PMATH 450 Final Exam Summary
Lebesgue Integration and Fourier Analysis

1 Riemann Integration

Definition 1.1. Let $[a,b] \subseteq \mathbb{R}$ compact and $f : [a,b] \mapsto \mathbb{R}$ be bounded. We say f is Riemann integrable if

$$\underline{\int_a^b} f = \overline{\int_a^b} f$$

and we denote this as $\int_a^b f$. Note that constant and continuous functions are Riemann integrable.

1.1 Riemann Sums on Vector Valued Functions

Definition 1.2. A real or complex vector space X is called a Banach space if it is a complete normed linear space, where completeness is when all Cauchy sequences in X converge.

Note 1. Recall the properties of a norm $\|\cdot\|$:

1) $\|x\| = 0 \iff x = 0$

2) $\|x+y\| \leq \|x\| + \|y\|$

3) $\|\alpha x\| = |\alpha|\|x\|$

Definition 1.3. For a given Banach space X, partition $P_r = \{t_i | t_0 = a < t_1 < ... < t_{n-1} < t_n = b, \max_i (t_i - t_{i-1}) \leq r\} \subseteq [a,b]$ and $f : [a,b] \mapsto X$, we define the Riemann sum over P_r for this Banach space valued function f as

$$S(f, P_r) = \sum_{i=1}^{n} \underbrace{f(t_i^*)}_{\in X} \underbrace{(t_i - t_{i-1})}_{\in \mathbb{R}} \in X$$

Definition 1.4. Let $f : [a,b] \mapsto X$ where X is a Banach space. We say that f is Riemann integrable if there is $x \in X$ such that $\forall \epsilon > 0$ there is P_ϵ with for any $P \supseteq P_\epsilon$ we have

$$\|S(f,P) - x\| < \epsilon$$

for any Riemann sum over P, independent of the $t_i^* s$.

Theorem 1.1. *(*Cauchy Criterion*) Let χ be a Banach space. A function $f : [a,b] \mapsto \chi$ is Riemann integrable $\iff$ $\forall \epsilon, \exists$ partition Q_ϵ such that for any $P, Q \supseteq Q_\epsilon$ and any Riemann sums over P, Q we have*

$$\|S(f,P) - S(f,Q)\| < \epsilon$$

Lemma 1.1. *Assume that $f : [a,b] \mapsto \chi$ is continuous. Let $\epsilon > 0$. Then $\exists \delta > 0$ such that if P is any partition with $\|P\| < \delta$ then for any $P_1 \supseteq P$ and any $S(f,P)$, $S(f,P_1)$ we have*

$$\underbrace{\|S(f,P) - S(f,P_1)\|}_{\textit{norm in } \chi} < \epsilon$$

Theorem 1.2. *Assume that $f : [a,b] \mapsto \chi$ is continuous. Then f is Riemann integrable.*

Example 1.1. Consider the function $\chi_{[0,\frac{1}{2})} : [0,1] \mapsto \mathbb{R}$ where χ_A is the characteristic/indicator function on some set A. Observe that $\int_0^1 \chi_{[0,\frac{1}{2})} = \frac{1}{2}$. Note that for any $[a,b] \subseteq [c,d]$ we have $\int_c^d \chi_{[a,b]} = b - c$.

Example 1.2. Consider the function $\chi_{\mathbb{Q}\cap[0,1]} : [0,1] \mapsto \mathbb{R}$. Let $P = \{x_i | 0 = x_0 < ... < x_n = 1\}$ be a any partition of $[0,1]$. Then for each $1 \leq i \leq n$,

$$\begin{aligned} M_i &= \sup\{\chi_{\mathbb{Q}\cap[0,1]}(t) : t \in [x_{i-1}, x_i]\} = 1 \\ m_i &= \inf\{\chi_{\mathbb{Q}\cap[0,1]}(t) : t \in [x_{i-1}, x_i]\} = 0 \end{aligned}$$

and so upper and lower Riemann sums will never converge ($1 = U(\chi_{\mathbb{Q}\cap[0,1]}, P) \neq L(\chi_{\mathbb{Q}\cap[0,1]}, P) = 0$) and the Riemann integral does not exist.

2 General Measures and Measure Spaces

Definition 2.1. Given a set X, we denote the power set of X as $\mathcal{P}(X)$. By definition, this is the set of all subsets of X.

Definition 2.2. Let X be a non-empty set. An algebra of subsets of X is a collection $A \subseteq \mathcal{P}(X)$ such that

1) $\emptyset$ and $X \in A$

2) If $E_1, E_2 \in A$ then $E_1 \cup E_2 \in A$

3) If $E \in A$ then $E^c = X \backslash E \in A$

Definition 2.3. A σ-algebra of subsets of $X$is a collection $A \subseteq P(X)$ such that

1) $\emptyset$ and $X \in A$

2) If $E_1, E_2, ... \in A$ then $\bigcup_{n=1}^{\infty} E_n \in A$

3) If $E \in A$ then $E^c = X \backslash E \in A$

Remark 2.1. All σ-algebras are algebras.

Note 2. Note that $E_1 \cap E_2 = (E_1^c \cup E_2^c)^c$ and so algebras are closed under finite intersections and σ-algebras are closed under countable intersections.

Example 2.1. Let X be an infinite set and let A be the collection of subsets $\{E_n\}_{n\in I}$ of X such that either E or E^C is finite. Then A is an algebra but not always a σ-algebra. This is due to the fact that the countable unions of sets may produce a set whose complement and itself is not finite.

Example 2.2. If $\{A_\alpha\}_{\alpha\in I}$ a family of algebras (σ-algebra) then $\bigcap_{\alpha\in I} A_\alpha$ is an algebra (σ-algebra).

Note 3. Given $S \subseteq \mathcal{P}(X)$, there exists a smallest algebra (σ-algebra) containing S which follows from the above example.

Notation 1. Let $S \subseteq \mathcal{P}(X)$. We denote:

$A(S)$: the algebra generated by S which is defined to be the smallest algebra containing S.

$\sigma(S)$: the σ-algebra generated by S which is the smallest σ-algebra containing S

Definition 2.4. Let $\mathcal{G}$ =$\{U \subseteq \mathbb{R} | U$ is open$\}$. The σ-algebra generated by $\mathcal{G}$, $\sigma(\mathcal{G})$, will be called the Borel σ-algebra of $\mathbb{R}$ and will also be denoted by $\mathcal{B}(\mathbb{R})$.

Remark 2.2. More generally, we may consider the Borel σ-algebra on any topological space. We will examine this shortly.

Given any set X and $M \subseteq \mathcal{P}(X)$, let

$$\begin{aligned} M_\delta &= \left\{ A \in \mathcal{P}(X) : A = \bigcap_{i=1}^{\infty} M_i, M_i \in M \right\} \\ M_\sigma &= \left\{ A \in \mathcal{P}(X) : A = \bigcup_{i=1}^{\infty} M_i, M_i \in M \right\} \end{aligned}$$

and G be the set of all open subsets of $\mathbb{R}$ and F be the set of closed subsets of $\mathbb{R}$

Then we have

$$\begin{aligned} \mathcal{G}_\delta &= \{\text{countable intersections of open sets of } \mathbb{R}\} \\ \mathcal{F}_\sigma &= \{\text{countable unions of closed sets of } \mathbb{R}\} \end{aligned}$$

and $\mathcal{G}_\sigma = G$, $\mathcal{F}_\sigma = F$. Therefore,

$$\begin{aligned} G &\subset \mathcal{G}_\delta \subset \mathcal{G}_{\delta\sigma} \subset \mathcal{G}_{\delta\sigma\delta} \subset \ldots \subset \mathcal{B}(\mathbb{R}) \\ F &\subset \mathcal{F}_\sigma \subset \mathcal{F}_{\sigma\delta} \subset \mathcal{F}_{\sigma\delta\sigma} \subset \ldots \subset \mathcal{B}(\mathbb{R}) \end{aligned}$$

and note that $\mathcal{G}_\delta$ sets are exactly the complements of $\mathcal{F}_\sigma$-sets. Note that none of these sets are equal.

Example 2.3. $\mathbb{Q}$ is $\mathcal{F}_\sigma$ but $\mathbb{Q} \notin F$. Similarly $\mathbb{R} \backslash \mathbb{Q}$ is G_δ (why?) but $\mathbb{R} \backslash \mathbb{Q} \notin G$.

Proposition 2.1. *$F \subset \mathcal{G}_\delta$ and $G \subset \mathcal{F}_\sigma$.*

Note 4. About the Borel $\sigma-$algebra:

$$\begin{aligned} \mathcal{B}(\mathbb{R}) &= \sigma(G) \\ &\subseteq \sigma\{(a,b) | a, b \in \mathbb{R}\} \\ &\subseteq \sigma\{(a,b] | a, b \in \mathbb{R}\} \\ &= \sigma\{[a,b) | a, b \in \mathbb{R}\} \\ &\subseteq \sigma\{[a,b] | a, b \in \mathbb{R}\} \end{aligned}$$

Remark 2.3. $\mathcal{G}_\delta = \mathcal{G}_{\delta\delta}$ and $\mathcal{F}_\sigma = \mathcal{F}_{\sigma\sigma}$ because the countable union and intersection of countable sets is countable.

2.1 Measures

Definition 2.5. The set $\mathbb{R}$ together with σ-algebra A, $(\mathbb{R}, A)$ is a called a measurable space. A (countably additive) measure on A is a function $\mu : A \mapsto \mathbb{R}^* := \mathbb{R} \cup \{\pm\infty\}$ with the properties:

1) $\mu(\emptyset) = 0$

2) $\mu(E) \geq 0$ for all $E \in A$

3) If $\{E_n\}_{n=1}^{\infty} \subset A$ is sequence of disjoint sets, then $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu\left(E_n\right)$

Definition 2.6. If we replace 3) by

3') If $\{E_n\}_{n=1}^{N} \subseteq A$ is a finite sequence of disjoint sets then $\mu\left(\bigcup_{n=1}^{N} E_n\right) = \sum_{n=1}^{N} \mu\left(E_n\right)$ where $N \in \mathbb{N}$.

then such a μ is called a finitely additive measure. Usually, we will assume a measure is countably additive unless otherwise specified.

Definition 2.7. We will call a measure μ finite if $\mu(\mathbb{R}) < \infty$ and call it σ-finite if there exists $\{E_n\}_{n=1}^{\infty} \subset A$ such that $\bigcup_{n=1}^{\infty} E_n = \mathbb{R}$ and each $\mu(E_n) < \infty$.

Definition 2.8. A triple $(\mathbb{R}, A, \mu)$ is called a measure space where A is a σ-algebra and μ is a measure on A. We also say that such a triple is complete if for any $E \in A$ with $\mu(E) = 0$ and $S \subset E$ we have $S \in A$. For $E \in A$ we call E a measurable set.

Proposition 2.2. *(MONOTONICITY) Let $(\mathbb{R}, A, \mu)$ be a measure space. If $E \subset F$ and $E, F \in A$ then $\mu(E) \leq \mu(F)$.*

Corollary 2.1. *If $\mu(E) < \infty$ then $\mu(F \backslash E) = \mu(F) - \mu(E)$.*

Note 5. If $\mu(E) = \infty$ then $\mu(F) = \infty$ and the difference $\mu(F) - \mu(E)$ is undetermined.

Proposition 2.3. *(Countable Subadditivity) Let $(\mathbb{R}, A, \mu)$ be a measurable space. Let $\{E_n\}_{n=1}^{\infty} \subset A$. Then $\mu(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu(E_n)$*

2.2 Lebesgue Outer Measure

Problem 2.1. We want to define a measure λ on $\mathcal{P}(\mathbb{R})$ such that

(1) $\lambda : \mathcal{P}(\mathbb{R}) \mapsto \mathbb{R}^{\geq 0} \cup \{\infty\} = [0, \infty]$

(2) If $I = (a, b)$ then $\lambda(I) = \lambda((a, b)) = b - a$

(3) λ is countably additive

(4) $\lambda(E + x) = \lambda(E)$, $E \subseteq \mathbb{R}$, $x \in \mathbb{R}$ (translation invariance)

Unfortunately, this is note possible. Thus, we relax our conditions by restricting our domain to a σ-algebra which is a proper subset of $\mathcal{P}(\mathbb{R})$. Still, we want to have $\mathcal{B}(\mathbb{R})$ to be contained in that σ-algebra.

