ACTSC 372 (Winter 2013 - 1135)

Corporate Finance II

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These notes are currently a work in progress, and as such may be incomplete or contain errors.

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Abstract

The purpose of these notes is to provide the reader with a secondary reference to the material covered in ACTSC 372. The formal prerequisite to this course is ACTSC 371 but this author believes that the overlap between the two courses is less than 5%. Readers should have a good background in linear algebra, basic statistics, and calculus before enrolling in this course.

Errata

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Office hours: M3 2018, T-Th from 4-5pm

Optional Textbook: Fahmy, H, Lecture Notes in the Theory of Finance

4 Assignments: 5% each, 1 Midterm (Friday, June 14th, 2013; 1:20 to 3:00 pm; M3 1006): 30%, 1 Final (Comprehensive): 50%

1 Microeconomic Principles

Definition 1.1. A *market* is defined uniquely through a demand function, a supply function and an equilibrium price.

Definition 1.2. The theoretical price is the point of intersection between the the demand and supply curves. In reality, prices are distorted, relative to a benchmark.

Remark 1.1. The supply of financial assets is fixed while the demand for assets is determined through methods in Chapter 2 + 3.

1.1 Consumer-Choice Theory under Choice

Suppose we have one consumer that must choose between two goods x_1 and x_2 such that the satisfaction (utility) from the consumption of both good is maximized. This utility is measured by utils. We subject this scenario to money income (m).

In this optimization problem, we consider two sides: ability and preference.

Under *ability*, the budget of the consumer is the constraint. Here are our assumptions:

i) 2 goods: x_1, x_2

ii) Rationality (consumer acts rationally)

iii) $m =$ money income, P_1 is the price of good 1, P_2 price of good 2 (short-run analysis assumes that these are constant)

Algebraically, the constraint is given by

$$m = P_1x_1 + P_2x_2$$

Under *preference*, for a good x, we should be given a total utility TU_x and marginal utility MU_x of our good x. Here, TU_x is the accumulation of utility for a given total number of goods consumed (i.e. $\mu(x) = TU_x$) and the marginal utility is the partial derivative of the total utility (i.e. $\frac{\partial}{\partial x}\mu(x) = TU_x$). Generally, in discrete terms, $MU_x|_{x=t} = TU_x|_{x=t} - TU_x|_{x=t-1}$. Also, MU_X is generally monotonically decreasing which is called the law of diminishing marginal returns.

Relationships: If $A, B \in \mathbb{R}$, then $A \geq B, A \leq B$ or $A \geq B$ and $A \leq B \implies A \sim B$. In preference, A and B are bundles (i.e. they are functions of x_1 and x_2). [1] [2]

Axiom 1. *For modeling preference, we have the following axioms.*

1) Completeness: The space of bundles is totally-ordered ($A \leq B$, $A \geq B$ or $A \sim B$ but nothing more) [allows comparisons]

2) Transitivity: For any 3 bundles A, B and C, if $A \geq B$ and $B \geq C$ then $A \geq C$ [allows rankings]

3) Continuity: We have $x \leq y$ given that $x_n \leq y_n$ for all $n \in \mathbb{P}$ and $x_n \to x$ and $y_n \to y$ AND x and y can be modeled as continuous functions $\mu(x)$ and $\mu(y)$ where $\mu(x) \leq \mu(y)$. This can also be generalized to bundles.

4) Monotonicity: $\frac{\partial}{\partial x_1}\mu(x_1, x_2), \frac{\partial}{\partial x_2}\mu(x_1, x_2) > 0$ [more is better]

5) Diminishing MRS (Marginal Rate of Substitution): (Example) Given $\mu(x_1, x_2) = x_1 \cdot x_2$, let $\mu = \bar{\mu} = 100$ utils. To keep a constant level of utils when one good is consumed, the other good must be reduced. The curve for a given level of utils is called an indifference curve, IC. The slope of the indifference curve is called the MRS. [convexity of IC]

[1] Notation: If bundle A is at least as preferred to bundle B, then we write $A \geq B$; if $A \leq B$ and $A \geq B$ then $A \sim B$ or A is indifferent to B.

[2] Note that our objective is to model the ranking of bundles and to create a utility from consumption as a function.

Axioms 1-5 give you *well-behaved preference* (Convex ICs)

Axioms 1-3 ensure the representation of preference in a $\mu(\cdot)$ function

Definition 1.3. The function μ that satisfies all 5 axioms is called the *Cobb-Douglas* utility function and is in the form

$$U(x_1, x_2) = x_1^{\alpha} \cdot x_2^{\beta}, \alpha > 0, \beta > 0$$

and for a constant utility J, the *convex* curve $J = x_1^{\alpha} \cdot x_2^{\beta}$ is called an indifference curve. Recall that the slope of the indifference curve is called the marginal rate of substitution (MRS). It can be shown that

$$MRS_{1,2} = -\frac{MU_1}{MU_2} = -\left(\frac{\partial U}{\partial x_1}\right)\left(\frac{\partial U}{\partial x_2}\right)^{-1}$$

The proof is trivial using multivariate calculus

$$\frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 = 0 = dJ$$

Problem 1.1. We want to solve the optimization problem

$$\max_{\{x_1, x_2\}} U(x_1, x_2), g(x_1, x_2) = m - P_1 x_1 - P_2 x_2 = 0$$

Using Lagrange multipliers

$$\nabla U(x_1, x_2) = \lambda \nabla g(x_1, x_2) \implies L(x_1, x_2, \lambda) = \nabla U(x_1, x_2) - \lambda \nabla g(x_1, x_2)$$

where we call g the *feasibility condition*. Using the above first order conditions (use $\frac{\partial L}{\partial x_1} = 0$, $\frac{\partial L}{\partial x_2} = 0$, $\frac{\partial L}{\partial \lambda} = 0$), we get that

$$\frac{MU_1}{MU_2} = \frac{P_1}{P_2}$$

which is known as the *tangency condition*.

2 Consumer Choice Theory

Notation 1. We denote all random variables with tildes (e.g. $\tilde{Z}$). Risky investments in finance are random variables that have value not equal to their statistical expectation but the expected utility from the outcome.

Example 2.1. People see financial gains relative to their own wealth. A poor student would value a \$100 gain much more than a rich millionaire would.

Example 2.2. Suppose that your initial wealth w is \$4000. You have merchandise abroad worth \$8000. You want to transport the goods here but the ship that carries the goods has $\frac{1}{10}$ chance of sinking. Let $\tilde{Z}$ be the risk of the gamble (disregarding wealth). We will generally use Greek letters to represent probabilities.

$$\tilde{Z} = \begin{cases} -8000 & \alpha = \frac{1}{10} \\ +8000 & (1-\alpha) = \frac{9}{10} \end{cases}$$

The risky wealth $\tilde{w}$ after the gamble is

$$\tilde{w} = \begin{cases} 4000 & \alpha = \frac{1}{10} \\ 12000 & (1-\alpha) = \frac{9}{10} \end{cases} \equiv (w_1, w_2, \alpha)$$

where $w_1 = 4000, w_2 = 12000$, and $\alpha = \frac{1}{10}$ or denoted by $\tilde{w}(4000, 12000; \frac{1}{10})$. By observation, $E(\tilde{w}) = \$11200$.

Now suppose that we have the alternative of shipping the goods through two ships carrying \$4000 worth of goods each. Our new wealth $\tilde{w}^*$ has expectation

$$E(\tilde{w}^*) = 4000 + 8000\left(\frac{81}{100}\right) + 4000\left(\frac{18}{100}\right) + 0\left(\frac{1}{100}\right) = \$11200$$

with $\tilde{w}^* = (12000, 4000, 8000, 8000; \alpha = \frac{81}{100})$.

2.1 The Expected Utility Theorem

Example 2.3. (St. Petersburg Paradox) Suppose you are tossing a coin with outcome H or T. Given n consecutive Hs, a monetary return of 2^n will be given with \$0 otherwise. For example, 1 H gives \$2 and 3 Hs gives 2^3=\$8. The expected wealth is

$$E[\tilde{w}] = \sum_{n=1}^{\infty} 2^n \left(\frac{1}{2^n}\right) = \infty$$

which does not make sense since no one is willing to pay $\$\infty$.

Bernoulli resolved this as follows. Any lottery should be valued according to the expected utility (EU) that it generates. So we treat wealth as a commodity or good so that we could have $U(w)$ such that

$U'(w) = MU(w) > 0$, $U''(w) = MU'(w) < 0$ (LAW OF DIMINISHING $MU(w)$)

Note that if $U(w)$ satisfies the above, then instead of using $E[\tilde{w}]$ we should use $E[U(\tilde{w})]$ and hence

$$E[U(\tilde{w})] = \sum_{n=1}^{\infty} U(2^n)\left(\frac{1}{2^n}\right) < \infty$$

since $U''(w) < 0$.

Building on this idea, for a gamble $\tilde{w}(w_1, w_2; \alpha)$ we need to construct a $U(\cdot)$ such that

$$EU(\tilde{w}) = \alpha U(w_1) + (1-\alpha)U(w_2)$$

which we call VN-M utility.

