CO 255 Final Exam Review

LaTeXer: W. Kong

1 Basic Linear Programming

Lemma 1.1. *Let M be a rational matrix. Then $\det M$ has size at most twice the size of M.*

Proof. Let $M = \left[\frac{p_{ij}}{q_{ij}}\right]$ and M has n rows. Suppose that $|\det M| = p/q$ where p and q are relatively prime. We first know that $|c(q)| \leq |c(M)|$. To see this, note that

$$q = \prod_{i,j} q_{i,j} < 2^{|c(M)|-1} \implies |c(q)| \leq \sum_{i,j} |c(q_{i,j})| < |c(M)|$$

where $c()$ is the encoding function. A similar result holds for p. To see this, note that $\det M$ is an alternating sum over all permutations, so

$$|\det M| = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \cdot \prod_{k=1}^{n} M_{k,\pi(k)} \leq \prod_{i,j}(|p_{ij}|+1|) \implies |p| = |\det M| \cdot q \leq \prod_{i,j}(|p_{ij}|+1|)q_{ij} < 2^{|c(M)|-1}$$
$$\implies |c(p)| < |c(M)|$$

and hence

$$|c(\det M)| = 1 + |c(p)| + |c(q)| < 2|c(M)|$$

□

Theorem 1.1. *If a rational system $Ax = b$ has a solution then it has one of size polynomially bounded by the size of $A|b$.*

Proof. We may assume rows of A are linearly independent By reordering the columns, we may write $A = [B\ N]$ where B is non-singular and called **basic** and N is **non-basic**. Then $\bar{x} = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}$ is a solution of $Ax = b$. Under Cramer's Rule,

$$B^{-1} = \left[\frac{(-1)^{j+i}\det(B_{ij})}{\det B}\right]$$

and from the above lemma, $\bar{x}$ is of polynomial size. □

Theorem 1.2. *(Edmonds 1967) If A and b are rational then Gaussian elimination is polynomial time.*

Proof. It suffices to show that all numbers that appear are of size polynomially bounded in the size of (A, b). During the execution of the algorithm, we find linear systems $A_k x = b_k$ where $0 \leq k \leq r$ and r is the rank of A. Consider this as working on matrices $E_k = [A_k|b_k]$. We may assume we need not permute any columns. We show all numbers in $(E_k : k = 0, ..., r)$ are of polynomial size by induction on k. The case of $k = 0$ is trivial since $A_0 = A$ and $b_0 = b$ and the result follows from the above theorem. Let $0 < k \leq r$ and suppose the sizes of $E_0, ..., E_{k-1}$ are polynomial in the size of $(A|b)$.

The matrix E_k is of the form $\begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$ where B is non-singular and upper triangular with k rows and k columns. The first k rows of E_k and E_{k-1} are identical. It remains to show the entries in D are small. Consider the entry d_{ij} of D. Let $(E_k)_{ij} = \begin{pmatrix} B & C \\ 0 & d_{ij} \end{pmatrix}$ and note that $|\det((E_k)_{IJ})| = |d_{ij} \det B|$ and hence

$$d_{ij} = \frac{\det(E_k)_{IJ}}{\det B} = \frac{\det(E_k)_{IJ}}{\det(E_k)_{KK}}$$

Now E_k arises from $(A|b)$ by adding multiples of the first k rows to other rows so $\det(E_k)_{IJ} = \det(A|B)_{IJ}$ and $\det(E_k)_{KK} = \det(A|b)_{KK}$ □

Theorem 1.3. *(Farkas' Lemma v1) $Ax \leq b$ has a solution if and only if $y^T b \geq 0$ for each vector $y \geq 0$ such that $y^T A = 0$.*

Proof. ($\implies$) Apply F-M. □

Theorem 1.4. *(Farkas' Lemma v2) Only one of the two systems holds:*

- There exists a solution to the system $Ax = b$ and $x \geq 0$
- There exists a vector y such that $y^T A \geq 0$ and $b^T y < 0$