Definition 2.9. A function $\mu^* : \mathcal{P}(\mathbb{R}) \to \mathbb{R}^*$ is a called an outer measure if

1) $\mu^*(\emptyset) = 0$

2) $\mu^*(A) \leq \mu^*(B)$ if $A \subseteq B \subseteq \mathbb{R}$

3) If $\{E_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathbb{R})$ then $\mu^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$

Definition 2.10. μ^* is finite if $\mu^*(\mathbb{R}) < \infty$ and is called σ-finite if $\mathbb{R} = \bigcup_{n=1}^{\infty}$ and $|\mu^*(E_n)| < \infty$.

Definition 2.11. (CADATHEODORY CRITERION) A set $E \in \mathcal{P}(\mathbb{R})$ is μ^*-measurable (measurable) if for any $A \subset \mathbb{R}$

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Note 6. By definition,

$$\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

so to prove measurability of E, it is enough to show that

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

for every $A \subset \mathbb{R}$. Furthermore, if $\mu^*(A) = \infty$ then the above trivially holds. So be only need to consider finite cases ($\mu^*(A) < \infty$).

Definition 2.12. Let $I = (a, b)$ and $l(I) = b - a$ with $l((a, \infty)) = +\infty$ and $l((-\infty, b)) = +\infty$. For any $E \subset \mathbb{R}$,

$$\lambda^*(E) = \inf \left\{ \sum_{n=1}^{\infty} l(I_n) : E \subset \bigcup_{n=1}^{\infty} I_n, I_n's \text{ are open intervals} \right\}$$

Remark 2.4. $\lambda^*(E) \geq 0$.

Proposition 2.4. *λ^* is an outer measure on $\mathbb{R}$.*

2.3 Lebesgue Measure

Definition 2.13. λ^* is called the Lebesgue outer measure on $\mathbb{R}$. We denote the σ-algebra of λ^*-measurable sets by $\mathcal{L}(\mathbb{R})$. Elements of $\mathcal{L}(\mathbb{R})$ are called Lebesgue measurable. $\lambda = \lambda^*\Big|_{\mathcal{L}(\mathbb{R})}$ is called the Lebesgue measure of $\mathbb{R}$.

Proposition 2.5. *If $a < b$ and are both in $\mathbb{R}$ and J is an interval of the form $(a, b), [a, b], (a, b], [a, b)$ then $\lambda^*(J) = b - a$.*

Theorem 2.1. *(Caratheodory's Theorem) The set $\mathcal{L}(\mathbb{R})$ of Lebesgue measurable sets is a σ-algebra and $\lambda^*\Big|_{\mathcal{L}(\mathbb{R})} = \lambda$ is a complete measure.*

Proposition 2.6. *λ is a measure.*

Proposition 2.7. *λ is complete. ($\lambda(E) = 0$ if $E \subseteq S$ with $\lambda(S) = 0$)*

Theorem 2.2. *Let μ^* be a non-negative outer measure on $\mathbb{R}$. Let $\mathcal{M}_{\mu^*}$ denote the μ^* measurable subsets of $\mathbb{R}$. Then $\mathcal{M}_{\mu^*}$ is a σ-algebra and $\mu^*\Big|_{\mathcal{M}_{\mu^*}} = \mu$ is a measure on $\mathcal{M}_{\mu^*}$ with the associated space $(\mathbb{R}, \mathcal{M}_\mu, \mu)$ being complete.*

Lemma 2.1. *Every bounded open interval $(a, b) \subset \mathbb{R}$ is in $\mathcal{L}(\mathbb{R})$*

Corollary 2.2. *$\mathcal{B}(\mathbb{R}) = \sigma(\{(a, b) : a, b \in \mathbb{R}\}) \subset \mathcal{L}(\mathbb{R})$ since $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra that is generated by open sets ($\mathcal{L}(\mathbb{R})$ is a larger σ-algebra that contains open sets).*

Remark 2.5. For $x \in \mathbb{R}$, $\{x\}$ is closed $\implies$ $\{x\} \in \mathcal{L}(\mathbb{R})$. We have

(i) $\lambda(\{x\}) = 0$

(ii) $\lambda(E) = 0$ for countable E

Problem 2.2. If $\lambda(E) = 0$ do we need $|E| \leq \aleph_0$? The answer is no!

Example 2.4. (Cantor set) Let $C_0 = [0, 1], C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right], ..., C_n = C_{n-1} \backslash (I_{n,1} \cup ... \cup I_{n,2^{n-1}})$ where $I_{n,k}$ is the open middle third of the k^{th} set from C_{n-1} and let

$$C = \bigcap_{n=1}^{\infty} C_n$$

where we call C the Cantor set.

Remark 2.6. $C \neq \emptyset$ due to the Cantor Intersection Theorem ($\{C_n\}$ has the finite intersection property).

Proposition 2.8. *(i) C is closed*

(ii) C is nowhere dense

(iii) $\lambda(C) = 0$

Proposition 2.9. *$|C| = c$ where c is the cardinality of the continuum.*

Definition 2.14. Let $E \subseteq \mathbb{R}, x \in \mathbb{R}$. We define the translate of E by x as

$$E + x = \{y + x : y \in E\}$$

Proposition 2.10. *(Translation Invariance of the Lebesgue Measure)*

(i) If $E \subseteq \mathbb{R}, x \in \mathbb{R}$ then $\lambda^*(x + E) = \lambda^*(E)$

(ii) If $E \in \mathcal{L}(\mathbb{R}), x \in \mathbb{R}$ then $x + E \in \mathcal{L}(\mathbb{R})$

(iii) If $E \subseteq \mathbb{R}, x \in \mathbb{R}$ then $\lambda(x + E) = \lambda(E)$

2.4 Non-Measurable Sets

Theorem 2.3. *There exist non-measurable subsets of $\mathbb{R}$. That is $\mathcal{P}(\mathbb{R})\backslash\mathcal{L}(\mathbb{R}) \neq \emptyset$. (Note that the proof will depend on the Axiom of Choice (AoC). Without it, it is possible to show $\mathcal{P}(\mathbb{R})\backslash\mathcal{L}(\mathbb{R}) = \emptyset$ (c.f. R.M. Solovay, 1970, Ann. of Math)).*

3 Measurable Functions

Definition 3.1. A function $f : \mathbb{R} \mapsto \mathbb{R}$ is called measurable if for every $\alpha \in \mathbb{R}$ we have

$$f^{-1}((\alpha, +\infty)) = \{x \in \mathbb{R} : f(x) > \alpha\}$$

is λ-measurable. f is called Borel measurable if $f^{-1}((\alpha, +\infty)) \in \mathcal{B}(\mathbb{R})$ for all $\alpha \in \mathbb{R}$.

Example 3.1. If $f : \mathbb{R} \mapsto \mathbb{R}$ is continuous, then $f^{-1}((\alpha, +\infty))$ is open and f is λ-measurable and Borel measurable.

Example 3.2. Let $A \subseteq \mathbb{R}$. Consider the characteristic function

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

We claim that χ_A is measurable. That is, $\chi_A \in \mathcal{M}(\mathbb{R}) \iff A \in \mathcal{L}(\mathbb{R})$. To prove this, let $\alpha \in \mathbb{R}$ and note that

$$\chi_A^{-1}((\alpha, \infty)) = \begin{cases} \emptyset & \alpha \geq 1 \\ A & 0 < \alpha \leq 1 \\ \mathbb{R} & \alpha \leq 0 \end{cases}$$

So χ_A is measurable if $A \in \mathcal{L}(\mathbb{R})$.

Proposition 3.1. *Let $f : \mathbb{R} \mapsto \mathbb{R}$. TFAE.*

(i) f is measurable (Borel measurability)

(ii) $\forall \alpha \in \mathbb{R}$, $f^{-1}((-\infty, \alpha])$ ($\in \mathcal{B}(\mathbb{R})$)

(iii) $\forall \alpha \in \mathbb{R}$, $f^{-1}((-\infty, \alpha))$ ($\in \mathcal{B}(\mathbb{R})$)

(iv) $\forall \alpha \in \mathbb{R}$, $f^{-1}([\alpha, \infty))$ ($\in \mathcal{B}(\mathbb{R})$)

Proposition 3.2. *A function $f : \mathbb{R} \mapsto \mathbb{R}$ is (Borel) measurable if and only if $f^{-1}(A)$ is (Borel) measurable for each Borel set A ($A \in \mathcal{B}(\mathbb{R})$)*

Let $f, g : \mathbb{R} \mapsto \mathbb{R}$ be measurable, $c \in \mathbb{R}$ and $\phi : \mathbb{R} \mapsto \mathbb{R}$ be continuous. Then

(i) cf is measurable

(ii) $f + g$ is measurable

(iii) $\phi \circ f$ is measurable, ϕ continuous

(iv) fg is measurable

Note that $(i), (ii)$, and (iv), as a corollary, tells us that $\mathcal{M}(\mathbb{R})$ is an algebra.

Corollary 3.1. *If $f : \mathbb{R} \mapsto \mathbb{R}$ is measurable, then so are $|f|$, f^+, f^- where*

$$f^+(x) = \max\{f(x), 0\}, f^-(x) = -\min\{f(x), 0\}$$

3.1 The Extended Reals

Definition 3.2. Define the extended real line $\mathbb{R}^*$ as

$$\mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\} = [-\infty, \infty]$$

(1) A function f on $\mathbb{R}$ is called extended real valued if $f : \mathbb{R} \mapsto \mathbb{R}^*$

(2) An extended real valued function is called measurable if $\forall \alpha \in \mathbb{R}$,

$$f^{-1}((\alpha, \infty]) \in \mathcal{L}(\mathbb{R})$$

Proposition 3.3. *An extended real valued function $f : \mathbb{R} \mapsto \mathbb{R}^*$ is measurable if and only if the following conditions are satisfied.*

1) $f^{-1}(\{-\infty\})$ and $f^{-1}(\{\infty\})$ are in $\mathcal{L}(\mathbb{R})$

2) The real valued function f_0 defined by

$$f_0(x) = \begin{cases} f(x) & f(x) \in \mathbb{R} \\ 0 & f(x) \in \{\pm\infty\} \end{cases}$$

is measurable (i.e. $f_0 \in \mathcal{L}(\mathbb{R})$)

Notation 2. The set of measurable extended $\mathbb{R}^*$ valued function are denoted by $\mathcal{M}^*(\mathbb{R})$.

Remark 3.1. Note that if $f, g \in \mathcal{M}^*(\mathbb{R})$ we could have that $f + g$ is indeterminate $(\infty - \infty)$ and so $\mathcal{M}^*(\mathbb{R})$ is not necessarily an algebra. Also, if $\phi : \mathbb{R} \mapsto \mathbb{R}$ is continuous, then $\phi \circ f$ may fail to make sense.

Proposition 3.4. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{M}^*(\mathbb{R})$. Then the following functions are also measurable:*

(i) $\sup_{n\in\mathbb{N}} f_n$ (pointwise infimum)

(ii) $\inf_{n\in\mathbb{N}} f_n$ (pointwise supremum)

(iii) $\limsup_{n\to\infty} f_n$ where $\left(\limsup_{n\to\infty} f_n\right)(x) = \inf_n \left(\sup_{k\geq n} f_k(x)\right)$

(iv) $\liminf_{n\to\infty} f_n$ where $\left(\liminf_{n\to\infty} f_n\right)(x) = \sup_n \left(\inf_{k\geq n} f_k(x)\right)$

Corollary 3.2. *If $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{M}^*(\mathbb{R})$ with pointwise limit $f(x)$ then $f \in \mathcal{M}^*$.*

4 Lebesgue Integration

Instead of partitioning the domain of a function, like in Riemann integration, we instead partition in the range. That is, we divide the range of f into a partition

$$y_0 < y_1 < \dots < y_n$$

and define

$$E_i = \{t \in A : y_{i-1} \leq f(t) < y_i\}$$

We then find the sized of $E_i = \lambda(E_i)$ and we will estimate $\int f$ by sums

$$\sum_{k=1}^{n} y_{i-1}\lambda(E_i)$$

4.1 Simple Functions

Definition 4.1. Let $A \in \mathcal{L}(\mathbb{R})$, a function $\phi : A \mapsto \mathbb{R}$ is called a simple function if $\phi(A) = \{\phi(x) : x \in A\}$ is a finite set.

