STAT 333 Final Exam Review

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Chapter 1

Indicator RV: $1_A(x) = \begin{cases} 1 & ,x \in A \\ 0 & ,x \notin A \end{cases}$ and $P(A) = E[I_A] = E[E[I_A|Y]]$

Bayes' Formula: $P(A|B) = \frac{P(B|A)P(A)}{\sum_i P(B|C_i)P(C_i)}$

Select properties of expectation: If X, Y independent, then $E(XY) = E(X)E(Y)$

We will not discuss the various r.v.'s that are discussed in class since their properties, means, and variance will be seen in a cheat sheet on the exam

Chapter 2

An waiting r.v. T_E is **proper** if $P(T_E < \infty) = 1$ or $P(T_E = \infty) = 0$. It is **short proper** if $E(T_E) < \infty$ and is **null proper** is $E(T_E) = \infty$

Chapter 3

Properties of joint pdf: $f_{X,Y}(x,y) \geq 0$, $\sum_x \sum_y f_{X,Y}(x,y) = 1$ and $f_X(x,y) = \sum_Y f_{X,Y}(x,y)$ with vice versa for f_Y (Same case in cts version)

The expectation is $E[h(x,y)] = \int_x \int_y h(x,y)f(x,y)\,dx\,dy$

If X, Y independent, then $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ and $g(X)$, $g(Y)$ independent

$f_{X|Y}(x,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$ is a pmf and if $X \perp Y$, then $E[X|Y=y] = E[X]$

Conditional Variance: $Var(X|Y) = E(X^2|Y) - [E(X|Y)]^2$

Double expectation: $E[E[X|Y]] = E[X]$, Double Variance: $Var(X) = E[Var(X|Y)] + Var(E[X|Y])$

Compound Variable Theorem: Let $W = \sum_{i=1}^N X_i$ with $X_i's$ being i.i.d., then $E(W) = E(N)E(X_1)$ and $Var(W) = Var(N)(E[X_1])^2 + E(N)Var(X_1)$

Chapter 4

Selected series: $\sum_{n=k}^\infty s^n = \frac{s^k}{1-s}$, $\sum_{n=k}^\infty (-1)^n s^n = \frac{(-1)^k s^k}{1-s}$, $\sum_{n=0}^\infty \frac{s^n}{n!} = e^s$, $\sum_{n=0}^\infty ns^n = \frac{1}{(1-s)^2} - \frac{1}{1-s}$

Properties of binary operations: $A(s) \pm B(s) = \sum_{n=0}^\infty (a_n \pm b_n)s^n$ with $|s| < \min(R_A, R_B)$, $A(s) \times B(s) = \sum_{n=0}^\infty \left(\sum_{k=0}^n a_k \cdot b_{n-k}\right) s^n$ with $|s| < \min(R_A, R_B)$

Combinatorial identity: $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$

Definition of PGF and MGF: $G_X(s) = \sum_{n=0}^\infty p_n s^n \triangleq E[s^X]$, $M_X(t) = E[e^{tX}]$

Properties of PGF and MGF: G_X is monotone increasing if $s \geq 0$, $G_X(1) = 1$, $G'_X(1) = E(X)$, $Var(X) = G''_X(1) + G'_X(1) - [G'_X(1)]^2 = E[X(X-1)]$

Selected PGFs: $X \sim Geo(p) \implies G_X(s) = \frac{ps}{1-(1-p)s}$

Also note that $G_{\sum_{i=1}^n X_i}(s) = \prod_{i=1}^n G_{X_i}(s)$.

Chapter 5

Classification of T_λ: A waiting time r.v. is **renewal** if $T_\lambda, T_\lambda^{(1,2)}, T_\lambda^{(2,3)}, ...$ is i.i.d. and is **delayed renewal** if $T_\lambda, T_\lambda^{(1,2)}$ are different distributions and $T_\lambda^{(1,2)}, T_\lambda^{(2,3)}, ...$ are i.i.d.

Note that after we observe λ once in a delayed renewal even, then the event becomes a renewal event called the **associated renewal event**, with our new event denoted by $\tilde{\lambda}$

Classification of λ: Let $f_\lambda = P(T_\lambda < \infty)$. We say λ is **transient** if $f_\lambda < 1$ and **recurrent** if $f_\lambda = 1$

If λ recurrent, we say it is **null recurrent** if $E(T_\lambda) = \infty$ and **positive recurrent** if $E(T_\lambda) < \infty$

Definition of v_λ: v_λ denotes the number of times we observe λ, ever. by definition it is $P(v_\lambda = k) = \begin{cases} f_\lambda^k(1-f_\lambda) & ,k = 0,1,... \\ f_\lambda^\infty & ,k = \infty \end{cases}$ and $E[v_\lambda] = \frac{f_\lambda}{1-f_\lambda}$, $Var(v_\lambda) = \frac{f_\lambda}{(1-f_\lambda)^2}$.

