PMATH 351 Final Exam Review

LaTeXer: W. Kong

Theorems and Statements

Proposition 0.1. [Statement] *The uniform limit of a sequence of continuous functions* $\{f_n : (X, d_X) \mapsto (Y, d_Y)\}$ *is continuous.*

Summary. Let $\epsilon > 0$. A sequence of functions $\{f_n\}$ is said to converge uniformly if there exists $N \in \mathbb{N}$ such that for $p, q \geq N$, we have that $d_Y(f_p(x), f_q(x)) < \epsilon$ for all $x \in X$.

Proposition 0.2. $C_b(X)$ *is complete.*

Proof. Let $\{f_n\} \subset C_b(X)$ be Cauchy. If $x \in X$, then $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty$. Hence $\{f_n(x)\}$ is Cauchy in $\mathbb{R}$ for each $x \in X$. Let $f_0(x) = \lim_{n\to\infty} f_n(x), \forall x \in X$. We claim that $f_0 \in C_b(X)$. Let $\epsilon > 0$. Then $\exists N_0 \in \mathbb{N}$ such that $n, m \geq N_0 \implies |f_n(x) - f_m(x)| < \frac{\epsilon}{2}$ for any $x \in X$. Let $n \geq N_0$ and $x \in X$. Then let $(*)$ be the statement that

$$|f_n(x) - f_0(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \leq \frac{\epsilon}{2} < \epsilon.$$

Hence, $f_n \to f_0$ uniformly on $X \implies f_0$ is continuous. Since $\{f_n\}$ is Cauchy, $\exists M \geq 0$ such that $\|f_n(x)\|_\infty < M, \forall n \in \mathbb{N}$. So

$$|f_0(x)| \leq |f_0(x) - f_{N_0}(x)| + |f_{N_0}(x)| < \epsilon + M \implies f_0 \in C_b(x)$$

By $(*)$, if $n \geq N_0$ then $|f_n(x) - f_0(x)| \leq \frac{\epsilon}{2}$ for all $x \in X \implies \|f_n - f_0\| \leq \frac{\epsilon}{2} < \epsilon \implies f_n \to f_0$ in $\|\cdot\|$. □

Theorem 0.1. [Statement] *(Generalized Weierstrass M-Test) Let* $(X, \|\cdot\|)$ *be a normed linear space (n.l.s.). Then TFAE:*

1) X is a Banach space.

2) X satisfies $(*) \to$ *If* $\{x_n\} \subset X$ *is such that* $\sum_{n=1}^\infty \|x_n\| < \infty$*, then* $\sum_{n=1}^\infty x_n$ *converges in X.*

Summary. A Banach space is a normed linear space that is complete under its norm.

Theorem 0.2. *(Baire Category Theorem I) Let* (X, d) *be a complete metric space. If* $\{U_n\}_{n=1}^\infty$ *is a sequence of open dense subsets of X, then* $\bigcap_{n=1}^\infty U_n$ *is also dense in X.*

Proof. Let W be open and non-empty. then $W \cap U_1$ is open and non-empty. So $\exists x_1 \in X$ and $r_1 \in [0, 1]$ with $B(x_1, r_1) \subset B[x_1, r_1] \subset W \cap U_1$. We can also find $x_2 \in X$ with $r_2 \in [0, \frac{1}{2}]$ such that $B(x_2, r_2) \subset B[x_2, r_2] \subset W \cap U_2$. We proceed inductively to get $\{x_n\} \in X$, $\{r_n\}$ with $r_n \in [0, \frac{1}{n}]$ such that $B[x_{n+1}, r_{n+1}] \subset B(x_n, r_n) \subset B[x_n, r_n] \subset W \cap U_n$. Let $F_n = B[x_n, r_n]$. Then $F_{n+1} \subset F_n$ and $diam(F_n) \leq \frac{2}{n} \to 0$. By the Cantor Intersection Theorem, $\{x_0\} = \bigcap_{n=1}^\infty F_n = \bigcap_{n=1}^\infty B[x_n, r_n] \implies x_0 \in B[x_1, r_1] \subset W \implies x_0 \in W$. Moreover $x_0 \in B[x_n, r_n] \subset U_n$ for all $n \in \mathbb{N}$. Hence $x_0 \in W \cap (\bigcap_{i=1}^\infty U_n)$. □

