfolio is negatively skewed, for example, its left tail is thicker than the normal distribution and VaR is underestimated if we use the normality assumption (e.g. a short call)
- The opposite is true for positively skewed distributions (e.g. a long call)
- Hence, we need some form of correction of this, which is what the Cornish-Fisher expansion provides
- Define

$$\xi = \frac{E[dV - \hat{\mu}]}{\hat{\sigma}^3}, z'_\alpha = z_\alpha - \frac{1}{6}(z_\alpha^2 - 1)\xi$$

- The second order, skewness-adjusted VaR is

$$VaR_{\alpha,n}^{CF} = z'_\alpha \hat{\sigma} - \hat{\mu}$$

where the volatility and mean are estimated under the delta-normal-gamma method with a normality assumption on dV

6.3 Simulation of VaR

The general algorithm is as follows:

1. For $i = 1, ..., N$ do the following:
   - Generate the value $\hat{f}_i$ for risk factor i at time τ
   - Compute the value at time τ of the portfolio corresponding to $\hat{f}_i$; compute $\hat{V}_i = V(\hat{f}_i)$
   - Determine the loss $L_i = V_0 - \hat{V}_i$ where V_0 is the initial value of the portfolio
2. Sort $\{L_i\}_{i=1}^N$ in increasing order and denote $L_{(1)} \le L_{(2)} \le ... \le L_{(N)}$
3. $VaR_{\alpha,\tau}$ is $L_{(\alpha(N+1))}$ if $\alpha(N+1) \in \mathbb{N}$, otherwise we interpolate between the suitable losses

Remark 6.2. The main problem here is that we need to generate $\hat{f}_i$, but luckily, there are 3 powerful tools to do this:

<u>Historical Simulation</u>

- Define f_{-i}^j as the value of j^{th} risk factor i periods ago for $i = 1, ..., N$ and $j = 1, ..., k$
- The time dependent historical change for factor j and past period i is

$$C_i^j = \frac{f_{-i+1}^j}{f_{-i}^j}$$

where we can now use the estimate

$$\hat{f}_j = f_0^j C_\tau^j = \frac{f_0^j f_{-\tau+1}^j}{f_{-\tau}^j}$$

Monte-Carlo Methods (full valuation)

- If one of the risk factors is a stock, we can assume that it is lognormal, which allows us to use the Black-Scholes model to get (with $f = S_t$):

$$\hat{f}_\tau = S_0 e^{(\mu - \sigma^2/2)\tau + \sigma\sqrt{\tau}z}$$

where z is a randomly generated normal variate that can be simulated using the Box-Muller algorithm or cdf inversion

Monte-Carlo Methods (partial valuation):

- This method uses the various n^{th} order Taylor approximations for $V_\tau - V_0$ to get an estimate for V_τ with dS (the instantaneous change in a single factor) as $(S_\tau - S_0)$
- The first order approximation is called the MC-Delta-Normal Approach and is

$$V_\tau = V_0 + \triangle(\hat{S}_{\tau,i} - S_0)$$

- The second order approximation is called the MC-Delta-Gamma-Normal Approach and is

$$V_\tau = V_0 + \triangle(S_\tau - S_0) + \frac{1}{2}\Gamma(S_\tau - S_0)^2$$

- Variances and expectations for $(V_\tau - V_0) \approx dV$ can be calculated through modeling S_τ and choosing a model that generates N observations $\{\hat{S}_{\tau,i}\}_{i=1}^N$ which will produce N observations $\{\hat{V}_{\tau,i}\}_{i=1}^N$
- VaR is then approximated using the Delta-Normal, Delta-Gamma-Normal, or Cornish-Fisher expansion adjustments using the above proxy for dV

6.4 Advantages and Disadvantages

(The following is taken *almost* verbatim from course notes)

Standard VaR

Advantages:

- The analytical formula for approximations are easy to compute
- Very simple and based on Markowtiz modern portfolio theory

Disadvantages:

- Requires volatility and correlation estimators
- Only gives a single value and not a confidence interval
- Cannot be used for sensitivity analysis

Historical Simulation

Advantages:

- Don't need to model the behaviour of the risk factor factor
- Widely accepted by management and trading community because it is easy to understand
- Fat tails, skewness, and others can be captures as long as they are in the data set