PGF relationship: $E[v_\lambda] = \sum_{n=1}^\infty r_n$ with $r_0 \triangleq 1$ and $r_n = P(\lambda \text{ occurs on trial } n)$, $R_\lambda(s) = \sum_{n=1}^\infty r_n s^n$

Note that $E[v_\lambda] = \infty$ recurrent, $E[v_\lambda] < \infty$ transient

$F_\lambda(s) = \frac{1}{1-R_\lambda(s)} = \sum_{n=1}^\infty f_n s^n$ and f_0, $f_n = P(\lambda \text{ occurs first on trial } n)$. $F'_\lambda(1) =$ Average time to see λfirst

Periodicity: If λ is a renewal event, define $d = gcd(\{n|r_n > 0, n \geq 1\})$. If $d = 1$, then λ is **aperiodic**, $d > 1$, then λ is **periodic**

If $d = 1 \implies E[T_\lambda] = \lim_{n\to\infty} \frac{1}{r_n}$

Also note that $F_\lambda(s) = \frac{D_\lambda(s)}{R_{\tilde{\lambda}}(s)}$

Gambler's Ruin: Let P_j be the probability of reaching k from j without touching 0. $P_j = \frac{1-(\frac{q}{p})^j}{1-(\frac{q}{p})^k}$ where q is left and p is right.

The solution involves telescoping or solving $P_i = pP_{i+1} + q + P_{i-1} \implies py + qy^{(2)} = y^{(1)}$.

Chapter 6

Markovian: $\{X_n\}_{n=0}^{\infty}$ is **Markovian** if $P_n(X_{n+1} = j|X_n = i, X_{n-1} = i_{n-1}, ..., X_0 = i_0) = P_n(X_{n+1} = j|X_n = i)$

Time Homogeneous: The process is T.H. if $P_n = P(X_1 = j|X_0 = i), \forall n \in \mathbb{N}$.

Transition Matrix: $p_{ij}^{(n)} = P(X_n = j|X_0 = i)$, $p_{ij} = P(i \to j) = P(X_1 = j|X_0 = i)$, $P^{(n)} = (p_{ij}^{(n)})_{i,j \in S}$ where $\sum_j p_{ij} = 1$ and it is **doubly stochastic** if $\sum_i p_{ij} = 1$ as well.

Absorbing States and Regularity: i **absorbing** if $p_{ii} = 1$, P regular if stochastic and $p_{ij} > 0, \forall i, j$

Chapman-Kolmogorov: $p_{ij}^{(n+m)} = \sum_{t \in S} p_{it}^{(n)} p_{tj}^{(m)}$ and $P^{(n+m)} = P^{(n)} P^{(m)} \implies P^{(n)} = P^n$

Distribution Vector: $\pi^{(n)}$ defined by $\pi_i^{(n)} = P(X_n = i)$ and $\pi^{(n)} = \pi^{(0)} P^n$

State Classification: State i classification is equivalent to classifying λ_{ii} ({transient, positive recurrent, null recurrent}~{t,pr,nr})

Periodicity: Period of state i is the period of λ_{ii}

Accessibility: Equivalence class where $i \sim j$ iff $\exists n \in \mathbb{N}$ such that $p_{ij}^{(n)} > 0$

Communicating States: **Open** if $\exists i \in C, \exists j \notin C$ such that $p_{ij} > 0$ and **closed** if $\forall i \in C, \forall j \notin C$ we have $p_{ij} = 0$

Important Properties: States in the same class $\to$ (1) States are one of {t,pr,nr} at the same time (2) States have the same period (3) If $\exists i \in C$ such that $p_{ii} > 0$ then the class period is 1 (4) Open class states are all transient (5) Finite closed classes are all positive recurrent (∞) Finite Markov chains are either all transient or recurrent but not both

Stationary Distribution: π defined by unique vector such that $\pi P = \pi$; If P^n is doubly stochastic for some n, then π is uniform

* Check Assignment 3 for types of problems (this is important); there will be systems of linear equations to solve

Chapter 7

Exponential Distribution: PDF = $\lambda e^{-\lambda x}$, CDF = $1 - e^{-\lambda x}$, $E(X) = \lambda^{-1}$, $Var(X) = \lambda^{-2}$, $P(X \geq t + s|X \geq s) = P(X \geq t)$, $X_{min} = \min(X_1, ..., X_m) \sim exp(\sum_{i=1}^{m} \lambda_i)$ if $X_i \sim exp(\lambda_i)$ and we have $P(X_{min} \leq x) = 1 - e^{-(\sum \lambda_i)x}$, $T_{max} = \max(T_1, T_2) \implies T_1 + T_2 = \max(T_1, T_2) + \min(T_1, T_2) \implies E(T_{max}) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2}$.

Poisson Process: A process $\{X_t = X(t)|t \geq 0\}$ is **Poisson** if it is counting and (1) $X(0) = 0$ (2) $X(t + s) - X(s) \sim Pois(\lambda t)$ (3) $0 \leq s_1 \leq s_2 \leq t_1 \leq t_2 \implies [X(t_2) - X(t_1)] \perp [X(s_2) - X(s_1)]$; $S_2 - S_1 = T_1$ is a positive r.v. implies $F_{T_2}(0) = 0$

Remarks and Properties: $\lim_{h \to 0} \frac{P(X(t+h) - X(t) \geq 2)}{h} = 0$, $\lim_{\triangle t \to 0} (X(t + \triangle t) - X(t) > 1 | X(t + \triangle t) - X(t) \geq 1) = 0$, if the waiting time between hits is $exp(\lambda)$ and i.i.d., then counting up to time t is a Poisson process, if $X(t) = n$ then $X(s) \sim Bin(n, \frac{s}{t})$, $s \leq t$, if $X = X_1 + X_2$ where $X_1 = Pois(\lambda pt)$, $X_2 = Pois(\lambda qt)$ and $p + q = 1$ then $P(X_1(t) = m, X_2(t) = n|X(t) = n + m) = \binom{n+m}{n} p^m q^n$