Theorem 0.3. *(Baire Category Theorem II) Let* (X, d) *be a complete metric space.. Then X is of* 2^{nd} *category itself.*

Proof. Assume that X was of 1^{st} category. then $X = \bigcup_{n=1}^\infty A_n = \bigcup_{n=1}^\infty \overline{A_n}$ where each A_n is nowhere dense (the closure argument works because X is the entire set). Let $U_n = (\overline{A_n})^c$. Then U_n is open and nowhere dense in X. However, $\bigcap_{n=1}^\infty U_n = X^c = \emptyset$ which contradicts the Baire Category Theorem I. □

Theorem 0.4. *(Banach Contractive Mapping Theorem) Let* (X, d) *be a compact metric space (c.m.s.). Let* $\Gamma : X \mapsto X$ *be contractive. Then* $\exists$ *unique* $x_0 \in X$ *with* $\Gamma(x_0) = x_0$.

Proof. Let $x_1 \in X, x_2 \in \Gamma(x_1), ..., x_n = \Gamma(x_n)$. This gives us a set $\{x_n\}$ which we claim is Cauchy. Using the recurrence relation $d(x_n, x_{n-1}) = d(\Gamma(x_{n-1}), \Gamma(x_{n-2})) \leq kd(x_{n-1}, x_{n-2})$, it is clear that $d(x_{n+2}, x_{n+1}) \leq k^n d(x_2, x_1)$. Let $m > n > 4$. Then

$$d(x_n, x_m) \leq \sum_{l=n}^{m-1} d(x_l, x_{l+1}) \leq k^{n-1} \left(\sum_{l=0}^{(m-2)-(n-1)} k^l \right) d(x_2, x_1) \leq \frac{k^{n-1}}{1-k} d(x_1, x_2)$$

and so given $\epsilon > 0$, we can choose N large enough so that $\frac{k^{n-1}}{1-k} d(x_1, x_2) < \epsilon \implies d(x_n, x_m) < \epsilon$. Thus, $\{x_n\}$ is Cauchy and since X is complete, $x_n \to x_0$. Since $d(\Gamma(x), \Gamma(y)) \leq kd(x, y)$, Γ is continuous and so $\Gamma(x_0) = \Gamma(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} x_{n+1} = x_0$.

To show uniqueness, suppose that $\Gamma(y_0) = y_0$. Since $d(y_0, x_0) = d(\Gamma(y_0), \Gamma(x_0)) \leq kd(y_0, x_0)$ and $k \in (0, 1)$ then $d(y_0, x_0) = 0 \implies x_0 = y_0$. □

Theorem 0.5. *(Arzela-Ascoli (1 $\implies$ 2)) Let (X, d) be a c.m.s.. Let $\mathcal{F} \subset C(X)$. Then TFAE:*

1) $\mathcal{F}$ is relatively compact

2) $\mathcal{F}$ is equicontinuous and pointwise bounded

Proof. Let ϵ >0. Since $\mathcal{F}$ is relatively compact, then $\mathcal{F}$ is bounded and clearly pointwise bounded. We show that $\mathcal{F}$ is equicontinuous. Let $\epsilon > 0$ and note that since $\mathcal{F}$ is totally bounded, then we can generate an ϵ-net $\{f_1, ..., f_n\}$ which is equicontinuous because it is a finite set. Thus, $\exists \delta > 0$ such that for $x, z \in X$, $d(x, z) < \implies |f_i(x) - f_i(z)| < \frac{\epsilon}{3}$ for any $i = 1, ..., n$. Also $\exists i_0 \in \{1, ..., n\}$ such that $\|f - f_{i_0}\|_\infty < \frac{\epsilon}{3}$. Choose such a δ and i_0 and let $x, z \in X$ with $d(x, z) < \delta$, and $f \in \mathcal{F}$. Thus, we have