Example 2.4. (Tutorial 2 Q3) Suppose the consumer preference is characterized by $U(w) = \ln w$. Suppose that you are facing a lottery

$$\tilde{L}_1\left(50000, 10000; \frac{1}{2}\right)$$

Determine the lottery

$$\tilde{L}_2(x, 0; 1)$$

such that $L_2 \sim L_1$. So what is x such that $U(L_1) = U(L_2)$? First, using VN-M utility,

$$\begin{aligned} U\left(50000, 10000; \frac{1}{2}\right) &= \frac{1}{2}(\ln 50000 + \ln 10000) \\ &= U(x) = \ln x \\ \implies & U(x) \approx 22360.68 \end{aligned}$$

Theorem 2.1. *(The Expected Utility Theorem) We first need the following axioms to construct the VN-M utility function*

(1) Completeness (2) Transitivity (3) Continuity

[These guarantee the existence of a real-valued continuous $U(\cdot)$]

(4) Independence of irrelevant alternatives

[E.g. Consider the lotteries $\tilde{w}_x(z, x; \alpha)$ and $\tilde{w}_y(z, y; \alpha)$. When given z, if $\tilde{w}_x \geq \tilde{w}_y \implies x \geq y$; the z's are called mutually exclusive outcomes]

(5) Ranking

[Consider 2 outcomes x, y such that $a \geq x, y \geq b$. Then if $x \sim \tilde{w}_x(a, b; \alpha)$ and $y \sim \tilde{w}_y(a, b; \beta)$, where $a \sim b$ is when $E[a] = E[b]$, then $x \geq y \implies \tilde{w}_x \geq \tilde{w}_y$ iff $\alpha \geq \beta$, i.e. $\tilde{\omega}_x \equiv \alpha$ and $\tilde{\omega}_y \equiv \beta$]

(6) Measurability

[An outcome could be expressed as a lottery; there exists α for an outcome $a < x < b$ such that $x \sim (a, b; \alpha)$

(7) Bounded Set

[For any lottery there exists a least preferred and most preferred outcome]

If Axioms 1-7 are satisfied then for any lottery $\tilde{w}(x, y; \alpha)$ we have $U(\tilde{w}) = U(x)\alpha + U(y)(1 - \alpha)$.

Remark 2.1. Consider two goods x_1, x_2. When $\uparrow x_1 \implies MU_1 \downarrow$ and $\downarrow x_2 \implies MU_2 \uparrow$ while keep the utility constant (i.e. this is a move along an IC), we have $\downarrow |MRS| = \frac{MU_1}{MU_2}$ which implies that it is convex.

Summary 1. The optimal choice between two goods satisfies two conditions:

1) Tangency: $\frac{MU_1}{MU_2} = \frac{P_1}{P_2}$

- The interpretations of this condition are as follows. The standard interpretation says that the left side is the marginal benefit (MB) of good 1 relative to good 2. The right side would be the marginal cost (MC) of the last unit of good 1 relative to good 2. If MC > MB, we consume more of good 1 with MU_1 decreasing until we reach equilibrium. The opposite is true for when MC < MB.

2) Feasibility: $m = P_1x_1 + P_2x_2$

2.2 The Financial System

Summary 2. The financial system is divided into supply and demand segments. On the supply side, we have borrowers/investors which we consider as the business sector or producers. On the demand side, we have lenders/savers which are the consumers or household. Stocks and bonds connect supply to demand and funds connect demand to supply. What results is the equilibrium price of securities.

Problem 2.1. Why consumers are engaged in such financial contracts (between supply and demand segments)?

Because we want to save today to be better off in the future. That is, we are sacrificing consumption today to increase consumption in the future. This concept is called *consumption smoothing*.

Example 2.5. Suppose we are living in a 2-period world. In period 1 we save s_0, and consume c_0 with income y_0. In period 2 we save 0, consume $s_0(1 + r) + y_1$ with income y_1. We require some smoothing to get a constant level of consumption. Note that this means we need to look for an equivalent bundle that has constant consumption with the same utility and the previous situation.

For example if $U(c_0, c_1) < U(9, 9)$ and $U(c_0, c_1) > U(8, 10)$ then $U(9) > \frac{1}{2}[U(8) + U(10)]$ and the utility function is thus concave. This would imply that the utility function of any saver has to be strictly concave to match with consumption smoothing.

We assume this is true based on the intuition that for a fixed level of total consumption, consumers prefer a steady consumption per period as opposed to a unevenly distributed one.

2.3 Certain Equivalence

Definition 2.1. (Risk aversion) Consider the risky alternative $\tilde{w}\left(8, 10; \frac{1}{2}\right)$. We have the following facts.

- The certainty wealth is $E[\tilde{w}] = 9$.
- The utility from certainty is $U\left(E\left[\tilde{w}\right]\right) = U(9)$
- The utility from uncertainty is a function of $U(8)$ and $U(10)$ which is $E[U(\tilde{w})] = \left[\frac{1}{2}U(8) + \frac{1}{2}U(10)\right]$

Any *risk adverse* consumer would favor certainty over uncertainty. So a formal definition of risk adverse is

$$U\left(E\left[\tilde{w}\right]\right) > E\left[U(\tilde{w})\right]$$

Similarly, a consumer is risk seeking/loving if

$$U\left(E\left[\tilde{w}\right]\right) < E\left[U(\tilde{w})\right]$$

and risk neutral if

$$U\left(E\left[\tilde{w}\right]\right) = E\left[U(\tilde{w})\right]$$

Definition 2.2. We call the *risk premium* π, the maximum amount that a consumer is willing to pay to avoid risk. To do this, we need to find the *certainty equivalence wealth* w_{CE} which is the wealth level if the gamble is avoided. To find this, you need to find the utility from the gamble and work backwards. So $\pi = E[\tilde{w}] - w_{CE}$.

Example 2.6. Suppose $U(w) = \ln(w)$ which ensures concavity, where you are faced with the following lottery:

$$\tilde{z} = \begin{cases} +1 & \alpha = \frac{1}{2} \\ -1 & 1-\alpha = \frac{1}{2} \end{cases}$$

where when we have $E[\tilde{z}] = 0$ we call such a lottery an *actuarially fair gamble*. Suppose your initial wealth if $w = \$10$. Then define $\tilde{w} = \tilde{z} + w$. We have the following

1) Certainty outcome is $E[\tilde{w}] = \$10$

2) Uncertainty outcomes are $\$11$ and $\$9$

3) Utility from uncertainty is $E\left[U(\tilde{w})\right] = \frac{1}{2}\ln 9 + \frac{1}{2}\ln 11 = 2.2976$ utils

Note that since the utility is convex, the consumer here is risk adverse. We take the utility from uncertainty, 2.2976 utils, and find the corresponding level of wealth:

$$2.2976 = \ln w_{CE} \implies w_{CE} = e^{2.2976} = \$9.95$$

Thus the risk premium is $\pi = 10 - 9.95 = \$0.05$.

Remark 2.2. Another way to compute π is to find the *certainty or cash equivalence* CE as follows. We first define the $CE(\tilde{z}, w)$ as the sure *increase* in wealth that has the same effect on welfare as bearing the risk of the gamble or equivalently, it is the asking price of the risk. To calculate it, recall that

$$\begin{aligned} E\left[U(\tilde{w})\right] &= U(w_{CE}) \\ &= U(E[\tilde{w}] - \pi) \\ &= U(w + \underbrace{E[\tilde{z}] - \pi}_{CE}) \end{aligned}$$

and so we can see that $CE = E[\tilde{z}] - \pi$ since this increase in wealth makes us indifferent between taking the increase and taking the gamble. Thus, we compute π by

$$\pi = E[\tilde{z}] - CE$$

Example 2.7. (p. 65) Suppose we have two risk adverse consumers U, V with utility functions

$$U(w) = \sqrt{w}, V(w) = \ln(w)$$

and same wealth of \$4000. They are offered a gamble

$$\tilde{z} = \begin{cases} -2000 & \alpha = \frac{1}{2} \\ +2000 & 1-\alpha = \frac{1}{2} \end{cases}$$

with $\tilde{w} = w + \tilde{z}$. We first verify that U, V are well behaved ($U'(\cdot), V'(\cdot) > 0, U''(\cdot), V''(\cdot) < 0$). We will check U first

$$MU = U'(w) = \frac{1}{2}w^{-\frac{1}{2}} > 0, U''(w) = -\frac{1}{4}w^{-\frac{3}{2}} < 0$$

and for V we have

$$MV = V'(w) = \frac{1}{w} > 0, V''(x) = -\frac{1}{w^2} < 0$$

Next, we compute π_U and π_V. Using

$$E\left[U(\tilde{w})\right] = U(w + E[\tilde{z}] - \pi)$$

we can solve the following equations

$$\begin{aligned} \frac{1}{2}\left(\sqrt{2000} + \sqrt{6000}\right) &= \sqrt{4000 + 0 + \pi_U} \\ \frac{1}{2}\left(\ln 2000 + \ln 6000\right) &= \ln\left(4000 + 0 + \pi_V\right) \end{aligned}$$

to get values for the risk premiums.

Definition 2.3. (Arrow & Pratt (A-P) Measure of Risk Aversion) We define here an approximation for π, the risk premium. Here is the set up

1) Assume $U(\cdot)$ is well behaved

2) w is the initial wealth

3) $E[\tilde{z}] = 0$, $var[\tilde{z}] = E[\tilde{z}^2] = \sigma^2$

The risk premium should satisfy $E\left[U(\tilde{w})\right] = U(\tilde{w} - \pi) \implies E\left[U(w + \tilde{z})\right] = U(\tilde{w} - \pi)$. The utility on the left side can be approximated with a 2nd order Taylor expansion about $\tilde{z}$ and the right hand side can be approximated with a 1st order Taylor expansion about π. This gives

$$E\left[U(w) + \tilde{z}U'(x) + \frac{1}{2}\tilde{z}^2 U''(w)\right] \approx U(w) - \pi U'(w)$$

which is accurate when $\tilde{z}^3$ and π^2 are very small. Simplifying we can get

$$U(w) + \frac{U''(w)}{2}\sigma^2 \approx U(w) - \pi U'(w) \implies \pi \approx \frac{1}{2}\sigma^2 - \frac{U''(w)}{U'(w)}$$

where we call $\frac{U''(w)}{U'(w)}$ the *absolute risk aversion* term or $ARA(w)$. This approximation is called the *Arrow-Pratt Measure (A-P measure)*.

Remark 2.3. We make a few remarks about the above.

(1) π from A-P is useful only for small $\tilde{z}$ is small. Otherwise, we use the *Markowitz definition of risk aversion*, which we known as

$$U\left(E[\tilde{w}]\right) > E\left[U(\tilde{w})\right]$$

(2) If $ARA(w) > 0 \implies \pi > 0 \implies$ Risk Averse and $ARA(w) < 0 \implies \pi < 0 \implies \implies$ Risk Seeking $\implies U(\cdot)$ is convex. Also if $ARA = 0 \implies \pi = 0 \implies$ Risk Neutral.

(3) Suppose we have two agents A, B with different utilities $\triangle U$ and same initial wealth w. For small $\tilde{z}$ if U_A is more concave (large magnitude of second derivative) than U_B then A is more risk adverse than B and $ARA_A \geq ARA_B \implies \pi_A \geq \pi_B$.