Disadvantages:

- Requires a large enough N to produce a "rich" sample
- Cannot accommodate change in the market structure
- Cannot be used for sensitivity analysis

Monte Carlo Simulation

Advantages:

- Flexible: can be applied even with complex models for the risk factors, including those that deal appropriately with skewness, kurtosis, variation in volatility, etc.
- Allows calculation of confidence intervals for VaR
- Fat tails, skewness, and others can be captures as long as they are in the data set
- Allows calculation of confidence intervals for VaR
- Sensitivity analysis can be performed and stress testing
- Can be used to compute other risk measures, such as the Conditional Tail Expectation (CTE)/Tail Var

Disadvantages:

- Computationally expensive
- Slow convergence of the error which is $O(1/\sqrt{N})$
- Need to estimate model's parameters.

6.5 Coherent Risk Measures

(Taken verbatim from notes)

Although VaR is popular in practice due to its simplicity, it has some drawbacks. An important one is that it does not satisfy the axioms of a coherent risk measure.

Definition 6.6. A risk measure ρ is a functional mapping of a given risk X to a non-negative real number.

Definition 6.7. Consider two arbitrary risks X and Y. A risk measure ρ is called a coherent risk measure if it satisfies the following axioms:

1. The risk measure must be bounded from above by the maximum loss: $\rho(X) \leq \max(X)$
2. The risk measure must be bounded below by the expected value of the loss: $E(X) \leq \rho(X)$
3. A risk measure should be scale invariant: $\rho(aX) = a\rho(X)$, for $a \geq 0$.
4. A risk measure should be scalar additive: $\rho(X + b) = \rho(X) + b$ for $b \geq 0$, and a degenerate risk should have a risk measure equal to its certain loss: if $P(X = b) = 1$, then $\rho(X) = b$, for $b \geq 0$.
5. A risk measure should be sub-additive: $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

One can find relatively simple counterexamples proving that VaR fails to satisfy property 5, the sub-additivity property. As an alternative, the conditional tail expectation (CTE) defined by $E\left(L|L > VaR_{\alpha,n}\right)$ satisfies all of the above axioms.

7 Credit Risk

- Remark that in computing probabilities, we tend to use the Black-Scholes formula that involves μ_V (Merton's model), but in pricing, we use the formula that involves the risk-free rate r (options pricing)

Types of models

- *Static v. Dynamic*: static models are for credit risk management while dynamic models are for pricing risky securities
- *Structural and Threshold v. Reduced-form*: Threshold models are when default occurs when a selected random process falls under a threshold; reduced form models are when the time to default is modeled as a non-negative random variable whose distribution depends on a set of economic variables

Challenges of Credit Risk Management

- *Lack of public information and data*; interpreted as-is
- *Skewed loss distributions*; problems of frequent small profits and occasional large losses
- *Dependence modeling*; defaults tend to happen simultaneously and this impacts the credit loss distribution

Structural Models of Default

- Let S_t, B_t be the equity and debt values and of a firm at time t respectively; these are modeled as stochastic processes
- Denote $V_t = S_t + B_t$ where V_t is the firm's value
- Assume that no dividends are paid and a payment B is paid at time T from the firm issuing a bond
- At time T we have
$$S_T = \max(0, V_T - B)$$
$$B_T = \min(V_T, B) = B - \max(0, B - V_T)$$
and so V_T is the payoff of a call option S_T of strike B, B units of a T year ZCB
  - This is because at time T, if $V_T < B$, the whole firm liquidates its assets to debtholders since it has defaulted and missed a payment
  - In the former case, since shareholders are paid last, they get nothing
  - Thus default occurs when $V_T < B$

Merton's Model

- Merton's model assumes V_t behaves as Brownian motion and implies
$$\begin{aligned} dV_t &= \mu_V V_t dt + \sigma_V V_t dB_t \\ \implies V_t &= V_0 e^{(\mu_V - \sigma_V/2)^2 + \sigma B_t} \end{aligned}$$
where $B_t \sim N(0, t)$.
- This implies that V_t is lognormally distributed and compute quantities like
$$\begin{aligned} P(\text{default}) &= P(V_T \leq B) = P(\ln V_T \leq \ln B) \\ &= P\left(\mathcal{N}(0,1) \leq \frac{\ln B - \ln V_0 - \left(\mu_V - \sigma_V^2/2\right) T}{\sigma_V \sqrt{T}}\right) \end{aligned}$$