$$\begin{aligned} |f(x) - f(z)| &\leq |f(x) - f_{i_0}(x)| + |f_{i_0}(x) - f_{i_0}(z)| + |f_{i_0}(z) - f(z)| \\ &< \underbrace{\frac{\epsilon}{3}}_{\epsilon-net} + \underbrace{\frac{\epsilon}{3}}_{cts} + \underbrace{\frac{\epsilon}{3}}_{\epsilon-net} = \epsilon \end{aligned}$$

□

Theorem 0.6. *(Weierstrass Approximation Theorem) Let $f \in C[a, b]$ and $\epsilon > 0$. Then $\exists p_n(x)$ such that $p_n \to f$ uniformly on $[a, b]$.*

Proof. First, we may assume that f is defined on $[0, 1]$ and that $f(0) = f(1) = 0$ because if f is not we know that we there transformations that can be made on f such that these properties are true, while preserving continuity. We can extend f into the domain of uniformly continuous functions on $\mathbb{R}$ by defining $f(x) = 0$ for $x \notin [0, 1]$. Now for each $n \in \mathbb{N}$, define $Q_n(t) = c_n(1 - t^2)^n$ where is c_n is defined such that $\int_{-1}^1 Q_n(t) = 1$. We then note that $\int_{-1}^1 (1 - x^2)^n \geq 2 \int_0^{\frac{1}{\sqrt{n}}} 1 - nx^2 \, dx = \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}}$ and so $c_n < \sqrt{n}$. If $0 < \delta < 1$ then for $x \in [-1, \delta] \cup [\delta, 1]$ we have $c_n(1 - x^2) \leq \sqrt{n}(1 - \delta^2)^n$. Let

$$p_n(x) = \int_{-1}^1 f(x + t) Q_n(t) \, dt = \int_{-x}^{1-x} f(x + t) Q_n(t) \, dt = \int_0^1 f(u) Q_n(u - x) \, du$$

where $u = x + t$. Using the Leibniz rule, we have that

$$\frac{d^{2n+1}}{dx^{2n+1}} p_n(x) = \int_0^1 f(u) \frac{d^{2n+1}}{dx^{2n+1}} Q_n(u - x) \, du = 0$$

and it follows that p_n is a polynomial of degree $2n+1$ or less. Let $\epsilon > 0$, $\|f\|_\infty = M$, and choose $\delta > 0$ so that $|x - y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{2}$. It is also the case that $\int_{-1}^1 Q_n(t) \, dt = 1 \implies f(x) = \int_{-1}^1 f(x) Q_n(t) \, dt = 1$.

Thus, if $x \in [0,1]$ we have

$$\begin{aligned} |p_n(x) - f(x)| &\leq \int_{-1}^{1} |f(x+t) - f(x)| Q_n(t)\, dt \\ &= \int_{-1}^{-\delta} |f(x+t) - f(x)| Q_n(t)\, dt + \int_{-\delta}^{\delta} |f(x+t) - f(x)| Q_n(t)\, dt + \int_{\delta}^{1} |f(x+t) - f(x)| Q_n(t)\, dt \\ &< 2M\sqrt{n}(1-\delta^2)^n + \frac{\epsilon}{2} + 2M\sqrt{n}(1-\delta^2)^n = \frac{\epsilon}{2} + 4M\sqrt{n}(1-\delta^2)^n. \end{aligned}$$

Hence, if we choose n large enough so that $4M\sqrt{n}(1-\delta^2)^n < \frac{\epsilon}{2}$, then $|p_n(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for all $x \in X$ and $\|p_n(x) - f(x)\|_\infty < \epsilon$. ☐

Theorem 0.7. [Statement] *(Stone-Weierstrass: Lattice) Let (X,d) be a c.m.s. and Φ be a linear subspace of $C(X)$ such that*

1) Φ is point separating

2) $1 \in \Phi$

3) If $f, g \in \Phi$ then $f \vee g = \max(f,g) \in \Phi$ (i.e. Φ is a lattice)

Then $\overline{\Phi} = C(X)$.