(4) Now suppose we have two agents A, B with different wealth levels $\triangle w$ ($w_A < w_B$) and same initial utility functions $U_A = U_B$. Both are facing the same $\tilde{z}$. Arrow (1963) argued that wealthier people are willing to pay *less* to avoid a risk in general. This means $ARA_A > ARA_B \implies \pi_A > \pi_B$.

These 3 are known as the *decreasing absolute risk aversion* (DARA) features.

Example 2.8. $U(w) = \ln(w), U'(w) = \frac{1}{w}, U''(w) = -\frac{1}{w^2} \implies ARA(w) = \frac{1}{w} > 0$. Now $\frac{d(ARA)}{dw} = -\frac{1}{w^2} < 0$ and so U is DARA. This is an example of a verification technique: $\frac{d[ARA]}{dw} < 0$.

2.4 Monotonic Transformations of Utility Functions

Example 2.9. Say $v = 2w$, $u = w$. We claim that v is monotonic transformation (m.t.) of u. To verify, we follow the following steps:

1) Express $v = f(u) \implies v = 2u$

2) Check the sign of $\frac{dv}{du} = 2 > 0$

So v is a monotonic transformation of u.

Remark 2.4. Note if a utility function v is a m.t. of another function u, then both represent the same preference. That is, 2

$$MRS^V = MRS^U$$

Example 2.10. Let $u = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$ and $v = \frac{1}{2}\ln(x_1) + \frac{1}{2}\ln(x_2)$. We claim that v is a m.t. of u and $MRS^V = MRS^U$. To see this, note that

$$\ln u = \ln x_1^{\frac{1}{2}} + \ln x_2^{\frac{1}{2}} = v \implies v = \ln u \implies \frac{dv}{du} = \frac{1}{u} > 0$$

so v is a m.t. of u. It is simple to verify that

$$MRS^V = MRS^U$$

so this will be left as an exercise.

Remark 2.5. The A-P approximation of π is equivalent to Markowitz analysis iff the risk $\tilde{z}$ is small.

Example 2.11. $u = \ln w$ with initial wealth $w = \$10$ and $\tilde{z}(+1, -1; \frac{1}{2}) \implies \tilde{w}(11, 9; \frac{1}{2})$. From Markowitz first,

$$E[U(\tilde{w})] = U(E[\tilde{w}] - \pi) \implies E[U(\tilde{z} + w)] = U(E[\tilde{z}] + w - \pi)$$

and putting in our values, we get

$$\frac{1}{2}\ln 9 + \frac{1}{2}\ln 11 = \ln(10 + \pi) \quad \implies \quad \pi = 0.05$$

Now from A-P, we get

$$\pi \approx \frac{1}{2}\sigma_{\tilde{z}}^2 ARA(w) = \frac{1}{2}ARA(w) = \frac{1}{2}(0.1) = 0.05$$

where

$$\sigma_{\tilde{z}}^2 = E\left[(\tilde{z} - E[\tilde{z}])^2\right] = E[\tilde{z}^2] = \frac{1}{2}(1)^2 + \frac{1}{2}(-1)^2 = 1$$

2.5 Relative Risk Aversion

Definition 2.4. This is a unitless measure of relative risk defined by

$$RRA(w) = -\frac{\%\triangle MU}{\%\triangle w} = -\frac{\frac{\triangle MU}{MU}}{\frac{\triangle w}{w}} = -\frac{\frac{dMU}{MU}}{\frac{dw}{w}} = -\frac{\frac{dU'(w)}{U'(w)}}{\frac{dw}{w}} = -w\frac{U''(w)}{U'(w)} = w \cdot ARA(w)$$

and this can be interpreted as the % of wealth that an investor is willing to pay to get rid of a *proportional* risk. To put this into perspective, let

$$\tilde{z}_R = \frac{\tilde{z}}{w} = \text{proportional risk}, \pi_R = \frac{\pi}{w} = \text{proportional premium}$$

and note that

$$\tilde{z} = w \cdot \tilde{z}_R \implies Var[\tilde{z}] = \sigma_{\tilde{z}}^2 = \sigma^2 = w^2 Var[\tilde{z}_R] = w^2\sigma_{\tilde{z}_R}^2 = w^2\sigma_R^2$$

Also recall that

$$\pi \approx \frac{1}{2}\sigma^2 ARA(w) \implies \pi_R = \frac{\pi}{w} \approx \frac{\frac{1}{2}\sigma^2 ARA(w)}{w} = \frac{1}{2}\sigma_R^2 RRA(w)$$

Remark 2.6. It can be shown that (Pratt's Argument, 1963) RRA might be constant or perhaps increasing. Recently, the evidence shows that it is high at low wealth, decreases from low to mid wealth, increases from mid to high wealth and is high at high wealth. The general consensus is that RRA is constant which is known as $CRRA$. In a time series approximation, we can use the regression

$$w_t = f(w_{t-1}, ..., w_{t-k})$$

where a $LSTR2$ (logistic smooth transition regression method) would be ideal (p. 71-72).

Conclusion 1. A good utility function $U(\cdot)$ should be

- well-behaved : $U'(\cdot) > 0$, $U''(\cdot) < 0$
- DARA
- CRRA

(Typo on p.76-77 in course notes; Assignment 1 due on Monday)

(A1 Q5 notation: $\pi_R \approx \frac{1}{2}\sigma^2_{\tilde{z}_R} RRA(w) = \frac{1}{2}\sigma^2_{\tilde{z}} RRA(w)$)

2.6 Stochastic Dominance

Example 2.12. Given two alternatives:

x_1	$Pr(x_1)$	x_2	$Pr(x_2)$
10	0.4	10	0.4
1000	0.6	1000	0.4
2000	0	2000	0.2

we usually use a method called the Mean-Variance (M-V) criterion to select a gamble. It states that if $\mu_A > \mu_B \implies$ select A or $\sigma_A < \sigma_B \implies$ select A (highest return for lowest risk). Now in this case:

Project 1: $\mu_1 = E[x_1] = 64$, $\sigma_1 = 44$

Project 2: $\mu_1 = E[x_1] = 444$, $\sigma_1 = 779$

but we have contradicting viewpoints from the M-V criterion. So instead we use method called *stochastic dominance* which is based on the concept of probability matching and exceeding (using the cdf).

Definition 2.5. We first define *first order-stochastic dominance* (FSD). That is, we say that for two gambles x_1 and x_2 with their respective cdfs F_1 ad F_2, F_2 FSD F_1 if and only if F_2 is everywhere below and to the right of F_1. In this example, it is the case that F_2 for x_2 FSD F_1 for x_1.

Theorem 2.2. *For random payoffs x_1, x_2, $F_2(x_2)$ FSD $F_1(x_1)$ implies*

$$E[U_2(x_2)] \geq E[U_1(x_1)]$$

for all non-decreasing utility function $U(\cdot)$. That is FSD is a good criterion to select projects if we don't known the (risk) attitude of the investor.

Definition 2.6. We define *second order-stochastic dominance* (SSD). *F_2 SSD F_1 if*

$$\int_0^x [F_1(t) - F_2(t)]\ dt \geq 0, \forall x$$

where this integral is monotone.

Theorem 2.3. *F_2 SSD F_1 implies*

$$E[U_2(\cdot)] \geq E[U_1(\cdot)]$$

for all utility functions that are not decreasing and concave. Also, the investor is risk averse. The SSD is computed by taking the difference of the integrals of the two cdfs of the projects. See p.77 for an example.

3 Markowitz Portfolio Theory

This is also known as Mean-Variance portfolio theory or Markowitz analysis of portfolios.

Notation 2. We define

$$\tilde{r}_i \equiv \text{uncertain rate of return on Asset } i$$

$$rf \equiv r_f \equiv \text{certain rate of return}$$

where $i = 1, ..., n$ and rf is usually Treasury bills or government bonds. The mean of a risky rate of return is denoted by $E[\tilde{r}_i] = \mu_i$ and the variance is denoted by $Var[\tilde{r}_i] = \sigma_i^2 = \sigma_{ii}$. The covariance between 2 risky rates of return is denoted by $Cov(\tilde{r}_i, \tilde{r}_j) = \sigma_{ij}$. The matrix notation or compactified notation of general portfolio is

$$\tilde{\mathbf{r}} = \begin{bmatrix} \tilde{r}_1 \\ \tilde{r}_2 \\ \vdots \\ \tilde{r}_n \end{bmatrix}, \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

where $\mathbf{w}$ is an $n \times 1$ vector of weights of the assets and $\sum w_i = 1$.

Next, we define the *mean of portfolio returns* as

$$E[\tilde{r}_P] = \mu_P = E\left[\sum_{k=1}^{n} w_k \tilde{r}_k\right] = \sum_{k=1}^{n} w_k \mu_k = \mathbf{w}^t E[\tilde{\mathbf{r}}]$$

and the *variance of portfolio returns* as

$$Var[\tilde{r}_P] \equiv \sigma_P^2 = \mathbf{w}^t \Omega \mathbf{w}$$

where Ω is the variance-covariance matrix of individual asset returns. In particular, $\Omega_{ij} = \sigma_{ij}$.

3.1 Link to Consumer Theory

Example 3.1. Given a two asset portfolio $\mathbf{w} = \begin{pmatrix} w_1 & w_2 \end{pmatrix}^t$, $\tilde{\mathbf{r}} = \begin{pmatrix} \tilde{r}_1 & \tilde{r}_2 \end{pmatrix}^t$ we have $\tilde{r}_P = \mathbf{w}^t \tilde{\mathbf{r}} \implies E[\tilde{r}_P] = \mathbf{w}^t E[\tilde{\mathbf{r}}]$.

Remark 3.1. We establish the connection between $\mu - \sigma^2$ analysis and EU theory. Assume that we have a 2-period world. Here is a summary table of the situation in the view of period 0.