Remark 4.1. If $\phi(A) = \{\alpha_1 < ... < \alpha_n\}$, define the preimage of α_i as $E_i = \phi^{-1}(\{\alpha_i\})$ for $1 \le i \le n$. Note that $E_i \cap E_j = \emptyset$ if $i \ne j$. So we have

$$\phi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$$

and we call it the *standard representation of the simple function* ϕ.

Proposition 4.1. *Let A be a measurable set and $\phi : A \mapsto \mathbb{R}$ be a simple function with $\phi(A) = \{\alpha_1 < ... < \alpha_n\}$. Then ϕ is measurable iff each $1 \le i \le n$ we have that the $E_i = \phi^{-1}(\{a_i\})$ are measurable.*

Definition 4.2. Let

$$\begin{aligned} S(A) &= \{\phi : A \mapsto \mathbb{R} : \phi \text{ is simple and measurable}\} \\ S^+(A) &= \{\phi \in S(A) : \phi(x) \ge 0\} \end{aligned}$$

for $A \in \mathcal{L}(\mathbb{R})$.

Proposition 4.2. *If $\phi, \psi \in S(A)$, $\alpha \in \mathbb{R}$ then $\alpha\phi$,$\phi + \psi$ and $\phi \cdot \psi$ are all in $S(A)$.*

Definition 4.3. If $\phi \in S^+(A)$ for $A \in \mathcal{L}(\mathbb{R})$ with $\phi(A) = \{\alpha_1 < ... < \alpha_n\}$ and for $1 \le i \le n$, $E_i = \phi^{-1}(\{a_i\})$ define

$$I_A(\phi) = \sum_{i=1}^{n} \underbrace{\alpha_i}_{\in \mathbb{R}} \underbrace{\lambda(E_i)}_{\in [0,\infty]} \in [0, \infty]$$

and if $\alpha_i > 0$ and $\lambda(E_i) = \infty$ then will define $\alpha_i \lambda(E_i) = \infty$. Also if $\alpha_i = 0$ then will set $\alpha_i \lambda(E_i) = 0$.

Proposition 4.3. *Let $A \in \mathcal{L}(\mathbb{R})$ and $\phi, \psi \in S^+(A)$, $c \ge 0$ then*

(i) $I_A(c\phi) = cI_A(\phi)$

(ii) $I_A(\phi + \psi) = I_A(\phi) + I_A(\psi)$

(iii) If $\phi \le \psi$ then $I_A(\phi) \le I_A(\psi)$

Notation 3. Let $A \in \mathcal{L}(\mathbb{R})$, $A \ne \emptyset$. We put

$$(\mathcal{M}^*)^+(A) = \{f : A \mapsto \mathbb{R} : f \text{ measurable}, f \ge 0\}$$

For $f \in (\mathcal{M}^*)^+(A)$ we define

$$S_f^+(A) = \{\phi \in S^+(A) : \phi \le f\}$$

4.2 The Lebesgue Integral

Definition 4.4. Let $A \in \mathcal{L}(\mathbb{R}), A \ne \emptyset$ and $f \in (M^*)^+(A)$. The Lebesgue integral of f is defined by

$$\int_A f = \sup_{\phi \in S_f^+(A)} \underbrace{I_A(\phi)}_{\in [0,\infty]} \in [0, \infty]$$

Exercise 4.1. If $f : \mathbb{R} \mapsto \mathbb{R}^*$ is measurable, then $f\Big|_A$ is measurable as a function on $A \subseteq \mathbb{R}$.

Proposition 4.4. *Let $A \subseteq \mathcal{L}(\mathbb{R})\backslash\{\emptyset\}$ and $f, g \in (M^*)^+(A)$. Then*

(i) If $f \leq g$ then $\int_A f \leq \int_A g$

(ii) If $\emptyset \neq B \subset A$, $B \in \mathcal{L}(\mathbb{R})$ then $\int_B f = \int_A f\chi_B$

(iii) If $\phi \in S^+(A)$ then $I_A(\phi) = \int_A \phi$

Problem 4.1. If $\{f_n\}_{n=1}^{\infty} \subset (\mathcal{M}^*)^+(A)$ and $f_n \to f$ pointwise, then $f \in (\mathcal{M}^*)^+(A)$. Can we have $\lim_{n\to\infty} \int_A f_n = \int_A f$? The answer is unfortunately no. We do have some theorems that allow convergence.

4.3 Monotone Convergence Theorem

Theorem 4.1. *(Monotone Convergence Theorem (MCT)) Let $A \in \mathcal{L}(\mathbb{R})\backslash\{\emptyset\}$ and $\{f_n\}_{n=1}^{\infty} \subset (\mathcal{M}^*)^+(A)$. Suppose that*

$$0 \leq f_1 \leq ... \leq f_n < ...$$

and

$$f = \lim_{n\to\infty} f_n$$

(pointwise). Then $f \in (\mathcal{M}^*)^+(A)$ with

$$\int_A f = \lim_{n\to\infty} \int_A f_n \in [0, \infty]$$

Lemma 4.1. *(Continuity of λ) If $A_1 \subset A_2 \subset A_3 \subset ... \in \mathcal{L}(\mathbb{R})$ then*

$$\lambda\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \lambda(A_n)$$

Corollary 4.1. *If $\sup_{n\in\mathbb{N}} \int_A f_n < \infty$ then $\int_A f < \infty$.*

Lemma 4.2. *Let $f : A \mapsto [0, \infty]$ where $A \in \mathcal{L}(\mathbb{R})\backslash\,\{\emptyset\}$. Then $f \in (\mathcal{M}^*)^+(A)$ if and only if there is a sequence $\{\phi_n\}_{n=1}^{\infty} \subset S^+(A)$ such that*

$$\lim_{n\to\infty} \phi_n = f$$

Moreover, we can choose $\phi_1 \leq \phi_2 \leq ... \leq f$ pointwise.

Corollary 4.2. *Let $A \in \mathcal{L}(\mathbb{R})\backslash\{\emptyset\}$. Then we have*

(i) If $f, g \in (\mathcal{M}^*)^+(A)$, $c \geq 0$ then

$$\int_A cf = c\int f \text{ and } \int_A (f + g) = \int_A f + \int_A g$$

(ii) If $\{f_n\}_{n=1}^{\infty} \subset (\mathcal{M}^*)^+(A)$ then

$$\int_A \sum_{i=1}^{\infty} f_i = \sum_{i=1}^{\infty} \int_A f_i$$

(iii) If $A_1, A_2, ... \subseteq A$ are measurable disjoint sets such that $\bigsqcup_{n=1}^{\infty} A_n = A$ and

$$\int_A f = \sum_{i=1}^{\infty} \int_{A_i} f$$

where $f \in (\mathcal{M}^*)^+(A)$.

Notation 4. Let $f \in \mathcal{M}^*(A) = \{f : A \to \mathbb{R}^* = [-\infty, \infty] : f \text{ is measurable}\}$ where $A \in \mathcal{L}(\mathbb{R}) \backslash \{\emptyset\}$. We have

$$\begin{aligned} f^+ &= \max\{f, 0\} \geq 0 \\ f^- &= \max\{-f, 0\} = -\min\{f, 0\} \geq 0 \end{aligned}$$

and $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

Definition 4.5. Let $A \in \mathcal{L}(\mathbb{R}) \backslash \{\emptyset\}$. We say $f : A \mapsto \mathbb{R}^*$ is (Lebesgue) integrable if $f \in \mathcal{M}^*(A)$ and $\left|\int_A f^+ - \int_A f^-\right| < \infty$. In this case, we define the (Lebesgue) integral of f as

$$\int_A f = \int_A f^+ - \int_A f^- \in \mathbb{R}$$

We define the set of $\mathbb{R}^*-$valued integrable functions by $L^*(A)$.

Lemma 4.3. *(i) $f \in L^*(A)$ implies $\lambda(f^{-1}(\{\pm\infty\})) = 0$.*

(ii) If $f \in \mathcal{M}^*(A)$ then $\int_A |f| = 0$ if and only if

$$\lambda\left(\{x \in A | f(x) \neq 0\}\right) = \lambda\left(f^{-1}([-\infty, 0)) \cup f^{-1}((0, \infty])\right) = 0$$

Definition 4.6. If $f, g \in \mathcal{M}^*(A)$ we say f and g are equal almost everywhere (a.e.) on A, written as $f = g$ a.e. (on A) if

$$\lambda\left(\{x \in A : f(x) \neq g(x)\}\right) = 0$$

Corollary 4.3. *(of Lemma (ii)) If $f, g \in \mathcal{M}^*(A)$ such that $f = g$ a.e. on A then*

$$\int_A |f - g| = 0$$

whenever $f - g$ makes sense.

Notation 5. Let

$$\begin{aligned} L(A) &= \{f \in L^*(A) : f \text{ is real valued}\} \\ &= \{f : A \mapsto \mathbb{R} : f \text{ measurable and integrable}\} \end{aligned}$$

Corollary 4.4. *(of Lemma (i)) If $f \in L^*(A)$, there is $f_0 \in L(A)$ such that $f = f_0$ a.e. on A. So,*

$$\int_A |f - f_0| = 0$$

The proof is done by considering

$$f_0(x) = \begin{cases} f(x) & f(x) \in \mathbb{R} \\ 0 & \textit{otherwise} \end{cases}$$

Theorem 4.2. *If $f, g \in L(A)$ and $c \in \mathbb{R}$, then*

(i) $cf \in L(A)$ and $\int_A cf = c \int_A f$

(ii) $f + g \in L(A)$ and $\int_A (f + g) = \int_A f + \int_A g$ (*)

(iii) $|f| \in L(A)$ and $\left|\int_A f\right| \leq \int_A |f|$

In fact, $f \in L(A) \iff f$ is measurable and $|f|$ is integrable.

Example 4.1. Let $E \in \mathcal{P}(\mathbb{R}) \backslash \mathcal{L}(\mathbb{R})$ bounded, say $E \subset (a, b)$. Define $f = \chi_{((a,b)\backslash E)} - \chi_E$ and clearly f is not measurable. However, $|f| = \chi_{((a,b))}$ is measurable and integrable.

Lemma 4.4. *(Fatou's Lemma) If $\{f_n\}_{n \in \mathbb{N}}$ is a sequence in $(\mathcal{M}^*)^+(A)$ then*

$$\int_A \liminf_{n \in \mathbb{N}} f_n \leq \liminf_{n \in \mathbb{N}} \int_A f_n$$

Definition 4.7. A sequence of $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}^*(A)$, $f_n : A \mapsto \mathbb{R}^*$ is said to converge to $f : A \mapsto \mathbb{R}^* \in \mathcal{M}^*(A)$ almost everywhere (on A), written $f_n \to f$ a.e. (on A) if

$$\lambda(\underbrace{\{x \in A : \lim_{n\to\infty} f_n(x) \neq f(x)\}}_{N}) = 0$$

Exercise. Why is $N \in \mathcal{L}(\mathbb{R})$?

Note 7. (1) If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{M}^*(A)$, $f = \lim_{n\to\infty} f_n$ a.e. on A then f is measurable on A. (Proof as an exercise)

(2) The MCT and Fatou's Lemma remain valid if pointwise convergence is replaced by a.e. convergence.

(3) Pointwise convergence $\implies$ a.e. convergence but the converge may fail.

(4) If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{M}(A)$ and $f = \lim_{n\to\infty} f_n \in \mathcal{M}^*(A)$. Furthermore, suppose that f is integrable ($f \in L^*(A)$). Then we replace f by $f_0 : A \mapsto \mathbb{R}$ such that $f = f_0$ a.e. on A. Then $f_0 \in L(A)$ and $f_n \to f_0$ a.e. on A.