Theorem 0.8. *(Stone-Weierstrass: Subalgebra) Let (X,d) be a c.m.s. and Φ be a linear subspace of $C(X)$ such that*

1) Φ is point separating

2) $1 \in \Phi$

3) Φ is an algebra

Then $\overline{\Phi} = C(X)$.

Proof. Since Φ satisfies 1),2),3) we can assume WLOG that Φ is closed. Let $f \in \Phi$. Then f is bounded and $\exists M$ such that $f(x) \in [-M, M]$ for $x \in X$. Let $\epsilon > 0$ and using the Weierstrass Approximation theorem, create a polynomial $p(t) = \sum_{k=0}^{n} a_k t^k$ with $||t| - p(t)| < \epsilon$ for all $t \in [-M, M]$. Let $p \circ f = \sum_{k=0}^{n} a_k f^k$ and note that $||t| - p(t)| < \epsilon$, $\forall x \in X \implies |f| \in \overline{\Phi} = \Phi$. Since $f \vee g = \frac{(f+g)-|f-g|}{2}$ then Φ is a lattice and is dense in $C(X)$ by the first Stone-Weierstrass Theorem. But Φ is closed and hence $\Phi = C(X)$. ☐

Exercises

Exercise 0.1. Show that $\mathbb{Q}$ is not a G_δ set.

Suppose that it is. Let $\mathbb{Q} = \bigcap_{n=1}^{\infty} U_n$ where each U_n is open. Since $\mathbb{Q} \subset U_n$ for all $n \in \mathbb{N}$, U_n is dense. Let $F_n = U_n^c$ which is closed and nowhere dense. We also have that $\mathbb{R} \backslash \mathbb{Q} = \bigcup_{n=1}^{\infty} F_n$. Let $\mathbb{Q} = \{r_1, r_2, ...\}$ and $F_n' = F_n \cup \{r_n\}$. F_n' is closed and nowhere dense and $\mathbb{R} = \bigcup_{n=1}^{\infty} F_n'$ which implies that $\mathbb{R}$ is of 1^{st} category which is clearly impossible.

Exercise 0.2. Show that (X,d) is a c.m.s. iff whenever $\Im$ is a family of closed sets with the Finite Intersection Property (FIP), then $\bigcap_{F \in \Im} F \neq \emptyset$.

($\implies$) Assume X is compact and take some collection of closed sets $\Im = \{F_\alpha\}_{\alpha \in I}$ with the FIP. If $\bigcap_{\alpha \in I} F_\alpha = \emptyset$ and $U_\alpha = F_\alpha^c$ then $\bigcup_{\alpha \in I} U_\alpha = X$ so $\{U_\alpha\}_{\alpha \in I}$ is a cover. Take a finite cover $\{U_i\}_{i=1}^{n}$ and note that $\bigcap_{i=1}^{n} F_i = \emptyset$ which contradicts the FIP.

($\Longleftarrow$) Suppose a collection $\Im$ of closed sets in X with the FIP is that $\bigcap_{F \in \Im} F \neq \emptyset$ for any $\Im$. Suppose that X is not compact. Take an open cover $\{U_\alpha\}_{\alpha \in I}$ that has no finite subcover and let $F_\alpha = U_\alpha^c$ and note that $\{F_\alpha\}_{\alpha \in I}$ has the FIP. So $\bigcap_{\alpha \in I} F_\alpha \neq \emptyset$ which contradicts the fact that $\{U_\alpha\}_{\alpha \in I}$ is a cover.

Exercise 0.3. If (X, d_1) is sequentially compact and $f : (X, d_1) \to (Y, d_2)$ is continuous, then $f(X)$ is sequentially compact.