	Period 1	Period 2
Wealth	y_0	$\tilde{y}_1$
Price of Asset i	$P_{i,0}$	$P_{i,1}$

and we also define

$$(0)\ \tilde{r}_i = \frac{P_{i,1} - P_{i,0}}{P_{i,0}}$$

At $t = 0$, y_0 is allocated on n assets such that

$$(1)\ y_0 = \sum_{k=0}^{n} a_k \cdot P_{k,0}$$

where a_k is the amount purchased of asset k at $t = 0$. Next, define

$$(2)\ w_i = \frac{a_i \cdot P_{i,0}}{y_0}$$

where (3) $\sum w_i = 1$. The return on the portfolio or the portfolio rate of return is (4) $r_p = \mathbf{w}^t \tilde{\mathbf{r}}$. The wealth of the consumer at period 1, $\tilde{y}_1$ is

$$(5)\ \tilde{y}_1 = \sum_{k=0}^{n} a_k P_{k,1} = \sum_{k=0}^{n} a_k (P_{k,1} - P_{k,0}) - \underbrace{\sum_{k=0}^{n} a_i P_{i,0}}_{y_0}$$

from equation (1). Multiplying by $P_{i,0}$ on both the numerator and denominator, we get

$$(6)\ \tilde{y}_1 = \sum a_i \cdot P_{i,0} \left(\underbrace{\frac{P_{i,1} - P_{i,0}}{P_{i,0}}}_{r_i} \right) + y_0 = y_0 \left(1 + \left(\frac{\sum a_i P_{i,0}}{y_0} \right) r_i \right) = y_0(1 + \mathbf{w}^t \tilde{\mathbf{r}})$$

and taking expectations and variances, we get

$$(7)\ E\left[\tilde{y}_1\right] = y_0(1 + w' E[\tilde{\mathbf{r}}]) = y_0(1 + \mu_P)$$

$$(8)\ Var[\tilde{y}_1] = y_0^2 \mathbf{w}^t \Omega \mathbf{w} = y_0^2 \sigma_P^2$$

Problem 3.1. (Agent Problem) For a given $\tilde{r}_p$, the consumer chooses $w_{n\times 1} = \begin{bmatrix} w_1 & w_2 & \dots & w_n \end{bmatrix}^t$ such that $Var(\tilde{r}_p) = \sigma_p^2$ is minimized and this can also $E[U(\tilde{y}_1)]$ is maximized. We claim that

$$\min \sigma_p^2 \iff \max E[U(\tilde{y}_1)]$$

Proof. Let $U(\tilde{y}_1)$ be well behaved. Perform a 2^{nd} order Taylor approximation about $E[\tilde{y}_1]$ to get

$$U(\tilde{y}_1) \approx U(E[\tilde{y}_1]) + U''(E[\tilde{y}_1])(\tilde{y}_1 - E[\tilde{y}_1]) + \frac{1}{2} U''(E[\tilde{y}_1])(\tilde{y}_1 - E[\tilde{y}_1])^2 + R$$

where the remainder R is 0 if the utility function is quadratic. Taking expectations from both sides, we get

$$E[U(\tilde{y}_1)] \approx U(E[\tilde{y}_1]) + \frac{1}{2} U''(E[\tilde{y}_1]) Var[\tilde{y}_1] + R'$$

which is a function of the expectation and variance of the end-of-period expectation and variance. That is

$$\underbrace{E[U(\tilde{y}_1)]}_{\text{Consumer choice theory}} = \underbrace{f(\mu_P, \sigma_P^2)}_{\mu-\sigma^2 \text{ portfolio theory}}$$

□

Remark 3.2. (1) $\min \sigma_p^2 \equiv \max E[U(\tilde{y}_1)]$ iff $U''(\cdot) < 0 \implies U$ is concave $\implies$ Risk averse consumer.

(2) R has to go to zero and $R = 0$ if $U(\cdot)$ is quadratic; but quadratic utility is not desirable $\implies$ an alternative way is putting a solution on $\tilde{r}_1$. That is if $\tilde{r}_1 \sim N \implies E[U(\tilde{y}_1)]$ can be maximized and the problem would be equivalent to $\min \sigma_P^2$. It then follows that we can represent $E[U(\tilde{y}_1)]$ in $(\mu_P - \sigma_P)$ space.

3.2 Markowitz Analysis

Problem 3.2. (Markowitz Problem) We want to choose optimal weights $\hat{\mathbf{w}}$ such that σ_P^2 is minimized, given $E[\tilde{r}_P] = \mu_P$ and asset means μ. Rewriting this in vector form, this is equivalent to

$$\min_{\{\mathbf{w}_{n\times 1}\}} \sigma_P^2 \equiv \min_{\{\mathbf{w}_{n\times 1}\}} \frac{1}{2} \mathbf{w}^t_{1\times n} \Omega_{n\times n} \mathbf{w}_{n\times 1}$$
$$\text{subject to} \quad \mathbf{w}^t \mu_{n\times 1} = (\mu_P)_{n\times 1}$$

where the sum of the weights is 1. The solution to this problem is a vector $(n \times 1)$ of optimum weights $\hat{\mathbf{w}}$ that defines the minimum variance portfolio given μ_P. The solution set of optimal weights, as a function of portfolio weights is called a *set of minimum variance frontier.* For a constant σ_P^2, there will either be 0,1 or 2 portfolio solutions.

We always pick the the one with a high mean return in the 2 portfolio case. We call the upper leg that contains all the choices in the 2 solution case the *efficient frontier*.

We define the portfolio with minimum variance (single solution on the frontier for a constant σ_P^2) the *minimum variance portfolio* (MVP). The (*set of*) efficient frontier can also be described as the set of portfolios going up along the parabola.

3.3 The Algebra of the Portfolio Frontier

Recall that we want to solve

$$\begin{aligned} \min_{\{\mathbf{w}_{n\times 1}\}} \sigma_P^2 \quad &\equiv \quad \min_{\{\mathbf{w}_{n\times 1}\}} \frac{1}{2}\mathbf{w}^t_{1\times n}\Omega_{n\times n}\mathbf{w}_{n\times 1} \\ \text{subject to} \quad & (1)\ \mathbf{w}^t\mu_{n\times 1} = (\mu_P)_{n\times 1} \\ & (2)\ \mathbf{w}^t\mathbf{1} = 1 \end{aligned}$$

To do this, define

$$\mathcal{L} = \frac{1}{2}\mathbf{w}^t\Omega\mathbf{w} + \lambda_1[\mu_P - \mathbf{w}^t\mu] + \lambda_2[1 - \mathbf{w}^t\mathbf{1}]$$

where $\mathcal{L}$ is 1×1. Using the first order conditions,

$$\begin{aligned} (1)\ \frac{\partial \mathcal{L}}{\partial w_{n\times 1}} &= 0_{n\times 1} \implies \Omega\mathbf{w} - \lambda_1\mu - \lambda_2\mathbf{1} = 0_{n\times 1} \\ (2)\ \frac{\partial \mathcal{L}}{\partial \lambda_1} &= 0 \implies \mu_P - \mathbf{w}^t\mu = 0 \implies \mathbf{w}^t\mu = \mu_p \iff \mu^t\mathbf{w} = \mu_P \\ (3)\ \frac{\partial \mathcal{L}}{\partial \lambda_1} &= 0 \implies \mathbf{w}^t\mathbf{1} = 1 \iff \mathbf{1}^t\mathbf{w} = 1 \end{aligned}$$

and from (1),

$$\Omega\mathbf{w} = \lambda_1\mu + \lambda_2\mathbf{1} \implies \mathbf{w} = \Omega^{-1}(\lambda_1\mu + \lambda_2\mathbf{1}),\ \frac{1}{\det(\Omega)}\text{Cofactor}^t$$

Assuming that Ω is invertible, then

$$(4)\ \mathbf{w} = \lambda_1\Omega^{-1}\mu + \lambda_2\Omega^{-1}\mathbf{1}$$

where we will need to define λ_1 and λ_2. Using equation (4) and do the following: $\times\mu^t$, $\times\mathbf{1}^t$. This gives

$$\begin{aligned} (5)\ \mu^t\mathbf{w} &= \lambda_1 \underbrace{\mu^t\Omega^{-1}\mu}_{a} + \lambda_2 \underbrace{\mu^t\Omega^{-1}\mathbf{1}}_{b} \\ (6)\ \mathbf{1}^t\mathbf{w} &= \lambda_1 \underbrace{\mathbf{1}^t\Omega^{-1}\mu}_{b} + \lambda_2 \underbrace{\mathbf{1}^t\Omega^{-1}\mathbf{1}}_{c} \end{aligned}$$

and setting $(2) = (5)$ and $(3) = (6)$ gives us, in matrix form:

$$\begin{pmatrix} \mu_P \\ \mathbf{1} \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \Psi \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

and since $\Psi^{-1} = \frac{1}{ac-b^2}\begin{pmatrix} c & -b \\ -b & a \end{pmatrix} = \frac{1}{d}\begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$ then

$$(8)\ \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \frac{1}{d}\begin{pmatrix} c & -b \\ -b & a \end{pmatrix}\begin{pmatrix} \mu_P \\ \mathbf{1} \end{pmatrix} = \begin{pmatrix} \frac{c\mu_P - b}{d} \\ \frac{a - b\mu_P}{d} \end{pmatrix}$$

Plugging (8) in (4), we solve for $\mathbf{w}$ to get

$$\begin{aligned} (9)\ \hat{\mathbf{w}} &= \left(\frac{c\mu_P - b}{d}\right)\Omega^{-1}\mu + \left(\frac{a - b\mu_P}{d}\right)\Omega^{-1}\mathbf{1} \\ &= \underbrace{\left[\frac{a\Omega^{-1}\mathbf{1} - b\Omega^{-1}\mu}{d}\right]}_{\Phi} + \underbrace{\left[\frac{c\Omega^{-1}\mu - b\Omega^{-1}\mathbf{1}}{d}\right]}_{\Theta}\mu_P \end{aligned}$$

and thus $\hat{\mathbf{w}} = \Phi + \Theta\mu_P$ for a given μ_P.

