AMATH 350 Midterm Exam Review

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Linear DEs Techniques

Separable equations: $\frac{dy}{dx} = g(x)h(y) \implies \int \frac{1}{h(y)} dy = \int g(x) dx$

Integrating Factor: $\frac{dy}{dx} + p(x)y = q(x) \implies \frac{d}{dx}[\mu(x)y(x)] = \mu(x)q(x)$ where $\mu(x) = e^{\int p(x)dx}$

Homogeneous Equations (Char. Eqns.)

1. Distinct roots $m = r_1, r_2 \implies y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
2. Complex conjugates $m = \alpha \pm i\beta$ (use 'complete the square') $\implies y = e^{\alpha x}[A\cos\beta x + B\sin\beta x]$
3. Repeated roots $m = r \implies y = c_1 e^{rx} + c_2 x e^{rx}$

Inhomogeneous Equations (Method of Undetermined Eqns)

Forcing Term	Trial Function
e^{kx}	Ae^{kx}
$\sin kx$,$\cos kx$	$A\cos kx + B\sin kx$
x^n	$\sum_{k=0}^{n} A_k x^k$
xe^x	$(Ax+B)e^x$

Inhomogeneous Equations (Variation of Params)

- In the 1st order equation $y' + p(x)y = F(x)$, suppose that we have the homogeneous solution $y_h = c_1 y_1$ then we try $y_p = u_1 y_1$ and substitute $y = y_p$ and $y' = y_p'$ in the original DE and solve for u. Include the coefficient of integration to get the general solution.
- In the 2nd order equation $y'' + p(x)y' + q(x)y = F(x)$, suppose that we have the homogeneous solution $y_h = c_1 y_1 + c_2 y_1$ then we try $y_p = u_1 y_1 + u_2 y_2$ and we need (1) $u_1' y_1 + u_2' y_2 = 0$, (2) $u_1' y_1' + u_2' y_2' = F(x)$.

Inhomogeneous Equations (Reduction in Order)

Suppose that you have found a solution $y = y_1$ to the original DE. Then one can guess another solution uy_1 to the DE and take derivatives up to the original order of the DE, one can solve for u when you plug these values back into the DE.

Special Substitutions

1. The form $y' = f(ax+by) \implies$ replace $y(x)$ with $u(x)$ where $u = ax + by$
2. The form $y' = f\left(\frac{y}{x}\right)$ or $y' = f\left(\frac{x}{y}\right) \implies$ use $u = \frac{y}{x}$ where $\frac{dy}{dx} = x\frac{du}{dx} + u$
3. The form $\frac{dy}{dx} + p(x)y = q(x)y^n \implies$ multiply the original by $y^{-(n-1)}$ and use $v = y^{1-n} = y^{-(n-1)}$ where $\frac{dv}{dx} = -(n-1)y^{-n}\frac{dy}{dx}$

Boundary Value Problems

- These are problems where we want to know what values of k a certain homogeneous DE, like $y'' + ky = 0, y(0) = 0, y(1) = 0$ has solutions
- To do this, you examine the characteristic equation as a function of k and check "interesting" cases for k
- We then calculate the eigenvalues (valid k values) and eigenfunctions (valid functions implied by the eigenvalues)

Graphing Solutions

- Solve DE if possible
- Identify any **exceptional solutions** which behave differently from the rest (usually set $C = 0$)
- Consider the behaviour of the other solutions as $x \to \pm\infty$ or near vertical asymptotes
- Set $\frac{dy}{dx} = 0$ in the DE to find the **horizontal isocline**
- Determine how $\frac{dy}{dx}$ behaves outside of the horizontal isocline

Models

Malthusian Model: $\frac{dP}{dt} = rP$

Logistic Model: $\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$