Let $\{y_n\} \subset f(X)$. Then $\exists \{x_n\} \subset X$ such that $y_n = f(x_n)$, $\forall n \in \mathbb{N}$. We get a subsequence $\{x_{n_k}\}$ with $x_{n_k} \to x_0$ for some $x_0 \in X$. Let $y_0 = f(x_0) \in f(X)$. Then $y_{n_k} = f(x_{n_k}) \to f(x_0) = y_0$.

Exercise 0.4. If (X, d_1) is compact and $f : (X, d_1) \to (Y, d_2)$ is continuous, then $f(x)$ is uniformly continuous.

Assume that f is not uniformly continuous. Then $\exists \epsilon_0 > 0$ and two sequences $\{x_n\}, \{z_n\} \subset X$ with $d_1(x_n, z_n) \to 0$ but $d_2(f(x_n), f(z_n)) \geq \epsilon_0$, $\forall n \in \mathbb{N}$. Since X is sequentially compact, $\exists \{x_{n_k}\}$ with $x_{n_k} \to x_0 \in X$. Similarly, $z_{n_k} \to z_0 = x_0$. But then $f(x_{n_k}) \to f(x_0)$ and $f(z_{n_k}) \to f(z_0) = f(x_0)$ which is clearly impossible.

Exercise 0.5. If (X, d) is compact then it also has the Bolzano-Weierstrass Property.

Let $A \subset X$ be infinite. Let $\{x_n\}$ be a sequence of distinct elements of A. Let $F_n = \overline{\{x_n, x_{n+1}, ...\}}$. Note that F_n has the FIP and so $\exists x_0 \in \bigcap_{n=1}^{\infty} F_n$. Hence, $\forall \epsilon > 0$ such that $B(x_0, \epsilon) \cap \{x_n, x_{n+1}, ...\} \neq \emptyset, \forall n \in \mathbb{N} \implies B(x_0, \epsilon_0) \cap A$ is infinite as well with $x_0 \in Lim(A)$.

Exercise 0.6. If (X, d) is sequentially compact, then it is totally bounded.

Suppose that X is compact by not totally bounded. Then $\exists \epsilon_0 > 0$ with no finite ϵ-net. From here we can construct $\{x_n\} \subset X$ such that $x_i \notin B(x_j, \epsilon_0)$ if $i \neq j$. Note that $d(x_i, x_j) \geq \epsilon_0$ if $i \neq j$ and so this sequence has no convergent subsequence which is impossible.

Exercise 0.7. If (X, d) is sequentially compact, then it is Heine-Borel compact.

Let $\{U_\alpha\}_{\alpha \in I}$ be a cover of X and ϵ_0 be the Lebesgue number for the cover. Let $0 < \delta < \epsilon_0$ and since X is totally bounded, $\exists x_1, ..., x_n \in X$ with $X = \bigcup_{i=1}^{n} B(x_i, \delta)$. Since $\delta < \epsilon_0$, $\exists \alpha_i \in I$ with $B(x_i, \delta) \subset U_{\alpha_i}$ and hence $X = \bigcup_{i=1}^{n} U_{\alpha_i}$.

Exercise 0.8. If (X, d_X) is a c.m.s. and $f : (X, d_X) \to (Y, d_Y)$ is continuous, 1-1 and onto, then f^{-1} is also continuous.

Since $(f^{-1})^{-1} = f$ it suffices to show that if $U \subset X$ is open, then $f(U)$ is open. Let $U \subset X$ be open and $F = U^c$ which we note is compact[1]. So $f(F)$ is compact and also closed. Thus, $f(U) = [f(F)]^c$ is open.

Exercise 0.9. A space (X, d) is a c.m.s. iff it is complete and totally bounded.

We already know the ($\implies$) direction so we prove the reverse. Let $\{x_n\} \subset X$ and since X is totally bounded, take an infinite ball $S_1 = B(y_1, 1)$ with radius 1 around some point y_1 that covers an infinite number of terms in the sequence. Similarly, we can construct $S_2 = B(y_2, \frac{1}{2})$ which contains an infinite number of terms in $\{x_n\} \cap S_1$ for some point y_2.