Problem 3.3. What is the minimum variance corresponding to $\hat{\mathbf{w}}$? Plugging in our $\hat{\mathbf{w}}$ (c.f. to p. 100, eq. 3.5.7) gives us

$$(10)\ \hat{\sigma}_P^2 = \hat{\mathbf{w}}^t \Omega \hat{\mathbf{w}} = \frac{c}{d}\left(\mu_P - \frac{b}{c}\right)^2 + \frac{1}{c}$$

which is a parabola in $\sigma^2 - \mu_P$ space. We define this equation as the *Mean-Variance Equation in Case of n risky assets.* So the vertex point has properties

$$(\sigma_P^2, \mu_P) = \left(\frac{1}{c}, \frac{b}{c}\right) = \left([\sigma_P^2]_{MVP}, [\mu_P]_{MVP}\right)$$

with

$$\hat{\mathbf{w}}_{MVP} = \Phi + \Theta(\mu_P)_{MVP}$$

Also note that at the portfolio R where $\mu_P = 0$, we have $\hat{\mathbf{w}} = \Phi + \Theta(0) = \Phi$.

Remark 3.3. In $\sigma - \mu_P$ space, this would look like a hyperbola.

Example 3.2. We are given the following data on $\tilde{r}_X$ and $\tilde{r}_Y$

Pr	$\tilde{r}_X$	$\tilde{r}_Y$
0.2	18	0
0.2	5	-3
0.2	12	15
0.2	4	12
0.2	6	1

Computing some key statistics gives us $\mu_X = 9$, $\mu_Y = 5$, $var(\tilde{r}_X) = \sigma_X^2 = 28$, $var(\tilde{r}_Y) = \sigma_Y^2 = 50.8$, $cov(\tilde{r}_X, \tilde{r}_Y) = \sigma_{XY} = -1.2$. We then define

$$\mu_{2\times1} = \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}, \mathbf{w}_{2\times1} = \begin{bmatrix} w_X \\ w_Y \end{bmatrix}, \mathbf{1}_{2\times1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \Omega = \begin{bmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{bmatrix} = \begin{bmatrix} 28 & -1.2 \\ -1.2 & 50.8 \end{bmatrix}$$

We want to compute our efficient frontier of weights in $\sigma^2 - \mu$ space. We know from above that

$$\hat{\mathbf{w}} = \Phi + \Theta\mu_P, \hat{\sigma}_P = \frac{c}{d}\left(\mu_P - \frac{b}{c}\right)^2 + \frac{1}{c}$$

and inverting Ω gives us

$$\Omega^{-1} = \frac{1}{\det(\Omega)}\text{Cofactor}^t = \frac{1}{28(50.8) - (-1.2)^2}\begin{bmatrix} 50.8 & 1.2 \\ 1.2 & 28 \end{bmatrix} = \begin{bmatrix} 0.03575 & 0.00084 \\ 0.00084 & 0.01970 \end{bmatrix}$$

So thus we have

$$\begin{cases} a = \mu^t\Omega^{-1}\mu = 3.46385 \\ b = \mu^t\Omega^{-1}\mathbf{1} = 0.43201 \\ c = \mathbf{1}^t\Omega^{-1}\mathbf{1} = 0.05713 \\ d = ac - b^2 = 0.01125711 \end{cases}$$

Thus, $\hat{\sigma}_p^2 = 0.5(\mu_p - 7.56)^2 + 17.5$ for any given mean μ_P. Notice that the vertex of the parabola is

$$(\hat{\sigma}_p^2, \mu_p) = \left(\frac{1}{c}, \frac{b}{c}\right) = (17.5, 7.56)$$

and this is the variance and mean of the MVP respectively. To find the weights, we solve $\hat{\mathbf{w}} = \Phi + \Theta\mu_P$ with

$$\begin{aligned} \Phi &= \frac{1}{d}\left[a\Omega^{-1}\mathbf{1} - b\Omega^{-1}\mu\right] \\ &= \begin{bmatrix} -1.251218 \\ 2.2495752 \end{bmatrix} \end{aligned}$$

and similarly $\Theta = \begin{bmatrix} 0.24999 \\ -0.2500 \end{bmatrix}$. Therefore,

$$\begin{bmatrix} \hat{w}_X \\ \hat{w}_Y \end{bmatrix} = \Phi + \Theta(\mu_P)_{MVP} = \begin{bmatrix} 0.64 \\ 0.36 \end{bmatrix}$$

Proposition 3.1. *The covariance between two portfolios p and q is $\sigma_{pq} = \mathbf{w}_p^t \Omega \mathbf{w}_q$. The covariance between the MVP and any portfolio c on the efficient frontier is (Merton, 1972, "An Analytic Derivation of the Efficient Set")*

$$\sigma_{MVP,c} = \mathbf{w}_{MVP}^t \Omega \mathbf{w}_c = k$$

for some constant k (Assignment 2, Question 1(g)).

Errata. Midterm is Friday, June 14, 2013 in M3 1006 @ 1:30-2:30 pm. Coverage is CH 1 (only what was in class), CH 2, CH 3 (Up to today). 60% Theory, 40% Practice Problems.

Problem 3.4. How can we compute the rate of return on an individual financial asset (stock)? And then minimize the σ_P^2 of a portfolio formed from n of these stocks?

Example 3.3. Suppose we have two stocks to invest in, Apple stock and Google stock ($n = 2$). Let $P\underbrace{1}_{Stock}\underbrace{0}_{Period}, P_{20}$ be the spot prices for Apple and Google stock for period 0 (today) prices. In the case of period 1 prices, we have what are called *state contingent (dependent) prices* (state of nature that could exist next period).

Say we have 3 states $\theta_1 \equiv Expansion$, $\theta_2 \equiv Steady$ and $\theta_3 \equiv Recession$. That is, if we denote $(P_{11})_{\theta_1}$ as Apple's stock price in period 1 contingent on θ_1, with similar definitions for $(P_{11})_{\theta_2}$, $(P_{11})_{\theta_3}$, $(P_{21})_{\theta_1}$, $(P_{21})_{\theta_2}$ and $(P_{21})_{\theta_3}$. Using historical data, suppose that we predict that

Pr	State	Stock 1	Stock 2
$\frac{1}{3}$	θ_1	$(P_{11})_{\theta_1}$=10.2	$(P_{11})_{\theta_1}$=15.5
$\frac{1}{3}$	θ_2	$(P_{11})_{\theta_2}$=10.4	$(P_{11})_{\theta_2}$=14
$\frac{1}{3}$	θ_3	$(P_{11})_{\theta_3}$=10.2	$(P_{11})_{\theta_3}$=15.8

and using the formula $(\tilde{r}_1)_\theta = \frac{(P_{11})_\theta - P_{11}}{P_{11}}$ then $\mu_1 = E[\tilde{r}_1] = \frac{8}{3}$ and similarly, $\mu_2 = E[\tilde{r}_2]$. We also have enough data to find Ω, our covariance matrix and with that we can compute

$$\sigma_P^2 = \frac{c}{d}\left(\mu_P - \frac{b}{c}\right)^2 + \frac{1}{c}$$

with $\hat{\mathbf{w}} = \Phi + \Theta\mu_P$.

3.4 Shape of the Portfolio Frontier

Case 1: $n = 2$ risky assets with $\rho_{12} = +1$. Note that this implies $\sigma_{12} = \sigma_{11}\sigma_{22}$ from the definition of ρ and given $w_1 + w_2 = 1 \implies w_2 = 1 - w_1$ we can write

$$\begin{aligned} E[\tilde{r}_P] &= \mu_P = \mu_1 + (1 - w_1)(\mu_2 - \mu_1) \\ Var[\tilde{r}_P] &= \sigma_P^2 = w_1^2\sigma_{11} + (1 - w_1)^2\sigma_{22} + 2w_1(1 - w_1)\sigma_{12} \end{aligned}$$

and from our equation about σ_{12} earlier we get

$$\begin{aligned} \sigma_P^2 &= w_1^2\sigma_{11} + (1 - w_1)^2\sigma_{22} + 2w_1(1 - w_1)\sigma_{12} \\ &= w_1^2\sigma_{11} + (1 - w_1)^2\sigma_{22} + 2w_1(1 - w_1)\sigma_{11}\sigma_{22} \\ &= (w_1\sigma_1 + (1 - w_1)\sigma_2)^2 \end{aligned}$$

and so σ_P is a perfect square with $\sigma_P = w_1\sigma_1 + (1 - w_1)\sigma_2$ and solving for w_1 gives

$$w_1 = \frac{\sigma_P - \sigma_2}{\sigma_1 - \sigma_2} \implies (1 - w_1) = \frac{\sigma_1 - \sigma_P}{\sigma_1 - \sigma_2} \implies \mu_P = \mu_1 + \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1}(\sigma_P - \sigma_1)$$

and the equation of the mean-variance frontier in mean variance space is a straight line with the slope

$$\frac{d\mu_P}{d\sigma_P} = \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1}$$

Note that if $w_1 = 1$ then $\sigma_P = \sigma_1$ and $\mu_P = \mu_1$ (Point A) and if $w_1 = 0$ then $\sigma_P = \sigma_2$ and $\mu_P = \mu_2$ (Point B). Also, If $w_1 \neq 0$ and $w_2 \neq 0$, in the case of $\rho_{12} = +1$, the Mean variance frontier and the efficient frontier are the same represented by line AB. If $-1 < \rho_{12} < +1$, then σ_P will be smaller, This implies the line AB will move to the left. If $|P_{12}| < 1$, then the mean-variance frontier will be different than the efficient frontier; instead, it curves to the left with A and B as two ending points.

Case 2: $-1 < P_{12} < 1$. Note in mean and variance space, the minimum-variance frontier is a parabola.

Case 3: $\rho_{12} = -1$. Perfectly negatively correlated. Again $\sigma_P^2 = (w_1\sigma_1 + (1-w_1)\sigma_2)^2 \implies \sigma_P = \pm(w_1\sigma_1 + (1-w_1)\sigma_2)$ solve for $w_1 = \frac{\pm\sigma_P + \sigma_2}{\sigma_1 + \sigma_2}$. Plug w_1 in the portfolio mean equation

$$\tilde{r}_P = w_1\tilde{r}_1 + (1-w_1)\tilde{r}_2$$

$$\mu_P = w_1\mu_1 + (1-w_1)\mu_2$$

Substitute w_1 in μ_P and simplify

$$\mu_P = \left(\frac{\sigma_2}{\sigma_1+\sigma_2}\mu_1 + \frac{\sigma_1}{\sigma_1+\sigma_2}\mu_2\right) \pm \frac{\mu_1 - \mu_2}{\sigma_1 + \sigma_2}\sigma_P$$

So the $\mu_P = \alpha \pm \beta\sigma_P$ is the equation of the minimum-variance frontier in the case $\rho_{12} = -1$. Note that the MVP in the case $\rho_{12} = -1$ is a risk-free portfolio ($\sigma_P = 0$).