- Going back to the first point of this section, let r be the risk-free rate. If a security has a payoff of $h(V_T)$ at time T, then its price is
$$E_Q(e^{-rT}h(V_T))$$
where this expectation is done under the risk-neutral measure.
- This is equivalent to
$$V_t = V_0 e^{(r-\sigma_V^2/2)t+\sigma_V B_t}$$
which is the Black-Scholes framework under r

Threshold Models

- Used to model default in the case of a portfolio of securities issued by a large number of obligors
- This is a generalization of Merton's model where firm i defaults if $V_{T,i} < B_i$
- In a general threshold model, firm i defaults if its associated "critical" random variable X_i falls below some threshold d_i

Threshold Model Notation:

- Let d_{ij} be the critical threshold of firm i at rating j (e.g. credit rating)
- Let $D = [d_{ij}]_{m\times n} \in \mathbb{R}^{m\times n}$ where $X_i < d_{i1}$ implies default
- Let S_i be the state of firm i with $S_i \in \{0,1,...,n\}$ and $S_i = j \iff d_{ij} < X_i \le d_{i(j+1)}$ with $d_{i,0} = -\infty$, $d_{i(n+1)} = \infty$
- $S_i = 0$ is true iff there is default
- Let $Y_i = \chi_{X_i(T)<d_{i1}}$, the default indicator variable for X_i
- We denote the marginal cdf of X_i through the following equivalent forms:
$$\bar{p}_i = P(X_i \le d_i) = F_{X_i}(d_i) = F_i(d_i) = P(Y_i = 1)$$
- $M = \sum_{i=1}^m Y_i$ is the number of obligors who have defaulted at time T
- $L = \sum_{i=1}^m \delta_i e_i Y_i$ is the overall loss of the portfolio where e_i is the exposure of firm i and δ_i is the fraction of money that is lost from default
- The default correlation is given as
$$\rho(Y_i, Y_j) = \frac{E(Y_iY_j) - \bar{p}_i\bar{p}_j}{\sqrt{(\bar{p}_i - \bar{p}_i^2)(\bar{p}_j - \bar{p}_j^2)}}$$

7.1 Copula-Based Models

Example 7.1. If $Y = 1 - e^{-\lambda X}$ where $X \sim Exp(\lambda)$, find the cdf of Y.

Solution. Remark that $y \in (0,1)$ and
$$P(Y \le y) = P\left(X \le -\frac{1}{\lambda}\ln(1-y)\right) = F_X\left(-\frac{1}{\lambda}\ln(1-y)\right) = 1 - e^{-\lambda\left[-\frac{1}{\lambda}\ln(1-y)\right]} = y$$
and so $Y \sim Unif(0,1)$.

Remark 7.1. The above holds for any arbitrary cdf. That is, $Y = F_X(x) \implies Y \sim Unif(0,1)$.

Example 7.2. Independence

Let $U_1 \sim Unif(0,1)$ and $U_2 \sim Unif(0,1)$ independent of U_1. Find the joint cdf of (U_1, U_2).

Solution. By independence,

$$\begin{aligned} F_{U_1U_2}(u_1,u_2) &= F_{U_1}(u_1)F_{U_2}(u_2) \\ &= u_1 \cdot u_2, 0 \le u_1, u_2 \le 1 \end{aligned}$$

Example 7.3. Perfect Negative Dependence

Suppose that $U_1 \sim Unif(0,1)$ and $U_2 = 1 - U_1$. Find the joint cdf of (U_1, U_2).

Solution. By direct evaluation,

$$\begin{aligned} F_{U_1,U_2}(u_1,u_2) &= P(U_1 \le u_1, 1 - U_1 \le u_1) \\ &= P(1 - u_2 \le U_1 \le u_1) \\ &= \begin{cases} 0 & u_1 < 1 - u_2 \\ u_1 + u_2 - 1 & u_1 \ge 1 - u_2 \end{cases} \\ &= \max(0, u_1 + u_2 - 1) \end{aligned}$$

Example 7.4. Perfect Positive Dependence

Suppose that $U_1 \sim Unif(0,1)$ and $U_2 = U_1$. Find the joint cdf of (U_1, U_2).