Inductively we can construct $\{S_k = B(y_k, \frac{1}{k}\}\}$ such that s_{k+1} has infinitely many terms in $\{x_n\} \cap S_1 \cap ... \cap S_k$. This means that we could also choose a sequence $n_1 < n_2 < ...$ such that $x_{n_k} \in S_1 \cap ... \cap S_k$. Since $diam(S_k) \to 0$ and $\exists N \in \mathbb{N}$ such that if $k, m \geq N$ then $x_{n_m}, x_{n_k} \in S_N$, then it follows that $\{x_{n_k}\}$ is Cauchy which by completeness, this sequence converges. Hence X is sequentially compact and therefore compact.

Exercise 0.10. If (X, d) is a c.m.s. and F is equicontinuous on X, then F is also uniformly equicontinuous on X.

Let $\epsilon > 0$ and $\forall x_0 \in X$, create a $\delta_{x_0} > 0$ such that $d(x, x_0) < \delta_{x_0} \implies |f(x) - f(x_0)| < \frac{\epsilon}{2}$ for all $f \in F$. Then $\{B(x_0, \delta_{x_0})\}_{x_0 \in X}$ with a Lebesgue number, say $\delta_1 > 0$. Let $\delta_0 \in (0, \delta_1)$ and note that for any $y \in X$, there is some $x_0 \in X$ such that $B(y, \delta_0) \subset B(x_0, \delta_{x_0})$. If $y, z \in X$ with $z \in B(y, \delta_0)$ then

$$|f(y) - f(z)| \leq |f(y) - f(x_0)| + |f(x_0) - f(z)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

[1] *Closed subsets of a compact set are compact.*

Summary. The following are results that were commented on during the tutorial on finite dimensional (f.d.) n.l.s.. Let $\Gamma(\vec{a}) = \vec{a} \cdot \vec{w}$ for a basis $\vec{w}$ of our n.l.s. W.

1+2+3) $U \subset W$ is open/closed/bounded $\Longleftrightarrow \Gamma^{-1}(U)$ is open/closed/bounded in $\mathbb{R}^n$

4) Heine-Borel compactness of $A \subset W$ in a n.l.s. $\Longleftrightarrow A$ is closed and bounded

5) $w_n \to w_0 \Longleftrightarrow \Gamma^{-1}(w_n) \to \Gamma^{-1}(w_0)$

6) $\{w_n\}$ Cauchy in $W \Longleftrightarrow \{\Gamma^{-1}(w_n)\}$ Cauchy in $\mathbb{R}^n \implies (W, \|\cdot\|_W)$ is always complete

7) $(V, \|\cdot\|_V)$ is a n.l.s. and $W \subset V$ is a f.d. n.l.s. $\implies W$ is closed and nowhere dense in V

8) If $(V, \|\cdot\|_V)$ is an infinite dimensional Banach space and $\{v_\alpha\}_{\alpha \in I}$ is a basis, then I is uncountable.

Review of Concepts and Select Topics

Cauchy Sequences

- If any subsequence of a Cauchy sequence converges, the whole sequence converges
- All Cauchy sequences in a complete space converge
- If a sequence of elements in a sequence space is Cauchy, then each of its component sequences is Cauchy

Uniform Convergence

- If a sequence of continuous functions converges uniformly, then its limit is also continuous

Inequalities

- Holder's Inequality: $\sum_{i=1}^{n} |a_i b_i| \leq (\sum_{i=1}^{n} |a_i|^p)^{\frac{1}{p}} (\sum_{i=1}^{n} |b_i|^p)^{\frac{1}{p}}$

Sequence Spaces

- $l_1 \subset l_2 \subset ... \subset l_p \subset ... \subset l_\infty$

Completeness

- A subset of a complete set is is complete in the induced metric if it is closed