PMATH 351 Midterm Exam Review

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1 Theorems and Statements

Axiom 1. *(Zorn's Lemma) Let $(X, \leq)$ be a non-empty poset. Assume that every chain $C \subset X$ has an upper bound. Then, $(X, \leq)$ has a maximal element.*

Axiom 2. *(Axiom of Choice) If a set X is non-empty, then there exists a function $f : \mathcal{P}(X)\backslash\emptyset \mapsto X$ such that $f(A) \in A$ for any $A \in \mathcal{P}(X)$.*

Theorem 1.1. *(Cantor-Schroeder-Bernstein Theorem) Assume that $A_2 \subset A_1 \subset A_0$. Then if $A_0 \sim A_2$ we have $A_1 \sim A_0$.*

Proposition 1.1. *A set $A \subseteq (X, d)$ is closed if and only if whenever $\{x_n\} \subseteq A$ is such that $x_n \to x_0$ we have $x_0 \in A$.*

Proof. Assume that A is closed. Let $\{x_n\} \in A$ with $x_n \to x_0$. If $x_0 \in A^c$, then $\exists\epsilon_0 > 0$ such that $B(x_0, \epsilon_0) \cap A = \emptyset$. This is impossible because $\{x_n\} \subset A$, $x_n \to x_0$ and hence $x_n \in B(x_0, \epsilon_0)$ if n is large enough.

Assume that A is not closed. Then $\exists x_0 \in Lim(A)$ but $x_0 \notin A$. But then $\exists\{x_n\} \subset A$ with $x_n \to x_0$ by the fact that $x_0 \in Lim(A) \Leftrightarrow \exists\{x_n\} \in A\backslash\{x_0\}$ with $x_n \to x_0$. This contradicts our assumption that if $\{x_n\} \subset A$ with $x_n \to x_0$, then $x_0 \in A$. □

Proposition 1.2. *A function $f : (X, d_X) \mapsto (Y, d_Y)$ is continuous iff when $x_n \to x_0$ in X, then $f(x_n) \to f(x_0)$ in Y.*

Proof. We first show that $f^{-1}(W)$ is open for any open set $W \subset Y$. Let $W \subset Y$ be open with $V = f^{-1}(W)$. Let $x_0 \in V$. Then $y_0 = f(x_0) \in W$ and V is a neighbourhood of $x_0 \implies x_0 \in int(V)$, $\forall x_0 \in V \implies V$ is open. Next, take some sequence $\{x_n\} \subset X$ with $x_n \to x_0$. Let $f(x_0) = y_0$ and $\epsilon > 0$. If $W = B(y_0, \epsilon)$, then since W is open, $V = f^{-1}(W)$ is as well. Since $x_0 \in V$, there exists $\delta > 0$ for which $B(x_0, \delta) \subseteq V$. Since $x_n \to x_0$, there is some $N \in \mathbb{N}$ such that for $n \geq N$ we have $d_X(x_n, x_0) < \delta$. It follows that if $n \geq N$, we have $d_Y(f(x_n), f(x_0)) < \epsilon$. That is $f(x_n) \to f(x_0)$.

The other direction follows trivially from the sequential definition of continuity. □

Proposition 1.3. *The uniform limit of a sequence of continuous functions $\{f_n : (X, d_X) \mapsto (Y, d_Y)\}$ is continuous.*

Proof. Let $\epsilon > 0$. We know that $f_n \to f_0$ uniformly so pick $N_0 \in \mathbb{N}$ such that $n \geq N_0 \implies d_Y(f_n(x_0), f_0(x_0)) < \frac{\epsilon}{3}$. We know that $f_{N_0}$is continuous at x_0. Hence, we find δ such that $d_X(x, x_0) < \delta \implies d_Y(f_{N_0}(x), f_{N_0}(x_0)) < \frac{\epsilon}{3}$ for any $x \in X$. If $d_X(x, x_0) < \delta$, then

$$\begin{aligned} d_Y(f_0(x), f_0(x_0)) &\leq d_Y(f_0(x), f_{N_0}(x)) + d_Y(f_{N_0}(x), f_{N_0}(x_0)) + d_Y(f_{N_0}(x_0), f_0(x_0)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