Case 4: n risky assets with $-1 < \rho_{ij} < 1$. No perfect correlation between any two assets i and j, $i \neq j$. Our general case:

$$\sigma_P^2 = \frac{c}{d}\left(\mu_P - \frac{b}{c}\right)^2 + \frac{1}{2}$$

This is the equation of the min-variance frontier in case of n risky assets.

Case 5: $n = 2$, 1 is risk free and 2 is risky. $\mu_1 = r_f$ and $E[\tilde{r}_2] = \mu_2$ such that $\mu_2 > r_f$. $\sigma_0 = 0$, $\sigma_2 > \sigma_1$ and $\sigma_{22} > \sigma_{11}$. We claim that in this case the minimum-variance frontier is a straight line starting the risk free rate on the vertical axis.

$$\tilde{r}_P = w_1\tilde{r}_1 + (1-w_1)\tilde{r}_2$$

$$Var(\tilde{r}_P) = \sigma_P^2 = w_1^2\sigma_{11} + (1-w_1)^2\sigma_{22} + 2w_1(1-w_1)\sigma_{12}$$

Hence $\sigma_P = (1-w_1)\sigma_2$. Solving for w_1 gives $w_1 = 1 - \frac{\sigma_P}{\sigma_2}$ and plugging in w_1 in μ_P of the portfolio gives

$$\mu_P = w_1 r_f + (1-w_1)\mu_2$$

$$\mu_P = \left(1 - \frac{\sigma_P}{\sigma_2}\right) r_f + \left(1 - 1 + \frac{\sigma_p}{\sigma_2}\right)\mu_2$$

$$\mu_P = r_f + \frac{\mu_2 - r_f}{\sigma_2}\sigma_P$$

Note that this is the line that is tangent to the frontier.

Case 6: n risky assets and 1 risk free asset. We claim that in this model, the minimum variance frontier is a hyperbola but the efficient frontier is a straight line. To prove this, let A be the point where 100% is held in the risk free asset, M be the point where $\overrightarrow{AM}$ is tangental to the risk asset frontier. Consider portfolios $E < F$ (σ_P wise) that lie between M and the MVP of the risky frontier.

Note that r_f + portfolio $E \implies \overrightarrow{AE}$ is the minimum (min) variance frontier and r_f + portfolio $F \implies \overrightarrow{AF}$ is the minimum variance frontier. $\overrightarrow{AF}$ is preferred to $\overrightarrow{AE}$ and so on. $\overrightarrow{AM}$ is the best constructed min-variance portfolio (it is the efficient frontier)

The implications of this go into *optimal market portfolio theory.*

Midterm Content

- **Ch 1: p. 5-17 [Consumer Choice under Certainty]**
- Optimal Choice Conditions
- $\frac{MU_1}{MU_2} = \frac{P_1}{P_2}$, $m = P_1x_1 + P_2x_2$ [Be able to draw the graph and find x_1^*, x_2^*]
- Explanation of the tangency condition
- Ex. in p. 18 is NOT required
- p. 19-48 NOT required
- **Ch 2: [Utility Theory]**
- Implications of the axioms
- No need for the VN-M EU function
- p. 80, 81 [Pratt's 3 propositions] are NOT required
- **Ch 3 [Markowitz Analysis]**
- p. 83-90 [Notation and main idea]
- Link between Ch. 2 and Ch. 3
- p. 90-94 [EU as a function of $(\mu_p - \sigma_p)$] Know the shapes of the IC's in $\mu - \sigma$ space; only the knowledge of the graph is required
- p. 95-102 [Risky assets]
- [Correlation] Case 1: 103,104 ONLY

Problem 3.5. What is the optimal portfolio that will maximize the investor's $(\mu - \sigma)$ utility?

To solve this problem, we make the following assumptions:

1. Borrowing rate = Lending rate
2. Investors have homogeneous beliefs

This implies that everyone would like to hold $r_f + M$ (where M is the point of tangency, the market portfolio, between r_f and the risky frontier) regardless of the degree of equilibria.

Theorem 3.1. *(Two-Fund Separation Theorem) Everyone will be on the efficient-frontier (line from r_f passing through M) regardless of their risk aversion.*

4 The Capital Asset Pricing Model (CAPM)

The CML gives the relationship between $E[\tilde{r}_p]$ and the risk of the portfolios *only* for efficient portfolios

In this analysis, risk is measured by σ. What about individual risky assets that are inefficient? CAPM is needed.

4.1 Standard Definition of CAPM

The risk of individual assets is measured by its Beta coefficient. The CAPM is

$$E[\tilde{r}_j] = r_f + \underbrace{\text{Asset } j's \text{ premium}}_{\text{Amount of Risk by } j \times \text{Price of Risk}}$$

where the amount of risk due to stock j is $\beta_j \cdot \sigma_M$ where σ_M is the market risk and β_j is the beta coefficient. The price of risk is

$$\frac{E[\tilde{r}_M] - r_f}{\sigma_M}$$

So the CAPM model can be rewritten as

$$E[\tilde{r}_j] = r_f + \beta_j \cdot \sigma_M \times \left(\frac{E[\tilde{r}_M] - r_f}{\sigma_M} \right)$$

Example 4.1. Suppose that the market consists of 2 stocks: 1 and 2. Then

$$\tilde{r}_M = w_1 \tilde{r}_1 + w_2 \tilde{r}_2 \implies \tilde{r}_M = w_1 E[\tilde{r}_1] + w_2 E[\tilde{r}_2]$$

$$\implies \tilde{r}_M = \sigma_M^2 = w_1 \sigma_{11} + w_2 \sigma_{22} + 2 w_1 w_2 \sigma_{12}$$

Say you want to hold asset $j = 1$. What is $E[\tilde{r}_1]$? By the CAPM model, this is

$$E[\tilde{r}_1] = r_f + (\text{Premium})$$

This risk contribution of asset 1 is $\triangle \sigma$ when there is an increase in the market share of 1. That is, this is

$$\frac{d\left[\sigma_M\right]}{dw_1} = \frac{1}{\sigma_M}\left(w_1 \sigma_{11} + w_2 \sigma_{12}\right)$$

from $w_2 = (1 - w_1)$. Note that

$$Cov(\tilde{r}_1, \tilde{r}_M) = Cov(\tilde{r}_1, w_1 \tilde{r}_1 + w_2 \tilde{r}_2) = w_1 \sigma_{11} + w_2 \sigma_{12}$$

and so

$$\frac{d\left[\sigma_M\right]}{dw_1} = \frac{Cov(\tilde{r}_1, \tilde{r}_M)}{\sigma_M} = \frac{Cov(\tilde{r}_1, \tilde{r}_M)}{\sigma_M^2} \cdot \sigma_M = \beta_{1,M} \cdot \sigma_M$$

Recalling that the general price of risk is

$$\frac{E[\tilde{r}_M] - r_f}{\sigma_M}$$

then we have

$$E[\tilde{r}_1] = r_f + \beta_{1,M}\left(E[\tilde{r}_M] - r_f\right)$$

Remark 4.1. In general, for any stock j such that the market consists of n stocks; then

$$\underbrace{E[\tilde{r}_j]}_{\text{Required rate of return}} = r_f + \beta_{j,M}\left(\underbrace{E[\tilde{r}_M] - r_f}_{\text{Market premium}}\right), \beta_{j,M} = \frac{Cov(\tilde{r}_1, \tilde{r}_M)}{\sigma_M^2}$$

Comparing to Markowitz Analysis:

CAPM (SML)	Markowitz (CML)
$\mu - \beta$ Space	$\mu - \sigma$ Space
Inefficient individual assets	Efficient portfolios
Risk is β	Risk is σ
SML = Security Market Line	CML = Capital Market Line
Slope = $E[\tilde{r}_M] - r_f$	Slope = $\frac{\mu_M - r_f}{\sigma_M}$ where M (point of tangency) depends on r_f

and when $\beta = 1$ then $\mu = E[\tilde{r}_M]$. We generally hold if $\beta > 1$ and short if $\beta < 1$.

Note 1. We also have

$$\beta_M = \frac{Cov(\tilde{r}_M, \tilde{r}_M)}{\sigma_M^2} = 1, \beta_{r_f} = \frac{Cov(r_f, \tilde{r}_M)}{\sigma_M^2} = 0$$

Remark 4.2. Here are some observations on CAPM:

1. Asset j's risk contribution is $\beta_j \cdot \sigma_M$
2. Asset j's systematic risk is β_j
3. Sign of Beta:
   (a) $\beta_j = 0 \implies E[\tilde{r}_j] = r_f$
   (b) $\beta_j = 1 \implies E[\tilde{r}_j] = E[\tilde{r}_M]$
   (c) $\beta_j > 1 \implies E[\tilde{r}_j] > E[\tilde{r}_M]$ (high risk is associated with high return)
   (d) $\beta_j < 1 \implies E[\tilde{r}_j] < E[\tilde{r}_M]$
   (e) $\beta_j < 0$ is appealing because everyone would demand that stock since it is ideal for hedging

In the *ideal* case, we want to find a stock with a negative beta and a positive $\tilde{r}$ where $\tilde{r}$ is the r.v. that represents the return on that stock. In practice, though, the betas of the stocks traded in the market range between 0.5 to 1.5.

If β_j, for stock j, is extremely negative, then $\tilde{r}_j < 0$ but investors still hold it. They do this to hedge against market risk by paying the premium associated with negative returns.