Solution. By direct evaluation,

$$\begin{aligned} F_{U_1,U_2}(u_1,u_2) &= P(U_1 \le u_1, U_2 \le u_2) \\ &= P(U_1 \le \min(u_1,u_2)) \\ &= \min(u_1,u_2) \end{aligned}$$

Definition 7.1. A 2-dimensional copula C is a joint distribution on $[0,1] \times [0,1]$ with uniform marginal distributions. That is, if $U_1, U_2 \sim Unif(0,1)$ then the copula is

$$C(u_1,u_2) = P(U_1 \le u_1, U_2 \le u_2)$$

Basically, a copula provides us with a joint distribution of uniform r.v.s.

Example 7.5. (Examples of copulas)

These have been showcased in the previous examples:

1) Independence Copula

2) Perfect Negative Dependence Copula

3) Perfect Positive Dependence Copula

Remark 7.2. Let's examine the 2nd type. Let

$$\begin{aligned} X_1, X_2 \sim F_{X_1,X_2}(x_1,x_2) &= P(X_1 \le x_1, X_2 \le x_2) \\ &= P\left(F_{X_1}(X_1) \le F_{X_1}(x_1), F_{X_2}(X_2) \le F_{X_2}(x_2)\right) \\ &= P\left(U_1 \le F_{X_1}(x_1), U_2 \le F_{X_2}(x_2)\right) \\ &= P\left(U_1 \le u_1, U_2 \le u_2\right), u_1 = F_{X_1}(x_1), u_2 = F_{X_2}(x_2) \end{aligned}$$

Theorem 7.1. *(Sklar's Theorem) Let $F_{X_1,X_2}(x_1,x_2)$ be the joint distribution of X_1 and X_2 where X_1 and X_2 have marginal distributions F_{X_1} and F_{X_2} respectively. Then there exists a copula $C(u_1,u_2)$ such that*

$$F_{X_1X_2}(x_1,x_2) = C\left(F_{X_1}(x_1), F_{X_2}(x_2)\right)$$

C is unique if the marginal distribution functions are continuous.

Corollary 7.1. *If C is a copula and F_{X_1}, F_{X_2} are the marginal distribution functions, then a joint for (X_1,X_2) is given by*

$$F_{X_1X_2}(x_1,x_2) = C\left(F_{X_1}(x_1), F_{X_2}(x_2)\right)$$

Example 7.6. Suppose that $X_i \sim Exp(\lambda_i)$ for $i = 1, 2$ and you want to model the dependence structure by a negative dependence copula. Then the joint distribution function of (X_1, X_2) is given by

$$F_{X_1X_2}(x_1, x_2) = C(1 - e^{-\lambda_1 x_1}, 1 - e^{-\lambda_2 x_2}) = \max(0, 1 - e^{-\lambda_1 x_1} - e^{-\lambda_2 x_2})$$

Example 7.7. Gauss Copula

Denote the joit cdf of $(X_1, ..., X_n)$ by $\Phi_\Sigma(x_1, ..., x_n)$. Usig Sklar's Theorem, there exists a copula $C_\Sigma(u_1, ..., u_n)$ (called the Gauss Copula) such that

$$\Phi_\Sigma(x_1, ..., x_n) = C_\Sigma(\Phi(x_1), ..., \Phi(x_n))$$

where $\Phi(x) = P(\mathcal{N}(0, 1) \leq x)$.

Example 7.8. Li's Model

Suppose that we have m firms and x_i is the time to default of firm i. Let $d_i = T$, the time horizon and assume that $X_i \sim Exp(\lambda_i)$ so that $F)_i(t) = 1 - e^{-\lambda_i t}$. Use the Gauss copula to model the dependence structure. That is

$$P(X_1 \leq t_1, ..., X_m \leq t_m) = C_\Sigma(F_1(t_1), ..., F_m(t_m))$$

Let

$$Y_i = \begin{cases} 1 & \text{firm defaults} \\ 0 & \text{otherwise} \end{cases}$$

Assume $m = 4$. Then we have

$$\begin{aligned} P(Y_1 = 1, Y_1 = 1) &= P(X_1 \leq d_1, X_2 \leq d_2, X_3 \leq \infty, X_4 \leq \infty) \\ &= C_\Sigma(F_1(d_1), F_2(d_2), 1, 1) \\ &= C_\Sigma(1 - e^{-\lambda_1 d_1}, 1 - e^{-\lambda_2 d_2}, 1, 1) \end{aligned}$$

and software is required at time point.