4.4 Lebesgue Dominated Convergence Theorem

Theorem 4.3. *(Lebesgue Dominated Convergence Theorem (LDCT)): If $\{f_n\}_{n=1}^{\infty} \subset L(A)$, $f : A \mapsto \mathbb{R}$ and $g \in L^+(A)$ are such that*

(i) $f = \lim_{n\to\infty} f_n$ pointwise a.e. on A

(ii) $|f_n| \leq g$ a.e. on A for all $n \in \mathbb{N}$ (g is called an integrable majorant for $\{f_n\}_{n\in\mathbb{N}}$)

Then $f \in L(A)$. That is, f is measurable and integrable with

$$\int_A f = \lim_{n\to\infty} \int_A f_n$$

Example 4.2. (Of necessary of existence of integrable majorant in LDCT) Let

$$f_n(x) = \begin{cases} n & x \in (0, \frac{1}{n}] \\ 0 & x \in (\frac{1}{n}, 1] \end{cases}, A = [0, 1]$$

Then if g is an integrable majorant of f_n we have for any m,

$$\int_A g \geq \int_{[\frac{1}{m},1]} g = \sum_{n=1}^{m-1} \int_{(\frac{1}{n+1},\frac{1}{n}]} g \geq \sum_{n=1}^{m-1} \int_{(\frac{1}{n+1},\frac{1}{n}]} n = \sum_{n=1}^{m-1} \frac{1}{n+1}$$

and taking $n \to \infty$, this is the harmonic series and g cannot be integrable. Remark that $\int_0^1 \liminf f_n = 0$ and $\lim_{n\to\infty} \int_A f_n = \lim_{n\to\infty} 1 = 1$.

5 L_p-Spaces

Let $A \in \mathcal{L}(\mathbb{R})\backslash\{\emptyset\}$ (usually $A = \mathbb{R}$ or $A = [a, b]$). Here are the cases for different values of p.

Summary 1. p=1: The space $L_1(A)$.

For $f \in L(A)$, define $\|f\|_1 = \int_A |f| \in \mathbb{R}^{\geq 0}$ and $\|\cdot\|_1 : L(A) \mapsto [0, \infty)$ is a seminorm, that is for any $f, g \in L(A)$, $c \in \mathbb{R}$,

(i) $\|cf\|_1 = |c|\|f\|_1$ (homogeneity)

(ii) $\|f+g\|_1 \leq \|f\|_1 + \|g\|_1$ (subadditivity)

The proof of this is straightforward. Note that we are lacking non-degeneracy. We say earlier that $\|f\|_1 = \int_A |f| = 0 \iff f = 0$ a.e. on A.

Remark 5.1. On $L(A)$ we define an equivalence relation $\sim$ as

$$f \sim g \iff f = g \text{ a.e. on } A \iff \|f-g\|_1 = 0$$

(proving that $\sim$ is an equivalence relation will be left as an exercise) We put $L_1(A) = L(A)/\sim$ and will think of $L_1(A)$ as the space of integrable functions and agree that $f = g$ in $L_1(A) \iff f = g$ a.e. on A. So $\|\cdot\|_1$ is a norm on $L_1(A)$.

Note 8. Since $\{x\}$ is a null set for $x \in A$, the value of '$f(x)$' is meaningless. That is, we lose the notion of pointwise convergence.

Fact 5.1. *(Convergence in $(L_1(A), \|\cdot\|_1)$)*

1) If $\{f_n\}_{n=1}^{\infty} \subset L_1(A)$ and $f \in L_1(A)$ such that $\lim_{n\to\infty} f_n = f$ a.e. on A and there is $g \in L_1^+(A)$ such that $|f_n| \leq g$ then we can conclude that $\lim_{n\to\infty} \|f_n - f\|_1 = 0$.

2) If $\{f_n\}_{n=1}^{\infty} \subset L_1^+(A)$ and $f \in L_1^+(A)$ such that $\lim_{n\to\infty} f_n = f$ a.e. and $f_1 \leq f_2 \leq \ldots$, then by the MCT we get

$$\lim_{n\to\infty} \|f_n - f\|_1 = 0$$

3) In general, a.e. convergence or pointwise convergence does not imply convergence w.r.t (with respect to) $\|\cdot\|_1$.

4) Can convergence w.r.t. $\|\cdot\|_1 \implies$ a.e. convergence or pointwise convergence? (Ans: No)

5.1 $0 < p < 1$: The Spaces $L_p(A)$

Definition 5.1. Let $0 < p < \infty$ and define the conjugate to p as the number q such that $\frac{1}{p} + \frac{1}{q} = 1 \implies q = \frac{p}{1-p}$. Note that if $p = 1$ then $q = +\infty$ and if $p = +\infty$ we put $q = 1$.

Definition 5.2. Let $1 \leq p < \infty$ and $f \in \mathcal{M}(A)$. Define $\|f\|_p = \left(\int_A |f|^p\right)^{\frac{1}{p}}$.

Definition 5.3. Let $1 \leq p < \infty$ and $\sim$ denote the almost everywhere equivalence relation. Define

$$L_p(A) = \{f \in \mathcal{M}(A) : |f|^p \in L(A)\}/\sim$$

Hence we think of $L_p(A)$ as the space of p-integrable functions on A and agree that

$$f = g \text{ in } L_p(A) \iff f = g \text{ a.e. on } A$$

We want to show that $\|\cdot\|_p : L_p(A) \mapsto [0,\infty)$ is a norm on $L_p(A)$.

Lemma 5.1. *If $1 < p < \infty$ and q is the conjugate to p. Suppose that $a, b \in [0,\infty)$. Then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

and equality holds if $a^p = b^q$.

5.2 Norm Inequalities

Proposition 5.1. *(Hölder's Inequality) If $f \in L_p(A)$ and $g \in L_q(A)$ where $1 < p < \infty$ and q is conjugate to p then fg is integrable and*

$$\|fg\|_1 = \int_A |fg| \leq \|f\|_p \|g\|_q$$

(that is, $fg \in L_1(A)$). Moreover, equality holds when

$$\|g\|_q^q|f|^p = \|f\|_p^p|g|^q \text{ a.e. on } A$$

Proposition 5.2. *(Minkowski's Inequality) If $1 < p < \infty$, $f, g \in L_p(A)$ ($A \in \mathcal{L}(\mathbb{R})\backslash\{\emptyset\}$) then $f + g \in L_p(A)$ and*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Moreover, the equality will hold only if there are $c_1c_2 \geq 0$, $c_1, c_2 \neq 0$ such that $c_1 f = c_2 g$ a.e. on A.

Corollary 5.1. *$\|\cdot\|_p$ is a norm on $L_p(A)$ where $1 < p < \infty$.*

Goal. For $A \in \mathcal{L}(\mathbb{R})$ and $\lambda(A) > 0$ we want to show that $(L_p(A), \|\cdot\|_p)$ is a Banach space (complete normed linear space) where $1 \leq p < \infty$.

5.3 Completeness

Lemma 5.2. *Let $(X, \|\cdot\|)$ be a normed vector space. Then X is complete w.r.t. $\|\cdot\| \iff$ for every sequence $\{x_n\}_{n=1}^\infty \subset X$ with $\sum_{n=1}^\infty \|x_z\| < \infty$ we have $\sum_{n=1}^\infty x_n = \lim_{n\to\infty} \sum_{k=1}^n x_n$ converges.*

Theorem 5.1. *Let $A \in \mathcal{L}(\mathbb{R})$ and $\lambda(A) > 0$. Then $(L_p(A), \|\cdot\|_p)$ is a complete space where $1 \leq p < \infty$.*

Corollary 5.2. *$A \in \mathcal{L}(\mathbb{R})$ with $\lambda(A) > 0$ and $1 \leq p \leq \infty$, $(L_p(A), \|\cdot\|_p)$ is a Banach space.*

5.4 The Space $L_\infty(A)$

Definition 5.4. If $f \in \mathcal{M}(A)$, let $\|f\|_\infty = \text{ess sup}_{x\in A}|f(x)| = \inf\{c > 0, \lambda(\{x \in A : |f(x)| > c\}) = 0\}$ where we call each c an essential upper bound for f.

Let $L_\infty(A) = \{f \in \mathcal{M}(A) : \|f\|_\infty < \infty\}$ where $\sim$ is the a.e. equivalence relation. Hence, $L_\infty(A)$ is the space of "essentially bounded functions" on A where $f = g$ in $L_\infty(A)$ iff $f = g$ a.e. on A.

Proposition 5.3. *$\|\cdot\|_\infty$ is a norm on $L_\infty(A)$. That is, for $f, g \in L_\infty(A)$ and $c \in \mathbb{R}$ we have*

(i) $\|f\|_\infty \geq 0$ *and* $\|f\|_\infty = 0 \iff f = 0$ *in* $L_\infty(A)$

(ii) $\|cf\|_\infty = |c|\|f\|_\infty$

(iii) $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$

Theorem 5.2. *$(L_\infty(A), \|\cdot\|_\infty)$ is complete and hence a Banach space.*

Remark 5.2. If $0 < p < 1$, the $\triangle \leq$ fails. (Exercise)

5.5 Containment Relations

We will consider $A = [a, b]$, $\lambda(a) < \infty$ and then $A = \mathbb{R}$ or $(0, \infty)$ where $\lambda(A) = \infty$. First, suppose that $A = [a, b]$, $a < b$, and let $1 \leq p < r < \infty$.

Theorem 5.3. *$L_r([a, b]) \subset L_p([a, b])$. Moreover, if $f \in L_r([a, b])$ then $\|f\|_p \leq \|f\|_r(b - a)^{\frac{r-p}{rp}}$.*

Note 9. 1) $L_\infty([a,b]) \subset L_p([a,b])$ for each $1 \leq p < \infty$. (Exercise)

2) If $\phi \in S([a,b])$ then $\lim_{p\to\infty} \|\phi\|_p = \|\phi\|_\infty$.

3) $\overline{S([a,b])} = L_\infty([a,b])$.

4) $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$ for and $f \in L_\infty([a,b])$.

Remark 5.3. $1 \leq p < r < \infty$ do we have $L_p([a,b]) \subset L_r([a,b])$? The answer is no! Let $A = [0,1]$. Then for any $1 \leq p < \infty$ consider $f(x) = \frac{1}{x^{1/r}}$ for a.e. $x \in [0,1]$. Since $\frac{p}{r} < 1$, $\int_{[0,1]} |f|^p = \underbrace{\int_0^1 x^{-p/r}\, dx}_{A3} = \frac{r}{r-p}$ while $\int_{[0,1]} |f|^r = \int_0^1 \frac{1}{x} = \infty$. So $L_p([0,1]) \not\subseteq L_r([0,1])$.

Exercise 5.1. $L_\infty([a,b]) \subset L_p([a,b])$ [ON THE MIDTERM]

Remark 5.4. If $A = \mathbb{R}$ or $[0,\infty)$ we ask what happens when $1 \leq r < p < \infty$.

Is $L_p(A) \subset L_r(A)$?

No! Consider the above given function f and define $g(x) = f(x)$ on $[0,1]$ and 0 elsewhere. Then $\int_A |g|^k = \int_A |f|^k$ if $k = p, r$

Is $L_r(A) \subset L_p(A)$?

No! Consider $h(x) = \min\left\{1, \frac{1}{x^{1/p}}\right\}$ to prove that $L_r([0,\infty)) \not\subseteq L_p([0,\infty))$. Check the details (Hint: you will need Q4 of A3).

Definition 5.5. A Banach space $(X, \|\cdot\|)$ is called separable if there is a countable subset $\{d_n\}_{n=1}^\infty$ which is dense (w.r.t. $\|\cdot\|$) in X. That is, given $x \in X$, $\epsilon > 0$, there is $n \in \mathbb{N}$ such that $\|x - d_n\| < \epsilon$.

Theorem 5.4. *If $A = [a,b]$ is a bounded interval and $1 \leq p < \infty$ then $L_p([a,b])$ is separable.*

For $1 \leq p < \infty$, $L_p(\mathbb{R})$ is separable.

Theorem 5.5. *$L_\infty([0,1])$ is not separable.*

5.6 Functional Analytic Properties of L_p-Spaces

Recall that for $1 \leq p \leq \infty$, $L_p(A)$ is a Banach space.