□

Proposition 1.4. *$C_b(X)$ is complete.*

Proof. Let $\{f_n\} \subset C_b(X)$ be Cauchy. If $x \in X$, then $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty$. Hence $\{f_n(x)\}$ is Cauchy in $\mathbb{R}$ for each $x \in X$. Let $f_0(x) = \lim_{n\to\infty} f_n(x), \forall x \in X$. We claim that $f_0 \in C_b(X)$. Let $\epsilon > 0$. Then $\exists N_0 \in \mathbb{N}$ such that $n, m \geq N_0 \implies |f_n(x) - f_m(x)| < \frac{\epsilon}{2}$ for any $x \in X$. Let $n \geq N_0$ and $x \in X$. Then let $(*)$ be the statement that

$$|f_n(x) - f_0(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \leq \frac{\epsilon}{2} < \epsilon.$$

Hence, $f_n \to f_0$ uniformly on $X \implies f_0$ is continuous. Since $\{f_n\}$ is Cauchy, $\exists M \geq 0$ such that $\|f_n(x)\|_\infty < M, \forall n \in \mathbb{N}$. So

$$|f_0(x)| \leq |f_0(x) - f_{N_0}(x)| + |f_{N_0}(x)| < \epsilon + M \implies f_0 \in C_b(x)$$

By $(*)$, if $n \geq N_0$ then $|f_n(x) - f_0(x)| \leq \frac{\epsilon}{2}$ for all $x \in X \implies \|f_n - f_0\| \leq \frac{\epsilon}{2} < \epsilon \implies f_n \to f_0$ in $\|\cdot\|$. □

Theorem 1.2. *(Cantor Intersection Theorem) Let (X, d) be a metric space. Then TFAE (the following are equivalent).*

1) (X, d) is complete.

2) (X, d) satisfies $(*) \to$ If $\{F_n\}$ is a sequence of non-empty closed subsets of X such that $F_{n+1} \subseteq F_n$ for all n and $diam(F_n) \to 0$, then $\bigcap_{n=1}^\infty F_n \neq \emptyset$.

Proof. 1) $\implies$ 2) Assume that $\{F_n\}$ is as in $(*)$. For each $n \in \mathbb{N}$, choose $x_n \in F_n$. Let $\epsilon > 0$ and choose N_0 such that $diam(F_{N_0}) < \epsilon$. If $n, m \geq N_0 \implies \{x_n\}$ is Cauchy. Because (X, d) is complete, $x_n \to x_0$ for some $x_0 \in X$. But $\{x_n, x_{n+1}, ...\} \subseteq F_n$ and $\{x_n, x_{n+1}, ...\}$ converges to x_0. Since F_n is closed $\implies x_0 \in F_n, \forall n \in \mathbb{N} \implies x_0 \in \bigcap_{n=1}^\infty F_n$.

2) $\implies$ 1) Assume that X satisfies $(*)$. Let $\{x_n\} \subset X$ be Cauchy. For each $n \in \mathbb{N}$, let $A_n = \{x_n, x_{n+1}, ...\}$. Let $F_n = \bar{A_n}$ and since $\{x_n\}$ is Cauchy, $diam(A_n) \to 0$. So $diam(F_n) \to 0$. Clearly $F_n \neq \emptyset$ and $F_{n+1} \subseteq F_n$. Then $\exists x_0 \in \bigcap_{n=1}^\infty F_n$. Let $\epsilon > 0$ and choose N_0 large enough such that $diam(F_{N_0}) < \epsilon$. Then,

$$\begin{aligned} A_{N_0} &= \{x_{N_0}, x_{N_0+1}, ...\} \subseteq F_{N_0} \subseteq B(x_{N_0}, \epsilon) \\ \implies& \text{If } n \geq N_0 \text{ then } d(x_n, x_0) < \epsilon \\ \implies& x_n \to x_0 \end{aligned}$$

□

Theorem 1.3. *(Generalized Weierstrass M-Test) Let $(X, \|\cdot\|)$ be a n.l.s. Then TFAE:*

1) X is a Banach space.