Example 7.9. Supose we have two firms for $i = 1, 2$ and X_i is the asset's value for firm i at $t = 1$ year. Suppose that $X_1 \sim Exp(10)$ and $X_2 \sim Exp(5)$. We have the following states.

States S_i	0	1	2
Credit Rating	D	B	A

The thresholds are $\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} = \begin{pmatrix} 8 & 11 \\ 3 & 7 \end{pmatrix}$ with $S_i = j \iff d_{i,j} \leq X_i < d_{i,j+1}$ for $j = 0, 1, 2$. Assume that $d_{i,0} = -\infty, d_{i,3} = \infty$. The copula is $C(u_1, u_2) = (u_1^{-1} + u_2^{-1} - 1)$. We want to find

1) P(Both firms default)

This is

$$P(X_1 < 8, X_2 < 3) = C\left[F_{X_1}(8), F_{X_2}(3)\right] = \left[F_{X_1}^{-1}(8) + F_{X_2}^{-1}(3) - 1\right] = 0.32978$$

2) P(Neither default)

This is

$$\begin{aligned} P(X_1 > 8, X_2 > 3) &= 1 - P(X_1 < 8) - P(X_2 < 3) + P(X_1 < 8, X_2 < 3) \\ &= 1 - F_{X_1}(8) - F_{X_2}(3) + C\left[F_{X_1}(8), F_{X_2}(3)\right] \\ &= 0.32792 \end{aligned}$$

3) P(Firm 1 has a credit rating of B while firm 2 defaults)

This is

$$\begin{aligned} P(8 < X_1 < 11, X_2 < 3) &= P(X_1 < 11, X_2 < 3) - P(X_1 < 8, X_2 < 3) \\ &= C\left[F_{X_1}(11), F_{X_2}(8)\right] - C\left[F_{X_1}(8), F_{X_2}(3)\right] \\ &= 0.03850 \end{aligned}$$

4) The loss given firm i defaults at $t=1$ is \$100 for $i=1,2$

(i) Construct the distribution of the overall loss L on the portfolio

(ii) Find the 60% VaR and 60% CTE

Solution. For part (i), Recall that $P(\text{both firms default})=0.32978$, $P(\text{neither firms default})=0.32792$. We then have

$$L=\begin{cases}0 & 0.32792\\ 100 & P(X_1<8,X_2>3)+P(X_1>8,X_2<3)=0.3423\\ 200 & 0.32978\end{cases}$$

and thus,

$$P(L\le l)=\begin{cases}0.32792 & l=0\\ 0.67022 & l=100\\ 1 & l=200\end{cases}$$

So

$$VaR_{0.6,1}=100, CTE_{0.6,1}=\frac{100(0.3423)+200(0.3297)}{0.3423+0.3297}\approx 149$$

Proposition 7.1. *In general,*

$$\begin{aligned}P(a<X_1<b,f<X_2<g) &= P(X_1<b,X_2<g)-P(X_1<b,X_2<f)-P(X_1<a,X_2<f)+P(X_1<a,X_2<f)\\ &= C\left[F_{X_1}(b),F_{X_2}(g)\right]-C\left[F_{X_1}(b),F_{X_2}(f)\right]-C\left[F_{X_1}(a),F_{X_2}(f)\right]+C\left[F_{X_1}(a),F_{X_2}(f)\right]\end{aligned}$$

Note 2. Not every function is a copula. This is because a copula is a joint distribution of uniform random variables so it must satisfy all of the properties of a joint distribution:

i) $C(u,0)=C(0,u)=0$

ii) $C(u,1)=C(1,u)=u$

iii) $C(u_1,u_2)$ is increasing in each of u_1 and u_2

Remark that that $C(u_1,u_2)=u_1+u_2$ is not a copula because at $u_1=u_2=1$ we have $C=2$.