Definition 5.6. Let X, Y be Banach spaces. A linear map $T : X \mapsto Y$ is *bounded* if the operator norm $\|\cdot\|$ of T, defined by

$$|||T||| = \sup\{\|T(x)\| : x \in X, \|x\| < 1\}$$

is finite ($< \infty$). If $Y = \mathbb{R}$ we call $f : X \mapsto \mathbb{R}$ a linear functional. Define

$$|||f||| = \|f\|_*$$

Proposition 5.4. *Let X, Y be Banach spaces and $T : X \mapsto Y$ linear. Then TFAE*

i) T is continuous

ii) T is bounded

iii) T is Lipschitz, with Lipschitz constant $|||T|||$

Aside. We say that a function $T : X \mapsto Y$ is Lipschitz if there is some constant $L > 0$ such that $\|T(x) - T(x')\| \leq L\|x - x'\|$ for $x, x' \in X$.

Theorem 5.6. *Let $A = [a, b]$ or $A = \mathbb{R}$ and $1 < p < \infty$. Let q be the conjugate of p. If $g \in L_q(A)$ then the map $\tau_g : L_p(A) \mapsto \mathbb{R}$ given by $f \mapsto \int_A fg$ is a bounded linear map (bounded functional) on $L_p(A)$ with norm $\|\tau_g\| = \|g\|_q$.*

Fact 5.2. *Any linear functional $\tau : L_p(A) \mapsto \mathbb{R}$ is of the form $\tau_g = \tau$ for some $f \in L_p(A)$. (PMATH 454)*

Theorem 5.7. *Let $A \in \mathcal{L}(\mathbb{R})$ be s.t. $0 < \lambda(A) < \infty$. Let ϕ. Define $\Gamma_\phi : L_1(A) \mapsto \mathbb{R}$ by $\Gamma_\phi(f) = \int_A f \cdot \phi$. Then Γ_ϕ is a bounded linear functional with $\|\Gamma_\phi\| = \|\phi\|_\infty$.*

Let $1 \le p < \infty$ and $A \in \mathcal{L}(\mathbb{R})$ with $\lambda(A) < \infty$. Let $\phi \in L_\infty(A)$. Define $M_\phi : L_p(A) \mapsto L_p(A)$ by $f \mapsto \phi \cdot f$. Then M_ϕ is a linear operator with $\|M_\phi\| = \|\phi\|_\infty$.

Theorem 5.8. *Let $a < b$ in $\mathbb{R}$. Then,*

(a) If $f \in L_1([a, b])$ then the functional $\Gamma_f : L_\infty([a, b]) \mapsto \mathbb{R}$ given by $\Gamma_f(\phi) = \int_{[a,b]} f \cdot \phi$ is linear and bounded with $\|\Gamma_f\| = \|f\|_1$.

(b) Furthermore we consider $\Gamma_f : \mathcal{C}([a, b]) \mapsto \mathbb{R}$. Then

$$\|\Gamma_f\| = \sup \{|\Gamma_f(h)| : h \in \mathcal{C}([a, b]), \|h\|_\infty \le 1\} = \|f\|_1$$

6 Fourier Analysis

Definition 6.1. A function on $A \in \mathcal{L}(\mathbb{R})$, $f : A \mapsto \mathbb{C}$ is said to be measurable if $\Im(f), \Re(f) : A \mapsto \mathbb{R}$ are both measurable. Furthermore, we say $f : A \mapsto \mathbb{C}$ is integrable if both $\Re(f)$ and $\Im(f)$ are integrable. In this case, we define

$$\int_A f = \int_A \Re(f) + i \int_A \Im(f)$$

Fact 6.1. *1) Let $A \in \mathcal{L}(\mathbb{R})$. Then*

$$\mathcal{M}_{\mathbb{C}}(A) = \{f : A \mapsto \mathbb{C} : f \text{ measurable}\} \supset \mathcal{M}(A)$$

is an algebra of functions w.r.t. pointwise operations.

2) MCT and Fatou's Lemma require the order structure of $\mathbb{R}$ and hence they are theorems about $\mathbb{R}-$valued functions. Still they may be applied to real and imaginary parts of $\mathbb{C}-$valued functions.

3) LDCT works for $\mathbb{C}-$valued functions but we need a proof without Fatou's Lemma (Exercise) [i.e. $f_n \mapsto f$ a.e. on A and $\underbrace{|f_n|}_{\mathbb{C}-\text{modulus}} \le g$ a.e. on A, $g \in L(A)$ then $\int_A f_n \to \int_A f$)

Remark 6.1. Furthermore, Hölder's and Minkwoski's Theorems are valid for $\mathbb{C}-$valued functions. To see this, consider $A = [a, b]$ a compact interval in $\mathbb{R}$ ($a < b$). Define

$$\mathcal{C}([a, b]) = \{f : [a, b] \mapsto \mathbb{C} : f \text{ is cts}\}$$

equipped with the uniform/infinity norm. For $1 \le p < \infty$, define

$$L_p([a, b]) = \{f : [a, b] \mapsto \mathbb{C} : f \text{ is measurable and } |f|^p \text{ is integrable}\}/\sim$$

$$L_\infty([a, b]) = \{f : [a, b] \mapsto \mathbb{C} : f \text{ is measurable and } |f| \text{ is essentially boune}\}/\sim$$

equipped with the $\|\cdot\|_p$ norm for $1 \le p \le \infty$.

Definition 6.2. A function $f : \mathbb{R} \mapsto \mathbb{C}$ is called $\theta-$periodic ($\theta \in \mathbb{R}$) if

$$f(t + \theta) = f(t), \text{ a.e. for } t \in \mathbb{R}$$

We make the following remarks with regards to this definition.

- Notice that if we define $e^n : \mathbb{R} \mapsto \mathbb{T}$ by $t \mapsto e^{i(nt)}$ with $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ then for each $n \in \mathbb{N}$, e^n is 2π periodic.

- If $f : \mathbb{R} \mapsto \mathbb{C}$ is 2π periodic, then so are $\Re(f)$ and $\Im(f)$
- Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Then the map $\mathbb{R} \mapsto \mathbb{T}$ defined by $t \mapsto e^{it}$ carries $\mathbb{R}$ onto $\mathbb{T}$. So we let

$$\begin{aligned} \mathcal{C}(\mathbb{T}) &= \{f : \mathbb{R} \mapsto \mathbb{C} : f \text{ is cts and } 2\pi\text{periodic}\} \\ &\approxeq \{f \in \mathcal{C}([-\pi,\pi]) : f(-\pi) = f(\pi)\} \end{aligned}$$

and for $1 \le p \le \infty$,

$$L_p(\mathbb{T}) = \left\{ f : \mathbb{R} \mapsto \mathbb{C} : f \text{is } 2\pi\text{periodic and } f\Big|_{[-\pi,\pi]} \in L_p([-\pi,\pi]) \right\}$$

- Note that $f \in L_p(\mathbb{T}) \not\Rightarrow f$ is integrable on $\mathbb{R}$ with $f\Big|_{[-\pi,\pi]} \in L_p([-\pi,\pi])$ meaning $\int_{[-\pi,\pi]} |f|^p < \infty$. In fact, $\int_{\mathbb{R}} |f|^p$ is ∞ if $f \neq 0$ as an element of L_p.
- If $1 \le p < \infty$ we equip $L_p(\mathbb{T})$ with the norm

$$\|f\|_p = \left(\frac{1}{2\pi} \int_{[-\pi,\pi]} |f|^p \right)^{1/p}$$

- If $p = \infty$ we equip $L_\infty(\mathbb{T})$ with $\|f\|_\infty = \text{ess sup}_{t \in [-\pi,\pi]} |f(t)|$. Note that

$$L_1(\mathbb{T}) \supset L_p(\mathbb{T}) \supset L_\infty(\mathbb{T}) \supset \mathcal{C}(\mathbb{T}), 1 < p < \infty$$

Problem 6.1. Given a 2π periodic function $f \in L(\mathbb{T})$ we want to represent this function as a Fourier series. That is, we want to find $\{c_n\}_{n\in\mathbb{Z}}$ such that

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

for a.e. $t \in [-\pi,\pi]$. If we allow interchanging of the sum and the integral (ignoring questions of convergence) we observe that for any $k \in \mathbb{Z}$,

$$\underbrace{\int_{[-\pi,\pi]}}_{\text{Lebesgue Integral}} f(t)e^{-ikt}dt = \sum_{n=-\infty}^{\infty} \int_{[-\pi,\pi]} e^{int}e^{-ikt}dt = \sum_{n=-\infty}^{\infty} \int_{[-\pi,\pi]} \underbrace{e^{i(n-k)t}}_{\text{cts fn}} dt$$

By Assignment 3, Question 3, Riemann integrals imply that

$$\int_{[-\pi,\pi]} e^{i(n-k)t}dt = \int_{[-\pi,\pi]} \cos((n-k)t)dt + i\int_{[-\pi,\pi]} \sin((n-k)t)dt = \begin{cases} 2\pi & n = k \\ 0 & n \neq k \end{cases}$$

Therefore, $\int_{[-\pi,\pi]} f(t)e^{-ikt}dt = 2\pi c_k$ for any $k \in \mathbb{Z}$.

Definition 6.3. If $f \in L(\mathbb{T})$ and $k \in \mathbb{Z}$ the k^{th} Fourier coefficient of f is given by

$$c_k(f) = \frac{1}{2\pi} \int_{[-\pi,\pi]} f(t)e^{-ikt}dt = \frac{1}{2\pi} \int_{[-\pi,\pi]} fe^{-k}$$

with the exponential function $e^k(t)$ as $t \mapsto e^{-ikt}$. Note that if $f = g$ a.e. on $[-\pi,\pi]$ then $fe^{-k} = ge^{-k}$. That is, c_k is well-defined on $L_1(\mathbb{T})$.

Goal. Let's restate our goal: Let $f \in L(\mathbb{T})$ or $L_p(\mathbb{T})$ or $C(\mathbb{T})$. Then does the following hold?

$$f = \sum_{n=-\infty}^{\infty} c_n(f)e^n = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n(f)e^n$$

Pointwise? A.e. ? In L_1? In L_p? Uniformly?

6.1 The Fourier Approximation

Definition 6.4. (Fourier Approximation) For $f \in L(\mathbb{T})$ define

$$S_n(f) = \sum_{k=-n}^{n} c_k(f)e^k, S_n(f,t) = S_n(f)(t) = \sum_{k=-n}^{n} c_k(f)e^{ikt}$$

where $S_n(f)$ is a continuous 2π periodic function.

Remark 6.2. We observe that

$$\begin{aligned} S_n(f,t) = \sum_{k=-n}^{n} c_k(f)e^{ikt} &= \sum_{k=-n}^{n} \left(\frac{1}{2\pi}\int_{[-\pi,\pi]} f(s)e^{-iks}ds\right)e^{ikt} \\ &= \frac{1}{2\pi}\int_{[-\pi,\pi]} f(s)\sum_{k=-n}^{n} e^{ik(t-s)}ds \end{aligned}$$

and let $D_n = \sum_{k=-n}^{n} e^k \implies D_n(x) = \sum_{k=-n}^{n} e^{ikx}$ which we call the *Dirichlet kernel of order* n. Then,

$$S_n(f,t) = \frac{1}{2\pi}\int_{[-\pi,\pi]} f(s)\sum_{k=-n}^{n} e^{ik(t-s)}ds = \frac{1}{2\pi}\int_{[-\pi,\pi]} f(s)D_n(t-s)ds$$

and setting $\sigma = s - t$ gives us, by translation invariance,

$$\begin{aligned} S_n(f,t) &= \frac{1}{2\pi}\int_{[-\pi-t,\pi-t]} f(\sigma+t)D_n(-\sigma)d\sigma \\ &= \frac{1}{2\pi}\int_{[-\pi,\pi]} f(\sigma+t)D_n(-\sigma)d\sigma \\ &= \frac{1}{2\pi}\int_{[-\pi,\pi]} f(t-s)D_n(s)ds, s=-\sigma \\ &:= D_n * f(t) \end{aligned}$$

which we will call the *convolution* of D_n with f. That is to study the behaviour of $S_n(f)$ we need to study the behaviour of D_n. Remark that inversion invariance follows from the symmetry of the domain.

We will first study the notion of "convolution" in a more rigourous and theoretical way.