2) X satisfies $(*) \to$ If $\{x_n\} \subset X$ is such that $\sum_{n=1}^\infty \|x_n\| < \infty$, then $\sum_{n=1}^\infty x_n$ converges in X.

Proof. 1) $\implies$ 2) Assume that $\sum_{n=1}^\infty \|x_n\|$ converges. Let $T_k = \sum_{n=1}^k \|x_n\| \implies \{T_k\}$ is Cauchy. Let $s_k = \sum_{n=1}^k x_n$, $\epsilon > 0$. We can find N_0 such that if $k > m \geq N_0$, then

$$|T_k - T_m| = \left| \sum_{n=1}^k \|x_n\| - \sum_{n=1}^m \|x_n\| \right| = \sum_{n=m+1}^k \|x_n\| < \epsilon$$

If $j > m \geq N_0$, then $\|s_j - s_m\| = \|\sum_{n=m+1}^j x_n\| \leq \sum_{n=m+1}^j \|x_n\| < \epsilon$ which implies $\{s_k\}$ is Cauchy.

2) $\implies$ 1) Assume that X satisfies $(*)$ and that $\{x_n\}$ is Cauchy. Then $\exists N_1 < ... < N_k < ... \in \mathbb{N}$ such that if $n, m \geq N_k$ then

$$\|x_n - x_m\| < \frac{1}{2^k}.$$

Let $g_k = x_{N_{K+1}} - x_{N_k}$.Then $\sum_{k=1}^\infty \|g_k\| < \sum_{k=1}^\infty \frac{1}{2^k} = 1 < \infty$. By $(*)$, if $s_j = \sum_{k=1}^j g_k$ then $\{s_j\}$ converges. But $s_j = \sum_{k=1}^j (x_{N_{k+1}} - x_{N_k}) = x_{N_{j+1}} - x_{N_1}$ by telescoping. So $\{x_{N_{j+1}} - x_{N_1}\}_{j=1}^\infty$ converges in $X \implies \{x_{N_{j+1}}\}_{j=1}^\infty$ converges in $X \implies \{x_n\}$ converges in . □

Proposition 1.5. *A subset of a complete metric space (X, d) is complete in the induced metric iff it is closed.*

Proof. Assume that A is closed. Let $\{x_n\} \subset A$ be Cauchy in $A \implies \{x_n\}$ is Cauchy in X and in A. Hence $x_n \to x_0 \in X$ and A is closed $\implies x_0 \in A$. Now let (A, d_A) be complete, $\{x_n\} \subset A$ with $x_n \to x_0 \implies \{x_n\}$ Cauchy in X and in A. By completeness, $x_n \to y_0 \in A$ and $x_0 = y_0$ and so A is closed. □

2 Review of Concepts and Select Topics

Cauchy Sequences

- If any subsequence of a Cauchy sequence converges, the whole sequence converges
- All Cauchy sequences in a complete space converge
- If a sequence of elements in a sequence space is Cauchy, then each of its component sequences is Cauchy

Uniform Convergence

- If a sequence of continuous functions converges uniformly, then its limit is also continuous

Inequalities

- Holder's Inequality: $\sum_{i=1}^{n} |a_i b_i| \leq (\sum_{i=1}^{n} |a_i|^p)^{\frac{1}{p}} (\sum_{i=1}^{n} |b_i|^p)^{\frac{1}{p}}$

Sequence Spaces

- $l_1 \subset l_2 \subset ... \subset l_p \subset ... \subset l_\infty$

Completeness

- A subset of a complete set is is complete in the induced metric if it is closed