Example 7.10. Suppose that

$$F_{X_1X_2}(x_1,x_2)=\begin{cases}0 & \text{otherwise}\\ 1-e^{-x_1}-e^{-x_2}+e^{-(x_1+x_2+ax_1x_2)} & x_1,x_2\ge 0\end{cases}$$

(i) What is the formula for the copula?

Solution. The copula is explicitly

$$C(u_1,u_2)=F_{X_1,X_2}(F_{X_1}^{-1}(u_1),F_{X_2}^{-1}(u_2))$$

where $u_1=F_{X_1}(x_1)$ and $u_2=F_{X_2}(x_2)$. Now

$$F_{X_1}(x_1)=F_{X_1X_2}(x_1,\infty)=1-e^{-x_1}\implies\implies F_{X_1}^{-1}(x_1)=-\ln(1-x_1)$$

$$F_{X_2}(x_2)=F_{X_1X_2}(\infty,x_2)=1-e^{-x_2}\implies F_{X_2}^{-1}(x_2)=-\ln(1-x_2)$$

and hence

$$\begin{aligned}C(u_1,u_2) &= F_{X_1X_2}(-\ln(1-u_1),-\ln(1-u_2))=1-(1-u_1)-(1-u_2)+(1-u_1)(1-u_2)e^{-a\ln(1-u_1)(1-u_2)}\\ &= u_1+u_2-1+(1-u_1)(1-u_2)e^{-a\ln(1-u_1)(1-u_2)}\end{aligned}$$

(ii) Find the value of a such that $C(u_1,u_2)$ is the independence copula?

Solution. By inspection, $a=0$ gives the independence copula.

(iii) Suppose you want to use the negative dependence copula $C(u_1, u_2) = \max(u_1 + u_2 - 1, 0)$ with exponential marginals. (Doesn't relate to the previous parts) Suppose that $X_i \sim Exp(\lambda_i)$ for $i = 1, 2 \implies F_{X_i}(x) = 1 - e^{-\lambda_i x}$ for $x \geq 0$. Construct the joint cdf for X_1 and X_2.

Solution. We have

$$F_{X_1X_2}(x_1, x_2) = C\left(F_{X_1}(x), F_{X_2}(x)\right) \implies F_{X_1X_2}(x_1, x_2) = \max(1 - e^{-\lambda_1 x_1} - e^{-\lambda_2 x_2}, 0)$$

(iv) Suppose that X_1 and X_2 (from (iii)) are critical r.v.s for firm 1 and 2 respectively. The default threshold for both firms is 10. That is, firm i defaults if $X_i < 10$ for $i = 1, 2$. How would you go about finding the default correlation between the two firms?

Solution. Let Y_i be the indicator of default for firm i for $i = 1, 2$. That is

$$Y_i = \begin{cases} 1 & X_i < 10 \\ 0 & X_i \geq 10 \end{cases}$$

Remark that because $Y_i \sim Ber(F_{X_i}(10))$ then $Var[Y_i] = F_{X_i}(10)\left[1 - F_{X_i}(10)\right]$ and $E[Y_i] = F_{X_1}(10)$ with $E[Y_1Y_2] = F_{X_1X_2}(x_1, x_2)$ and hence

$$\begin{aligned} Cor(Y_1, Y_2) &= \frac{F_{X_1X_2}(10, 10) - F_{X_1}(10)F_{X_2}(10)}{\sqrt{F_{X_1}(10)\left[1 - F_{X_1}(10)\right]F_{X_2}(10)\left[1 - F_{X_2}(10)\right]}} \\ &= \frac{\max(1 - e^{-10\lambda_1} - e^{-10\lambda_2}, 0) - (1 - e^{-10\lambda_1})(1 - e^{-10\lambda_2})}{\sqrt{(1 - e^{-10\lambda_1})(e^{-10\lambda_1})(1 - e^{-10\lambda_2})(e^{-10\lambda_2})}} \end{aligned}$$

Final Exam Review

- 1 question from unit 5/6 -> Immunization/Duration (Most difficult) with concept questions

- Maybe a concept question from unit 7 (dedication)

- Rest will be on VaR, Binomial trees, Merton's Model, Credit Risk, and Copulas

- A few concept questions

- There will be a threshold model question

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