6.2 Convolution

Definition 6.5. A *homogeneous Banach space* over $\mathbb{T}$ is a Banach space $B \subset L_1(\mathbb{T})$ which is equipped with its own norm $\|\cdot\|_B$ (Note that $(B, \|\cdot\|)$ is a Banach space) if the following conditions hold

1. $\text{span}\{e^k\}_{k=-\infty}^{\infty} \subset B$ where we denote $\text{span}\{e^k\}_{k=-\infty}^{\infty} = \text{Trig}(\mathbb{T})$ with elements called the *trigonometric polynomials.*
2. If $s \in \mathbb{R}$, $f \in B$ then $s * f \in B$ where $s * f(t) = f(t-s)$
3. $\|\cdot\|_B$ satisfies:
   (a) $\|s * f\|_B = \|f\|_B$ for all $s \in \mathbb{R}$, $f \in B$
   (b) The mapping $\mathbb{R} \mapsto (B, \|\cdot\|_B)$ given by $s \mapsto s * f$ is continuous for any $f \in B$

Example 6.1. $(\mathcal{C}(\mathbb{T}), \|\cdot\|_\infty)$ is a homogeneous Banach space over $\mathbb{T}$.

Example 6.2. For $1 \le p < \infty$, $L_p(\mathbb{T})$ is a homogeneous Banach space over T.

Example 6.3. $(L_\infty(\mathbb{T}), \|\cdot\|_\infty)$ is NOT a homogeneous Banach space over $\mathbb{T}$.

Remark 6.3. Let $B \subset L_1(\mathbb{T})$ be a homogeneous Banach space over $\mathbb{T}$. Let $h \in \mathcal{C}(\mathbb{T})$, $f \in B$. Define the convolution of h and f as

$$h * f = \frac{1}{2\pi} \int_{[-\pi,\pi]} \underbrace{h(s)}_{\in \mathbb{C}} \underbrace{(s * f)}_{t \mapsto f(t-s)} \, ds$$

which is a vector valued Riemann integral. If we put $F(s) = \frac{1}{2\pi} h(s)(s * f)$ which is a function $\mathbb{R} \mapsto L(\mathbb{T})$. In Assignment 4, we will show:

1) $f \in B \implies F(s) \in B$

2) $F(s)$ is a vector-valued continuous function on $[-\pi, \pi]$

Therefore, $h * f$ is well defined and we have for a.e. $t \in \mathbb{R}$,

$$\begin{aligned} h * f(t) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} h(s) f(t-s) \, ds \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} h(s+t) f(-s) \, ds \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} h(t-s) f(s) \, ds \end{aligned}$$

by translation invariance and inversion invariance. For any $h \in \mathcal{C}(\mathbb{T})$ we can define

$$\begin{aligned} C(h): \quad & B \mapsto B \\ & f \mapsto h * f \end{aligned}$$

that is $C(h)_f = h * f$ for all $f \in B$.

Proposition 6.1. *If $h \in \mathcal{C}(\mathbb{T})$ and $C(h) : B \mapsto B$ denotes the convolution operator, then $C(h)$ is a bounded linear operator with*

$$|||C(h)|||_B \leq \|h\|_1$$

Note 10. We will see that if $B = L_1(\mathbb{T})$ or $\mathcal{C}(\mathbb{T})$ then $|||C(h)|||_B = \|h\|_1$, but it can be smaller in general.

Theorem 6.1. *Let $h \in \mathcal{C}(\mathbb{T})$ then*

(i) $|||C(h)|||_{\mathcal{C}(\mathbb{T})} = \|h\|_1$

(ii) $|||C(h)|||_{L_1(\mathbb{T})} = \|h\|_1$

6.3 The Dirichlet Kernel

Theorem 6.2. *(Properties of Dirichlet Kernel)*

The Dirichlet kernel (of order n) satisfies the following properties:

(1) D_n is real-valued, $2\pi-$periodic and even

(2) $\frac{1}{2\pi} \int_{-\pi}^{\pi} D_n = 1$

(3) For $t \in [-\pi, \pi]$, $D_n = \begin{cases} \frac{\sin\left[\left(n+\frac{1}{2}\right)t\right]}{\sin\left[\frac{1}{2}t\right]} & t \neq 0 \\ 2n+1 & t = 0 \end{cases}$

(4) Let $L_n = \|D_n\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n|$ which we call the Lebesgue constant. *Then* $\lim_{n\to\infty} L_n = \lim_{n\to\infty} \|D_n\|_1 = +\infty$

Corollary 6.1. $|||C(D_n)|||_{L_1(\mathbb{T})} = |||D_n|||_1 = L_n \to \infty$ *and* $|||C(D_n)|||_{\mathcal{C}(\mathbb{T})} = |||D_n|||_1 = L_n \to \infty$ *as* $n \to \infty$*. We want to use* $\lim_{n\to\infty} L_n$ *to show that if* $f \in \mathcal{C}(\mathbb{T})$ *then* $S_n(f,t) \nrightarrow f$ *as* $n \to \infty$ *in the uniform sense.*

Theorem 6.3. *(Banach -Steinhaus Theorem) Let* X, Y *be Banach spaces (usually* $Y = X$ *or* $Y = \mathbb{C}$*),* $\mathcal{F}$ *be a family of bounded linear operators from* X *to* Y*. Suppose that* U *is a set of second category in* X *(So* U *is not* 1^{st} *category, i.e.* U *cannot be written as a countable union of nowhere dense sets. Also note that since* X *is a Banach space, then any open subset of* X *is of second category by the Baire category theorem).*

Theorem 6.4. *If for each* $x \in U$ *we have* $\sup\{\|Tx\| : T \in \mathcal{F}\} < \infty$ *where* $T(x) = Tx$ *and* T *is linear, then* $\sup\{|||T||| : T \in \mathcal{F}\} < \infty$*.*

Corollary 6.2. *If* X, Y *are Banach spaces,* $\{T_n\}_{n\in\mathbb{N}}$ *is sequence of bounded linear maps from* X *to* Y *s.t.* $\sup_{n\in\mathbb{N}} |||T_n||| = \infty$*, then there is a non-empty set* $U \subseteq X$ *whose complement is first category s.t.* $\sup_{n\in\mathbb{N}} \|T_n x\| = \infty$ *for any* $x \in U$*.*

Note 11. If $F_1, F_2, ...$ are sets of first category, then $\bigcup_{n=1}^{\infty} F_n$ is also first category. Hence, if $U_1, U_2, ...$ are sets whose complements are of first category then $\bigcap_{n=1}^{\infty} U_n$ is also of second category.

Theorem 6.5. *Consider* $\{C(D_n)\}_{n\in\mathbb{N}}$*. We have the following results.*

1) There is a set $U \subset L_1(\mathbb{T})$ *whose complement is of first category such that* $\sup_{n\in\mathbb{N}} \|S_n(f)\|_1 = \infty$ *for any* $f \in U$*.*

2) There is $U \subset \mathcal{C}(\mathbb{T})$ *whose complement is of first category such that* $\sup_{n\in\mathbb{N}} \|S_n(f)\|_\infty = \infty$ *for* $f \in U$*.*

In light of the above theorem, there are two ways we can proceed:

- (An idea due to Fejer) We can average te Fourier series
- (Dini's Theorem) We can look at specific functions where convergence holds

6.4 Averaging Fourier Series

Definition 6.6. If X is a vector space and $x = \{x_n\}_{n=1}^{\infty} \subseteq X$ we let the n^{th} Cesaro mean (average) of X be defined by

$$\sigma_n(x) = \frac{x_1 + ... + x_n}{n}$$

Proposition 6.2. *If* X *is a normed vector space and* $x = {x_n}_{n=1}^{\infty}$ *is sequence converging to* $x_0 \in X$ *then the sequence of Cesaro means* $\{\sigma_n(X)\}_{n=1}^{\infty}$ *converges to* x_0 *too.*

Definition 6.7. If $f \in L(\mathbb{T})$ we define

$$\sigma_n(f) = \frac{1}{n+1}\sum_{j=0}^{n} S_j(f) = \frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=-j}^{j} c_k(f)e^k$$

called the n^{th} *Cesaro mean* of f. Note that

$$\begin{aligned}\sigma_n(f) &= \frac{1}{n+1}\left(S_0(f) + ... + S_n(f)\right)\\ &= \frac{1}{n+1}\left(D_0 * f + ... + D_n * f\right) = \left(\frac{1}{n+1}\sum_{j=0}^{n} D_j\right) * f\end{aligned}$$

Thus, if we let $K_n = \frac{D_0 + ... + D_n}{n+1}$ we have $\sigma_n(f) = K_n * f$ for each $n \in \mathbb{N}$. We call each K_n the n^{th} *Ferjer Kernel*.

Theorem 6.6. *(Properties of the Fejer Kernel) The Ferjer Kernel of order* n*,* K_n *satisfies the following:*

(i) K_n *is real-valued,* 2π*-periodic and even.*

(ii) We have

$$K_n(t) = \begin{cases} \frac{1}{n+1}\left(\frac{\sin\left[\frac{1}{2}(n+1)\right]t}{\sin\left[\frac{1}{2}t\right]}\right)^2 & t \neq 0 \\ n+1 & t = 0 \end{cases}, t \in [-\pi, \pi]$$

(iii) $\|K_n\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |K_n| = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n = 1$

(iv) If $0 < |t| \leq \pi$ *then* $0 \leq K_n(t) \leq \frac{\pi^2}{(n+1)t^2}$

Definition 6.8. A summability kernel is a sequence $\{k_n\}_{n=1}^{\infty}$ of 2π periodic bounded and piecewise continuous functions such that

(i) $\frac{1}{2\pi}\int_{-\pi}^{\pi} k_n = 1$

(ii) $\sup_{n\in\mathbb{N}} \|k_n\|_1 < \infty$

(iii) For any $0 < \delta \leq \pi$ we have $\lim_{n\to\infty}\left(\int_{-\pi}^{-\delta} |k_n| + \int_{\delta}^{\pi} |k_n|\right) = 0$ (as $n \to \infty$, the mass k_n concentrates at 0).

Example 6.4. The Fejer Kernel $\{k_n\}_{n=1}^{\infty}$ is a summability kernel.

The Diriclet Kernel $\{D_n\}_{n=1}^{\infty}$ is a not a summability kernel since (ii) fails. That is, $L_n = \|D_n\|_1 \to \infty$.

Example 6.5. (a) The sequence $\{k_n\}_{n=1}^{\infty} = \left\{n\pi\chi_{\left[-\frac{1}{n},\frac{1}{n}\right]}\right\}_{n=1}^{\infty}$ on $[-\pi,\pi]$, extend 2π periodically to $\mathbb{R}$. Then $\{k_n\}$ is a summability kernel.

(b) Similarly, $\{k_n\}_{n=1}^{\infty} = \left\{2n\pi\chi_{[0,\frac{1}{n}]}\right\}$, extend 2π periodically, is a measurability kernel

Theorem 6.7. *(Abstract Summability Kernel Theorem (ASKT)) Let B be a homogeneous Banach space over $\mathbb{T}$. If $\{k_n\}_{n=1}^{\infty}$ is a summability kernel, then*

$$\lim_{n\to\infty} \|k_n * f - f\|_B = 0$$

for any $f \in B$.

Corollary 6.3. *(1) For $f \in \mathcal{C}(\mathbb{T})$ we have*

$$\lim_{n\to\infty} \|\sigma_n(f) - f\|_\infty = 0$$

That is $\sigma_n(f) \to f$ uniformly as $n \to \infty$.

(2) If $1 \leq p < \infty$, for $f \in L_p(\mathbb{T})$ we have

$$\lim_{n\to\infty} \|\sigma_n(f) - f\|_p = 0$$

Fact 6.2. *Note that $f = g$ a.e. on $[-\pi,\pi] \implies c_n(f) = c_n(g)$ for all $n \in \mathbb{Z}$ in $L(\mathbb{T})$.*

Corollary 6.4. *Suppose that $f, g \in L(\mathbb{T})$ and $c_k(f) = c_k(g)$ for each $k \in \mathbb{Z}$. then $f = g$ a.e. on $[-\pi,\pi]$.*

Problem 6.2. If $f \in L(\mathbb{T})$ and $t \in \mathbb{R}$ (or $t \in [-\pi,\pi]$) then do we have $\sigma_n(f,t) \to f(t)$ pointwise as $n \to \infty$?