If $\beta \approx 0$ does that mean that volatility is 0? No. The volatility is σ_j which is the measure of the total risk. This is broken down into:

- Systematic/Non-diversifiable/Market Risk (Captured by beta)
- Non-Systematic/Firm-Specific/Idiosyncratic Risk

Example 4.2. We are given two stocks, $A =$ Apple and $G =$ Gillette. Suppose that $\beta_A = 1.4$, $\beta_G = 0.6$ and $r_f = 5\%$. The market premium is $E[\tilde{r}_M] - r_f = 6\%$. What is the required rate of return for each stock? From CAPM,

$$E[\tilde{r}_A] = r_f + \beta_A(E[\tilde{r}_M] - r_f) = 5\% + 1.4(6\%) = 13.4\%$$

Decision:

1) If $13.4\% <$ Actual $\tilde{r}_A$ then buy

1) If $13.4\% >$ Actual $\tilde{r}_A$ then short

This is similarly done for G.

CAPM Applications

1. CAPM can be used to find the required return on risky portfolios instead of one single asset.
   (a) Suppose that $\tilde{r}_p = \sum_{i=1}^n w_i \tilde{r}_i$. So we then have

$$\beta_p = \frac{Cov(\tilde{r}_p, \tilde{r}_M)}{Var(\tilde{r}_M)} = \frac{\sum_{i=1}^n w_i Cov(\tilde{r}_i, r_M)}{Var(\tilde{r}_M)} = \sum_{i=1}^n w_i \beta_i$$

   and so we can model $E[\tilde{r}_p]$ and β_p using CAPM. That is, for any portfolio p,

$$E[\tilde{r}_p] = r_f + \beta_p(E[\tilde{r}_M] - r_f)$$

   which is the CAPM for any portfolio p.

2. Measuring the performance of the portfolio using CAPM [using the alpha of the portfolio]

   (a) *Example.* Suppose the average rate of return (annually) on fund p is $r_p = 14.85\%$ with $\beta_p = -0.025$. Using historical data, $r_f = 5\%$ and the market premium is 6%. Thus,

   $$E[\tilde{r}] = 5\% + \beta \cdot 6\%$$

   Note that $E[\tilde{r}_M] = 11\%$ and $E[\tilde{r}_p] = 4.85\%$. The alpha is $|r_p - E[\tilde{r}_p]| = 10\%$.

3. Using the beta of the company to compute the NPV

Regression and CAPM

Take the *ex-ante CAPM* model

$$E[\tilde{r}_i] - r_f = \beta_i \left(E[\tilde{r}_M] - r_f\right)$$

On average, $\tilde{r}_{it} \approx E[\tilde{r}_i]$ for $t = 1, ..., T$ which we call the *fair game equation*. It is alternatively defined as

$$\tilde{r}_{i,t} = E[\tilde{r}_i] + \beta_i \left(\tilde{r}_{M,t} - E[\tilde{r}_{M,t}]\right)$$

where $\tilde{r}_{M,t}$ is the realized return on the market at time t. Taking expectations will give

$$E[\tilde{r}_{i,t}] = E[\tilde{r}_i]$$

Where the left is the average realized return and the right is the expected rate of return. Putting this with the CAPM gives us

$$\tilde{r}_{i,t} = r_{f,t} + \beta_i \left(E[\tilde{r}_M] - r_{f,t}\right) + \beta_i \left(\tilde{r}_{M,t} - E[\tilde{r}_{M,t}]\right)$$

and so

$$\tilde{r}_{i,t} - r_{f,t} = \beta_i \left(\tilde{r}_{M,t} - r_{f,t}\right)$$

which we call the *ex-post CAPM. We now add α_i as an intercept and $\tilde{\epsilon}_i$ to render the ex-post CAPM stochastic:*

$$\tilde{r}_{i,t} - r_{f,t} = \alpha_i + \beta_i \left(\tilde{r}_{M,t} - r_{f,t}\right) + \tilde{\epsilon}_{i,t}$$

To test the validity of the model, we use hypothesis testing and various test statistics.

CAPM in Practice

We will examine α, β and σ.

1. *Beta:* For a portfolio p of n risky assets,

   $$\tilde{r}_{p,t} - r_{f,t} = \alpha + \beta_p \left(\tilde{r}_{M,t} - r_{f,t}\right) + \tilde{\epsilon}_{p,t}$$

   where $\beta_p = \sum w_i \beta_i$. and using this model, the manager can use this β_p to measure the systemic risk of the portfolio.

2. *Alpha: (This is not from the the above equation)*

   (a) Find β_p from the above equation.

   (b) Plug β_p i in the CAPM (original equation and compute $E[\tilde{r}_p]$, the required rate of return of the portfolio. That is,

   $$E[\tilde{r}_p] = r_f + \beta_p (E[\tilde{r}_M] - r_f)$$

   e.g. Suppose that $E[\tilde{r}_p] = 5\% + 1.5(6\%) = 14\%$

   (c) Compute the average rate of return of the portfolio over the life time of the fund and take it as the actual rate of return.
   e.g. Suppose that $(r_p)_{Actual} = 18\%$. Then $\alpha_p =$ Actual - Required $= 4\%$ which is the measure of performance.

3. *Sigma:* This is the volatility of the portfolio. Assuming that r_f is fixed, then

$$r_{p,t} = \Gamma + \underbrace{\beta_p \tilde{r}_{M,t}}_{\text{systematic risk}} + \underbrace{\tilde{\epsilon}_{p,t}}_{\text{idiosyncratic risk}} = \Gamma + \beta_p \tilde{r}_{M,t} + \sum w_i \tilde{\epsilon}_{i,t}$$

where the systematic risk is due to market common factors but the idiosyncratic risk is firm specific. Now if w_i is very small and n is very large, we can get rid of $\tilde{\epsilon}_{p,t}$.
Proof. (p. 142) We have

$$Var(\tilde{r}_p) = \beta_p^2 \sigma_M^2 + Var\left[\sum w_i \tilde{\epsilon}_i\right]$$

and if $w_i \approx \frac{1}{n}$ then

$$\begin{aligned} Var(\tilde{r}_p) &= \beta_p^2 \sigma_M^2 + Var\left[\sum \frac{1}{n}\tilde{\epsilon}_i\right] = \beta_p^2 \sigma_M^2 + \frac{1}{n^2} Var\left[\sum \tilde{\epsilon}_i\right] \\ &= \beta_p^2 \sigma_M^2 + \frac{1}{n}\frac{\sum Var\left[\tilde{\epsilon}_i\right]}{n} \\ &= \beta_p^2 \sigma_M^2 + \frac{1}{n}\overline{\sigma}^2 \to \beta_p^2 \sigma_M^2 \end{aligned}$$

as $n \to \infty$ and

$$\tilde{r}_p \to \Gamma + \beta_p \tilde{r}_{M,t}$$

So our sigma is $\sigma_p = \beta_p \sigma_M$.

   (a) Note that although we diversified away the non-systematic risk, there are still other factors that are note captured by $\beta_p \tilde{r}_{M,t}$ that affect $\tilde{r}_p$. Those factors are captured by Γ.

<u>Formal Derivation of the CAPM</u>

See p. 142 in the Course Notes.

Also remark that an alternative formulation is to maximize the slope of the CML or the Sharpe ratio. That is

$$\begin{aligned} \max_{\{w_i\}} \frac{(\mu_p)_T - r_f}{\sigma_p} &= \frac{w^t \mu - r_f}{(w^t \Omega w)^{1/2}} \\ \text{subject to} \quad & w^t 1 = 1 \end{aligned}$$

after some tedious algebra,

$$(\hat{w})_T = \frac{\Omega^{-1}(\mu - r_f \mathbf{1})}{\mathbf{1}^t \Omega^{-1}(\mu - r_f \mathbf{1})}$$

4.2 CAPM Variants

<u>Assumptions:</u>

1. The market is in **equilibrium**
2. Investors are **risk averse**, living in a **two period world**, and want to **maximize their end of period 1 wealth** $E[U(\tilde{y}_1)]$
3. All investors can **borrow and lend at the risk free rate** rf
4. There is a **frictionless market** with no transaction costs
5. Investor's beliefs regarding the asset rate of returns and joint probability distributions are **homogeneous**

Standard Version (Ex-Ante Form) ; SML Line:

$$\begin{aligned} E[\tilde{r}_j] &= rf + \beta_{j,M}\sigma_M \left[\frac{E[\tilde{r}_M]-rf}{\sigma_M}\right] \\ &= rf + \rho_{j,M}\sigma_j \left[\frac{E[\tilde{r}_M]-rf}{\sigma_M}\right] \\ &= rf + \beta_{j,M}\left[E[\tilde{r}_M]-rf\right] \end{aligned}$$

where the systematic risk = $\beta_{i,M} = \frac{Cov(\tilde{r}_j,\tilde{r}_M)}{Var(\tilde{r}_M)} = \frac{Cov(\tilde{r}_j,\tilde{r}_M)}{\sigma_M^2}$, price of risk = $\frac{E[\tilde{r}_M]-rf}{\sigma_M}$, Sharpe ratio = $\frac{E[\tilde{r}_p]-rf}{\sigma_p}$, risk due to asset $j = \beta_{j,M}\sigma_M = \rho_{j,M}\sigma_j = \frac{d[\sigma_M]}{dw_j}$, required rate of return = $rr_p = rf + \beta_p\left(E[\tilde{r}_M]-rf\right)$, performance alpha = $\alpha_p = E[\tilde{r}_p] - rr_p$

Fair Game Equation:

$$\tilde{r}_{it} = E[\tilde{r}_{it}] + \beta_i\left(\tilde{r}_{Mt} - E[\tilde{r}_{it}]\right)$$

Time Series Version (Ex-Post Form):

Substitute the fair game equation into the Ex-Ante CAPM (last version) to get

$$\tilde{r}_{it} - rf_t = +\beta_i\left(\tilde{r}_{Mt} - rf_t\right)$$

Here, $\tilde{r}_{it} = \frac{\tilde{P}_{it}-\tilde{P}_{i(t-1)}+\tilde{D}_{it}}{\tilde{P}_{i(t-1)}}$.