Definition 6.9. Consider $f \in L(\mathbb{T})$ (or $f \in L_1(\mathbb{T}) = L(\mathbb{T})/\infty$) and $s \in \mathbb{R}$ (usually $s \in [-\pi,\pi]$). We let

$$w_f(s) = \frac{1}{2}\lim_{h\to0^+}[f(s+h) + f(s-h)]$$

This limit may fail to exist (note that the limit can be $+\infty$ or $-\infty$). If $w_f(s)$ exists, thorugh, we call it the mean value of f at s.

Note 12. If $s \in \mathbb{R}$ is a point of continuity for $f \in L(\mathbb{T})$ then clearly $w_f(s)$ exists and $w_f(s) = f(s)$.

Theorem 6.8. *(Fejer's Theorem) There are two parts:*

(1) If $f \in L(\mathbb{T})$ and $x \in [-\pi, \pi]$ such that $w_f(x)$ exists, then $\lim_{n\to\infty} \sigma_n(f, x) = w_f(x)$. In particular, $\lim_{n\to\infty} \sigma_n(f, x) = f(x)$ if f is continuous at x.

(2) If I is an open interval on which f is continuous then for any closed and bounded subinterval $J$of I we have

$$\lim_{n\to\infty} \sup_{t\in J} |\sigma_n(f, t) - f(t)| = 0$$

that is $\lim_{n\to\infty} \sigma_n(f, t) = f(t)$ uniformly on J.

Corollary 6.5. *Suppose $f \in L(\mathbb{T})$, $x \in [-\pi, \pi]$ and $w_f(x)$ exists. Then if $\lim_{n\to\infty} S_n(f, x)$ exists, we have*

$$\lim_{n\to\infty} S_n(f, x) = w_f(x)$$

Definition 6.10. If $f \in L([a, b])$ a point $x \in (a, b)$ is called a *Lebesgue point* of f if

$$\lim_{h\to 0} \frac{1}{h} \int_0^h \left| \frac{f(x+s) + f(x-s)}{2} - f(x) \right| ds = 0$$

Fact 6.3. *For any $f \in L([a, b])$, it is the case that almost every $x \in (a, b)$ is a Lebesgue point.*

Theorem 6.9. *If $x \in [-\pi, \pi]$ is a Lebesgue point for some $f \in L(\mathbb{T})$ then $w_f(x) = \lim_{n\to\infty} \sigma_n(f, t)$. In particular, for a.e. $x \in [-\pi, \pi]$, $\sigma_n(f, x) \to w_f(x)$ in $\mathbb{C}$.*

In short, given $f \in L(\mathbb{T})$ ($L_1(\mathbb{T})$) f has Fourier series defined as

$$\sum_{-\infty}^{\infty} c_k(f) e^k$$

Note 13. (Abel means and Abel summation) The idea is to consider a series of complex numbers $\sum_{k=0}^{\infty} c_k$ where $c_k \in \mathbb{C}$. We say that such a series is *Abel summable* to $s \in \mathbb{C}$ if for every $0 \le r < 1$ the series

$$A(r) = \sum_{k=0}^{\infty} c_k r^k$$

which we call an *Abel mean* for some r, converges and $\lim_{r\to 1} A(r) = s$. Note that if $\sum_{k=0}^{\infty} c_k$ converges to some s then $A(r) \to s$ as $r \to 1$.

Definition 6.11. We define

$$A_r(f)(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} c_n(f) e^{in\theta}, f \in L(\mathbb{T})$$

We easily see that

$$A_r(f) = \left(\sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} \right) * f = P_r(\theta)$$

which we call the *Poisson Kernel*.

Fact 6.4. *A given series converges $\implies$ Cesero summable $\implies$ Abel summable. However, NONE of the converse statements hold. (cf. Stein & Shakarchi, "Fourier Analysis", Section 2.5.)*

6.5 Fourier Coefficients

Suppose that we are given $f \in L(\mathbb{T})$, $\{c_k(f)\}_{k=-\infty}^{\infty}$ a sequence of $\mathbb{C}$-numbers. We will study the behaviour between the two.

Problem 6.3. Now suppose that we are viven a sequence $\{a_n\}_{n=-\infty}^{\infty}$. Is there a function $f \in L(\mathbb{T})$ such that $f \sim \lim_{n\to\infty} \sum_{k=-n}^{n} a_k e^k$? Or $c_k(f) = a_k$ for each $k \in \mathbb{Z}$? (The answer is: No!)

Lemma 6.1. *If $f \in L_1(\mathbb{T})$ then for all $k \in \mathbb{Z}$, $|c_k(f)| \leq \|f\|_1$.*

Notation 6. Let $c_0(\mathbb{Z})$ denote the Banach space of all sequences (indexed by $\mathbb{Z}$), $\{a_n\}_{n\in\mathbb{Z}}$ such that

$$\lim_{|n|\to\infty} |a_n| = 0$$

(with pointwise operations and norm $\|\{a_k\}_{k\in\mathbb{Z}}\| = \sup_{k\in\mathbb{Z}} |a_k|$)

Theorem 6.10. *(Riemann-Lebesgue Lemma) If $f \in L_1(\mathbb{T})$ then $\lim_{|n|\to\infty} |c_n(f)| = 0$. From our above notation, this theorem says that $\{c_k(f)\}_{k\in\mathbb{Z}} \in c_0(\mathbb{Z})$ for $f \in L_1(\mathbb{T})$.*

Corollary 6.6. *Let $f \in L(\mathbb{T})$. Then,*

1) $\lim_{n\to\infty} \int_{-\pi}^{\pi} f(t)\cos(nt)dt = 0$

2) $\lim_{n\to\infty} \int_{-\pi}^{\pi} f(t)\sin(nt)dt = 0$

Theorem 6.11. *(Open Mapping Theorem) Suppose that X, Y are Banach spaces and $T : X \mapsto Y$ is a bounded linear map. If T is surjective, then T is "open" (i.e. if $U \subset X$ open, then $T(U)$ is open in Y).*

Corollary 6.7. *(Inverse Mapping Theorem) Let X, Y be Banach spaces and $T : X \mapsto Y$ be linear and bounded. If T is bijective then $T^{-1} : Y \mapsto X$ is bounded.*

Corollary 6.8. $A(\mathbb{Z}) \subsetneq c_0(\mathbb{Z})$

6.6 Localization and Dini's Theorem

Recall that in $(L_1(\mathbb{T}), \|\cdot\|_1)$ we have on U (whose complement is of first category) that $\|S_n(f) - f\|_1 \not\to 0$. Before we used averaging to study this. Now, we will consider another method. In particular, we will find elements in $L(\mathbb{T})$ where $S_n(f) \mapsto f$.

If $f \in L(\mathbb{T})$ and $t \in [-\pi, \pi]$ we have

$$\begin{aligned} \sum_{j=-n}^{n} c_j(f)e^{int} &= S_n(f,t) = D_n * f(t) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(s)f(t-s)ds \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{\frac{\sin\left(n+\frac{1}{2}\right)s}{\sin\frac{1}{2}s}}_{\text{even}} f(t-s)ds \end{aligned}$$

and we apply inversion invariance to get

$$\sum_{j=-n}^{n} c_j(f)e^{int} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)s}{\sin\frac{1}{2}s} f(t+s)ds$$

which we will call (*).

Lemma 6.2. *If $f \in L(\mathbb{T})$ with $\int_{-\pi}^{\pi} \left|\frac{f(t)}{t}\right| dt < \infty$ then $\lim_{n\to\infty} S_n(f,0) = 0$.*

Theorem 6.12. *(Localization Principle) If $f \in L(\mathbb{T})$ and I is an open interval in $[-\pi,\pi]$ on which $f(t) = 0$ a.e. $t \in I$, then for any $t \in I$ we have*

$$\lim_{n\to\infty} S_n(f,t) = 0$$

Corollary 6.9. *If $f, g \in L(\mathbb{T})$ and I is an open subinterval in $[-\pi,\pi)$ on which $f(t) = g(t)$ a.e. $t \in I$. Then for any $t \in I$*

$$\lim_{n\to\infty} S_n(f,t) \textit{ exists iff } \lim_{n\to\infty} S_n(g,t) \textit{ exists}$$

and the two limits coincide when they exist.

Theorem 6.13. *(Dini's Theorem for differentiable functions) If $f \in L(\mathbb{T})$ and f is differentiable at $t \in [-\pi, \pi]$ then $\lim_{n\to\infty} S_n(f,t) = f(t)$.*

Theorem 6.14. *(Dini's Theorem for Lipschitz functions) Suppose $f \in L(\mathbb{T})$ and f is Lipschitz on an open interval. That is there is some $M > 0$ such that*

$$|f(s) - f(t)| \leq M|s-t|$$

for all $t, s \in I$. Then for $t \in I$ we have $\lim_{n\to\infty} S_n(f,t) = f(t)$.

7 Hilbert Spaces

Definition 7.1. Let X be a complex vector space. An inner product $\langle , \rangle : X \times X \mapsto \mathbb{C}$ is a map such that for $f, g, h \in X$ and $\alpha \in \mathbb{C}$ then

(1) $\langle f, f \rangle \geq 0$

(2) $\langle f, f \rangle = 0 \implies f = 0$

(3) $\langle f, g \rangle = \overline{\langle g, f \rangle}$

(4) $\langle \alpha f, g \rangle = \alpha \langle f, g \rangle$

(5) $\langle f + g, g \rangle = \langle f, h \rangle + \langle g, h \rangle$

We call $(X, \langle , \rangle)$ an inner product space. That that (3) and (5) gives

$$\langle f, g + h \rangle = \langle f, g \rangle + \langle f, h \rangle$$

while (3) and (4) give

$$\langle f, \alpha h \rangle = \bar{\alpha} \langle f, h \rangle$$

Furthermore, we define the induced norm for $f \in X$ by $\|f = \sqrt{\langle f, f \rangle}$ (we can check that is a norm).

Proposition 7.1. *(Cauchy-Schwarz) If $f, g \in (X, \langle , \rangle)$ we have $|\langle f, g \rangle| \leq \|f\|\|g\|$. Moreover, $|\langle f, g \rangle| = \|f\|\|g\|$ iff $g = tf$ for some $t \geq 0$.*

Example 7.1. (Kolmogorov's Function) Continuity $\not\Rightarrow$ Pointwise convergence of $S_n f(, x)$. Consider

$$f(x) = \prod_{k=1}^{\infty} \left(1 + i\frac{\cos 10^k x}{k}\right)$$

Here, f is continuous everywhere but for all $x \in [-\pi, \pi]$, $\{S_n(f,x)\}_{n\in\mathbb{N}}$ is unbounded.

Proposition 7.2. *If $(X, \langle , \rangle)$ is an i.p. sp. (inner product space) the $\|f\| = \sqrt{\langle f, f \rangle}$ defines a norm on X.*

Definition 7.2. A *Hilbert space* $\mathcal{H}$ is an inner product space which is complete w.r.t. $\|\cdot\|$.

Example 7.2. (1) $\mathbb{C}^n$, $\langle x, y \rangle = \sum_{i=1}^n x_i \bar{y}_i \implies \|x\|_2 = \sqrt{\sum_{i=1}^\infty |x_i|^2}$

(2) Let $A \in \mathcal{L}(\mathbb{R})$, $\lambda(A) > 0$. Then $L_2(A)$ has inner product

$$\langle f, g \rangle = \int_A f\bar{g} \, (= \Gamma_f(\bar{g}) = \Gamma_{\bar{g}}(f))$$

If $f, g \in L_2(A) \implies \bar{f} \in L_2(A)$ ($|\bar{g}|^2 = |g|^2$) which implies that $f\bar{g} \in L_1(A)$ (by Hölder's Inequality for $p = q = 2$). Hence $\langle , \rangle$ is well defined. The norm on $L_2(A)$ determined by $\langle , \rangle$ then gives

$$\|f\| = \left(\int_A f\bar{f}\right)^{\frac{1}{2}} = \left(\int_A f^2\right)^{\frac{1}{2}} = \|f\|_2$$

and since $(L_2(A), \|\cdot\|_2)$ is complete then $(L_2(A), \langle,\rangle)$ is a Hilbert space. Similarly,

$$L_2(\mathbb{T}) = \left\{ f : \mathbb{R} \mapsto \mathbb{C} : f \in \mathcal{M}_{\mathbb{C}}(\mathbb{R}), 2\pi - \text{periodic}, \int_{-\pi}^{\pi} |f|^2 < \infty \right\} \approxeq L_2([-\pi, \pi])$$

together with the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\bar{g}$$

is a Hilbert space.

(3) $\mathcal{C}([a,b])$ can be equipped with

$$\langle f, g \rangle = \int_A f\bar{g}$$

but it is NOT a Hilbert space. This is due to $\mathcal{C}([a,b]) \subsetneq L_2([a,b])$ which is dense in $L_2([a,b])$. This implies that it cannot be complete.

(4) Define the set

$$l_2 = l_2(\mathbb{N}) = \left\{ x = \{x_n\}_{n=1}^{\infty} : \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\}$$

The inner product on l_2 is defined by

$$\langle x, y \rangle = \sum_{n=1}^{\infty} x_n \bar{y}_n \implies \|x\|_2 \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2}$$

Note that

$$\begin{aligned} \sum_{n=1}^{\infty} |x_n \bar{y}_n| &= \lim_{N\to\infty} \sum_{n=1}^{N} |x_n||y_n| \\ &\leq \lim_{N\to\infty} \left(\sum_{n=1}^{N} |x_n|^2 \right)^{1/2} \left(\sum_{n=1}^{N} |y_n|^2 \right)^{1/2} \\ &= \|x\|_2 \|y\|_2 < \infty \end{aligned}$$

So $\sum_{n=1}^{\infty} |x_n \bar{y}_n|$ is convergent. Furthermore, $l_2(\mathbb{N})$ is a vector space. Observe that

$$\begin{aligned} \sum_{n=1}^{\infty} |x_n + y_n|^2 &\leq \sum_{n=1}^{\infty} (|x_n| + |y_n|)^2 \\ &= \sum_{n=1}^{\infty} \left(|x_n|^2 + 2|x_n||y_n| + |y_n|^2 \right) \\ &= \|x\|_2^2 + 2\sum_{n=1}^{\infty} |x_n||y_n| + \|y_2\|^2 \\ &\leq \|x\|_2^2 + 2\|x_n\|\|y_n\| + \|y_2\|^2 \\ &= (\|x\|_2 + \|y\|_2)^2 < \infty \end{aligned}$$

(5) Define

$$l_2 = l_2(\mathbb{Z}) = \left\{ x = \{x_n\}_{n\in\mathbb{Z}} : \sum_{n=-\infty}^{\infty} |x_n|^2 < \infty \right\}$$

We will show that $l_2(\mathbb{Z})$ s a Hilbert space isomorphic of $L_2(\mathbb{T})$. (*Plancherel's Theorem*)

Definition 7.3. Let $(X, \langle,\rangle)$ be an i.p. sp. A family of vectors $\{e_i\}_{i\in I} \subseteq X$ is called orthogonal if $\langle e_i, e_j \rangle = 0$ for all $i, j \in I$

and $i \neq j$. Moreover, $\{e_i\}_{i\in I}$ is called orthonormal if

$$\langle e_i, e_j \rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Proposition 7.3. *(Pythagorean Principle) If $\{f_1, ..., f_n\}$ is an orthogonal set in an i.p. sp. X, then*

$$\|f_1 + ... + f_2\| = \|f_1\|^2 + ... + \|f_n\|^2$$

Remark 7.1. Recall that in a normed vector space X,

$$\text{dist}(f, E) = \inf\left\{\left\|f - \sum_{i=1}^{n} \alpha_i e_i\right\| : \alpha \in \mathbb{C}\right\}$$

where $f \in X$ and $E = \text{span}\{e_1, ..., e_n\}$.

Lemma 7.1. *(Linear Approximation Lemma (LAL)) Suppose that $\{e_1, ..., e_n\}$ is an orthonormal set in an i.p. sp. X. Let $E = span\{e_1, ..., e_n\}$. Then for $f \in X$,*

$$dist(f, E)^2 = \left\|f - \sum_{i=1}^{n} \langle f, e_i \rangle e_i\right\|^2 = \|f\|^2 - \sum_{i=1}^{n} |\langle f, e_i \rangle|^2$$

Moreover, $\sum_{i=1}^{n} \langle f, e_i \rangle e_i$ is the unique vector $e \in E$ s.t. $dist(f, E) = \|f - e\|$.

Proposition 7.4. *Let X be an i.p. sp. and $g \in X$. Then*

$$\Gamma_g : X \mapsto \mathbb{C}$$

given by $\Gamma_g(f) = \langle f, g \rangle$ is linear and bounded with $\||\Gamma|\| = \|g\|$.

Remark 7.2. (Riesz Representation Theorem) If $\mathcal{H}$ is a Hilbert space, then every bounded linear functional $\Gamma : \mathcal{H} \mapsto \mathbb{C}$ is of the form $\Gamma = \Gamma_g$ where $g \in \mathcal{H}$.

Theorem 7.1. *(Orthonormal Basis Theorem (OBT)) Let X be an inner product space and $\{e_i\}_{i=1}^{\infty}$ be an orthonormal sequence. Then the following are equivalent.*

(1) span$\{e_i\}_{i=1}^{\infty} = \{\sum_{i=1}^{n} \alpha_i e_i : n \in \mathbb{N}, \alpha_i \in \mathbb{C}\}$ is dense in X.

(2) (Bessel's equality) For every $f \in X$, we have $\|f\|^2 = \sum_{i=1}^{\infty} |\langle f, e_i \rangle|^2$ in $\mathbb{C}$.

(3) For every $f \in X$ we have $f = \lim_{n\to\infty} \sum_{i=1}^{n} \langle f, e_i \rangle e_i = \sum_{n=1}^{\infty} \langle f, e_i \rangle e_i$, w.r.t. $\|\cdot\|$.

(4) (Parseval's Identity) For every $f, g \in X$, $\langle f, g \rangle = \sum_{n=1}^{\infty} \langle f, e_i \rangle \langle e_i, g \rangle$ in $\mathbb{C}$.

Remark 7.3. By (3) we are justified to call $\{e_i\}_{i=1}^{\infty}$ an orthonormal basis.

Definition 7.4. Any sequence satisfying conditions of the OBT is called an orthonormal basis for X.

Remark 7.4. (Bessel's Inequality) Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal (o.n.) sequence in an i.p. sp. X. Then for $f \in X$, we have

$$\langle f, f \rangle = \|f\|^2 \geq \sum_{i=1}^{\infty} |\langle f, e_i \rangle|^2$$

Note 14. Equality above holds if $f \in \overline{\text{span}\{e_1, e_2, ...\}}$ closed w.r.t. $\|\cdot\|$.

Theorem 7.2. *Let X be an i.p. sp. and $\{e_i\}_{i=1}^{\infty} \subset X$ be an orthonormal basis in X. Then the operator $U : X \mapsto l_2(\mathbb{N})$ given by $U_f = \{\langle f, e_i \rangle\}_{i=1}^{\infty}$ is an isometry preserving the inner product. That is, $\underbrace{\|U_f\|}_{\text{in } l_2} = \underbrace{\|f\|}_{\text{in } X}$ and $\underbrace{\langle U_f, U_g \rangle}_{\text{in } l_2} = \underbrace{\langle f, g \rangle}_{\text{in } X}$ for $f, g \in X$.*

Example 7.3. Here are some examples of orthonormal bases.

1. Let $X = l_2(\mathbb{Z})$ with the i.p. $\langle x, y\rangle = \sum_{n=-\infty}^{\infty} x_n \overline{y_n}$. Consider for each $n \in \mathbb{Z}$, the element

$$e_n = (..., 0, \underbrace{1}_{n^{th} \text{ entry}}, 0, ...)$$

   Then we have:

   (a) $\langle e_n, e_m\rangle = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$

   (b) If $x \in l_2(\mathbb{Z})$ then $\langle x, e_n\rangle = e_n$ (n^{th} entry in X)

   (c) If $x \in l_2(\mathbb{Z})$ then $\left\| x - \sum_{k=-n}^{n} \langle x, e_k\rangle\, e_k \right\|^2 \to 0$ as $n \to \infty$.
   So span$\{e_k\}_{k\in\mathbb{Z}}$ is dense in l_2 and $\{e_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis (o.n.b.) for $l_2(\mathbb{Z})$.

2. Consider $X = L_2(\mathbb{T})$ with $\langle f, g\rangle = \int_{\mathbb{T}} f\bar{g}$ for $f, g \in L_2(\mathbb{T})$. Consider $\{e^k\}_{k\in\mathbb{Z}} \subset L_2(\mathbb{T})$ where $e^k(t) = e^{ikt}$. Then we have:

   (a) $\{e^k\}_{k\in\mathbb{Z}}$ is orthonormal in $L_2(\mathbb{T})$

   (b) The Abstract Summability Theorem implies that $\{e^k\}_{k\in\mathbb{Z}}$ is an o.n.b for $L_2(\mathbb{T})$

Corollary 7.1. *(L_2 Convergence of Fourier Series) Let $f \in L_2(\mathbb{T})$. Then $\lim_{n\to\infty} \|f - S_n(f)\|_2 = 0$.*

Remark 7.5. Let's examine the convergence of Fourier series in various Banach spaces.

(1) Suppose that $f \in L(\mathbb{T})$. In $L_1(\mathbb{T})$, $S_n(f) \to f$ rarely w.r.t. $\|\cdot\|_1$. That is, from the properties of the $D_n's$ (Dirichlet Kernel), $\lim_{n\to\infty} \|S_n(f) - f\|_1 \neq 0$ on $U_1 \subseteq L_1(\mathbb{T})$ where U_1^c is of 1st category.

Suppose that $f \in \mathcal{C}(\mathbb{T})$. Then $\lim_{n\to\infty} \|S_n(f) - f\|_\infty \neq 0$ on $U_\infty \subseteq \mathcal{C}(\mathbb{T})$ where U_∞^c is of 1st category.

(2) Consider $\sigma_n(f, t) = \frac{1}{n+1}\left(\sum_{k=0}^{n} D_k\right) * f(t) = K_n * f(t)$. By the Abstract Summability Kernel Theorem, if $f \in L_p(\mathbb{T})$ for $1 \leq p < \infty$ then $\lim_{n\to\infty} \|\sigma_n(f) - f\|_p = 0$.

(3) For $p = 2$, $L_2(\mathbb{T})$ is a Hilbert space. By L_2 convergence of Fourier series, if $f \in L_2(\mathbb{T})$ then $\lim_{n\to\infty} \|S_n(f) - f\|_2 = 0$. To see this, recall that $\||C(D_n)|\|_{L_1(\mathbb{T})} = \|D_n\|_1 \to \infty$ as $n \to \infty$. In L_2, by Bessel's Inequality, $\||C(D_n)|\|_{L_2(\mathbb{T})} \leq 1$ for all n (this is in fact, an equality, which is left to be shown as an exercise) on $[-\pi, \pi]$, which implies that $L_2(\mathbb{T}) \subseteq L_1(\mathbb{T})$.

Theorem 7.3. *(Riesz-Fischer Theorem) Let $f \in L_1(\mathbb{T})$. Then $f \in L_2(\mathbb{T})$ if and only if $\sum_{n=-\infty}^{\infty} |c_k(f)|^2 < \infty$*

Theorem 7.4. *(Abstract Plancherel's Theorem) The map $U : L_2(\mathbb{T}) \mapsto l_2(\mathbb{Z})$ given by $f \mapsto U(f) = \{c_n(f)\}_{n\in\mathbb{Z}}$ is a surjective isometry with $\langle Uf, Ug\rangle_{l_2(\mathbb{Z})} = \langle f, g\rangle_{L_2(\mathbb{T})}$.*

Corollary 7.2. *$l_2(\mathbb{Z})$ is complete $\implies$ It is a Hilbert space.*

Summary 2. Let's examine the spaces of (almost everywhere equivalent classes of) functions by:

$$\begin{array}{ccccccc} A(\mathbb{T}) \subset & & \mathcal{C}(\mathbb{T}) \subset & & L_1(\mathbb{T}) \subset & & L_1(\mathbb{T}) \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ l_1(\mathbb{Z}) \subset & & C^*(\mathbb{Z}) \subset & & l_2(\mathbb{Z}) \subset & & A(\mathbb{Z}) \subsetneq c_0(\mathbb{Z}) \end{array}$$