Stochastic LS Version:

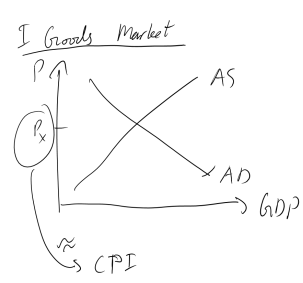
$$(\tilde{r}_{it} - rf_t) = \alpha_i + \beta_{i,M}\left(\tilde{r}_{Mt} - rf_t\right) + \tilde{\epsilon}_{it}$$

4.3 The Markets

Here, we will *briefly* discuss the 3 major markets in economic theory.

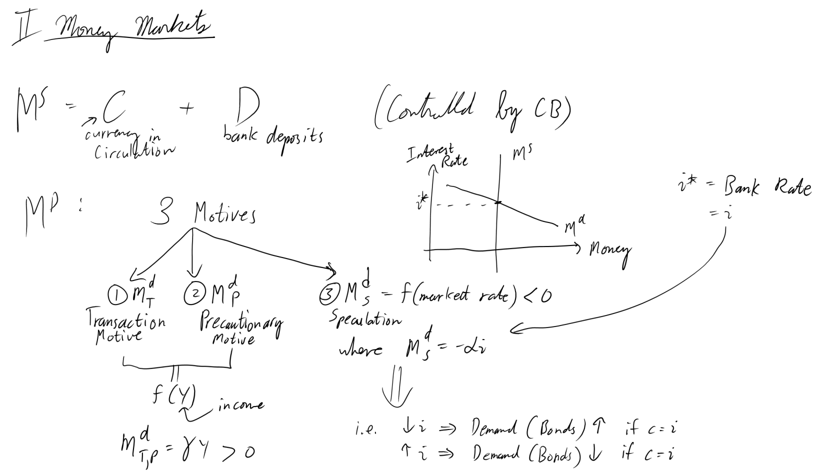
Goods Market

The goods market in this course is considered as an aggregate and viewed in terms of aggregate demand and aggregate supply on a GDP v. Price axis. The equilibrium price can be approximated using a proxy such as the CPI.



Money Market

Below is a summary of the money markets:



Here, the money supply M^s is divided between currency in circulation C and bank deposits D. It is well accepted that the money supply is inelastic.

In terms of the demand for money M^d, we assume that consumers have 3 motives:

- *Transaction motive* M_T^d: The motive to spend money on essential and non-essential goods
- *Precautionary Motive* M_P^d: The motive to save money for the future.
- *Speculation Motive* M_S^d: The motive to invest in the financial markets.

The first two are modeled together as a linear function of income Y,

$$M_{T,P}^d = f(Y) = \gamma Y > 0$$

and the last motive is modeled as a linear function of market (bank) rate i,

$$M_S^d = f(i) = -\alpha i$$

Here, there is a negative correlation between market (bank rate) and the rate of return in the in the financial markets. Together, $M^d = \gamma Y - \alpha i$.

Financial Market

This is the standard market that was discussed in ACTSC 371. It consists of all financial instruments and securities existing in the market, which include bonds and stocks.

5 Arbitrage Pricing Theory (APT)

Recall the assumptions for CAPM

1. Everyone calculates the same market portfolio
2. No friction in the market (no transaction costs)
3. Supply = Demand (market is always in equilibirum)
4. Everyone has access to the risk-free rate

and the only factor that explains the variations in $\tilde{r}_{it}$ is the market return $\tilde{r}_{Mt}$.

5.1 Standard Definition of the APT

In the *Arbitrage Pricing Theory,* we assume that $k > 1$ factors can explain the variation in $\tilde{r}_{it}$. Examples can include changes in

- GDP
- Inflation
- Interest Rate
- Cost of Labour

Example 5.1. Let's say that the GDP decreases. Then economic activities decrease, aggregate demand decrease, and sales of corporations will drop. This implies that future expected cash flows will drop. So the net present value of corporations will drop and their overall values drop. The price of a corporation will then drop and so will $\tilde{r}_{it}$.

Example 5.2. Consider utility companies and transportation companies. As GDP goes down, the effect is more profound on transportation companies because the demand for utility will remain relatively constant.

However, with regards to a change in interest rates (bank rates), if it decreases our demand for transportation will increase (lower borrowing costs) however, utility companies will not be affect much due to stable demand.

In conclusion, we have k factors affecting $\tilde{r}$ in different magnitudes according to the sector in which the corporation is operating.

Remark 5.1. Assets with the same beta can be compared by their expected return (the larger one should have a long position taken). This can be proven with the single factor model. Thus, any portfolios (that is well divisified; no systematic risk) with the same beta mus be on the SML (same point).

Fact 5.1. *If a portfolio lies above the SML, take a long position in that portfolio and if it is under, take a short position.*

Remark 5.2. We can make the CAPM applicable to efficient portfolios by first extending it to portfolios, then M (the market portfolio) from Markowitz with rf to construct the efficient SML. This implies that no arbitrage conditions exist.

Remark 5.3. The APT is a actually a generalization of the CAPM by extending the one-factor model in CAPM to a multifactor model. That is,

$$\tilde{r}_i = E[\tilde{r}_i] + \sum_{j=1}^{k} b_{ji} F_j + \tilde{\epsilon}_i$$

where $\tilde{r}_i$ is generally for a one asset case, but can be extended to cover a portfolio p. The general assumptions for the APT are

1. The financial market is in equilibrium
2. Homogeneous beliefs in the agents of the market
3. The market is *rich* (large n) for portfolios

5.2 Lambda Models

For portfolios in this market, the construction should follow three properties:

1. They should be self-financed; zero-cost
2. They should be well diversified; as $n \to \infty$, $\tilde{\epsilon}_p \to 0$.
3. There should be zero sensitivity to factor loading: $\sum_{i=1}^{n} w_i b_{ik} = 0$ for all k.

In summary, (1) $\implies w^t\mathbf{1} = 0$, (2) $\tilde{\epsilon}_p = 0$, and (3) $\implies w^t\mathbf{b}_k = 0$. Since $\tilde{r}_p = \sum_{i=1}^n w_i r_i$, we can define $\tilde{r}_p$ as

$$\tilde{r}_p = \sum_{i=1}^n \left(w_i E[\tilde{r}_i] + \sum_{j=1}^k w_i b_{ij}\tilde{F}_j + w_i\tilde{\epsilon}_i \right) = \sum_{i=1}^n w_i E[\tilde{r}_i]$$

using properties (2) and (3). However, by the no-arbitrage condition, this must be 0 $\implies w^t\mu = 0$. Hence we have:

$$w^t\mathbf{1} = 0, w^t\mathbf{b}_k = 0, w^t\mu = 0$$

Now if k is large enough relative to n, then $\mu \in \text{span}\{1, b_1, ..., b_k\}$ and so we write

$$\mu = \lambda_0 \cdot 1 + \lambda^t\mathbf{b} \implies E[\tilde{r}_i] = \lambda_0 + \sum_{j=1}^k \lambda_j b_{ij}$$

Here, the lambdas can be explicitly written as

$$\lambda_j = E[\tilde{r}_{p_j}] - rf$$

where p_j is the *pure factor portfolio* with unit factor sensitivity to the common factor j.

Why is APT more robust than CAPM?

1. APT makes no assumption about the distribution of asset returns
2. APT only requires risk aversion and makes no strong assumptions about individuals' utility functions
3. APT asserts that the rate of return is based on many factors rather than 1
4. The market portfolio is not special as it is CAPM (it is efficient in CAPM)
5. The APT can be extended to a multi-period framework

6 Arrow Debreu Economies

Here, we discuss a method to find the competitive equilibrium of an A-D ecomony

6.1 CE in Uncertainty

Let $A_{n\times m}$ be a matrix that represents the return of m assets over n states of the economy. That is, the columns are representative of assets and the rows are states of the economy. If $n = m$ then $X_{m\times n} = A^{-1}$ is the matrix of weights where the columns are representative of the states of the economy and the rows correspond to assets.

Competitive Equilibrium in an A-D Economy (Assumptions):

- Agents will live for only 2 periods
- We have a *complete market* (the number of states of nature is equal to the number of linearly independent assets)
- The set of states of nature are *collectively exhaustive* and *mutually exclusive*
- We assume only one *perishable* consumption good

Competitive Equilibrium in an A-D Economy (Goal):

Given $(\pi_1, ..., \pi_n, P_1, ..., P_n, \delta^i)$, the competitive equilibrium is a set of decision rules and prices such that:

(1) The values (c_0^A, c_1^A) and (c_0^B, c_1^B) for agents A and B respectively are chosen such that their respective utilities are maximized subject to the initial budget constraints (IBC). In other words, the pairs are the solutions to the following program for $i = 1, 2$:

$$\max_{\{c_0^i, c_1^i\}} \quad U(c_0^i, c_1^i) = U_0^i(c_0^i) + \delta^i \sum_{\theta=1}^{n} \pi_\theta U^i(c_\theta^i)$$

$$\text{s.to} \quad y_0^i + \sum_{\theta=1}^{n} P_\theta y_\theta^i \geq c_0^i + \sum_{\theta=1}^{n} P_\theta c_\theta^i$$

(2) All markets clear:

(i) $C_0 = Y_0$

(ii) $C_\theta = Y_\theta$ for $\theta = 1, ..., n$

Solution to our goal:

$$P_\theta = \frac{\delta^i \pi_\theta MU_\theta^i}{MU_0^i}$$

Final Exam Notes

- Final will be 3 questions; Q1 $\implies$ Markowitz Analysis and CAPM (calculations and theory), Q2 $\implies$ CAPM and APT, Q3 $\implies$ CE with certainty and uncertainty
  - There will be a total of 20 parts with 30 marks in total
- Chapter 1:
  - CE under certainty (p. 31-39)
- Chapter 3:
  - Markowitz: concepts (no proofs)
    - n risky assets
    - n risky assets and one risk free
- Chapter 4:
  - Everything should be included (from part II)
- Chapter 5:
  - Everything, including APT, should be included (from part II)
  - Typo: p. 167 (last equation)
    - It should be $E[\tilde{r}_i] - \lambda_0 = \sum_{j=1}^{k} \lambda_j \beta_{ij}$
- Chapter 6:
  - A-D Economy (under uncertainty); Excluding section 6.1.2. (p. 173-mid 174)

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