Theorem 1.5. *The system $Ax = b, x \geq 0$ has a solution if and only if $y^T b \geq 0$ for each vector y such that $y^T A \geq 0$.*

Proof. Write $Ax = b, x \geq 0$ as $Ax \leq b, -Ax \leq -b, -Ix \leq 0$ or

$$\begin{bmatrix} A \\ -A \\ -I \end{bmatrix} X \leq \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix}$$

So $Ax = b, x \geq 0$ has a solution

$$\iff \begin{bmatrix} y' \\ y'' \\ z \end{bmatrix} \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix} \geq 0 \text{ for each } \begin{bmatrix} y' \\ y'' \\ z \end{bmatrix} \geq 0 \text{ such that } \begin{bmatrix} y' \\ y'' \\ z \end{bmatrix}^T \begin{bmatrix} A \\ -A \\ -I \end{bmatrix} = 0 \iff$$

$$\iff (y' - y'')^T b \geq 0 \text{for each } \begin{bmatrix} y' \\ y'' \\ z \end{bmatrix} \geq 0 \text{such that } (y' - y'')^T A - z^T I = 0$$

$$\iff (y' - y'')^T b \geq 0 \text{ for each } y', y'', z \geq 0 \text{ such that } (y' - y'')^T A = z$$

$$\iff y \equiv y' - y'' \text{ and } y^T b \geq 0 \text{ for each } y \text{ such that } y^T A \geq 0$$

□

Summary 1. In summary, the previous sections say:

1. $Ax = b$ has a solution $\iff \nexists y$ such that $y^T A = 0$, $y^T b = 1$
2. $Ax = b$ with x integral has a solution $\iff \nexists y$ such that $y^T A$ integral, $y^T b$ non-integral
3. $Ax \leq b$ has a solution $\iff \nexists y$ such that $y^T A = 0$, $y^T b < 0$, $y \geq 0$

2 Basic Integer Programming

Theorem 2.1. *(Farkas, Minkowski, Weyl) A cone is polyhedral $\iff$ it is finitely generated.*

(Sketch) The idea behind the proof is that $b \in cone\{a_1, ..., a_m\} \iff \exists a$ solution to $y^T A = b, y \geq 0$, $A = [a_1 ... a_m]^T \iff b^T x \geq$ for all solutions to $Ax \geq 0$. Since there are infinitely $x's$, we need to choose a finite subset. So we need a sharper version of Farkas.

Theorem 2.2. *(Fundamental Theorem of Linear Inequalities, [Schrijver, p. 85]) Let* $a^1, ..., a^M \in \mathbb{R}^n$ *and let* $t = rank\{a^1, .., a^M, b\}$ *where* $b \in \mathbb{R}^n$*. Then exactly one of the two statements is true.*

1. b *is a non-negative linear combination of linearly independent vectors from* $a^1, ..., a^M$

2. There exists a hyperplane $\{x : C^T x = 0\}$ *containing* $(t-1)$ *linearly independent vectors from* $a^1, ..., a^M$ *such that* $C^T b < 0$ *and* $C^T a^1, ..., C^T a^M \geq 0$*.*

Proof. We may assume $a^1, ..., a^M$ span $\mathbb{R}^n$. Otherwise, use a transformation to map the space into a subspace with some $x_j = 0$. We first show that we cannot have both (1) and (2). Indeed, let $b = \lambda_1 a^1 + ... + \lambda_M a^M$ for some $\lambda_i \geq 0$ and suppose we have C as in (2). Then

$$\begin{aligned} C^T b < 0 \quad &\Longrightarrow \quad C^T(\lambda_1 a^1 + ... + \lambda_M a^M) < 0 \\ &\Longrightarrow \quad \lambda_1 \underbrace{C^T a^1}_{\geq 0} + ... + \lambda_M \underbrace{C^T a^M}_{\geq 0} < 0 \end{aligned}$$

which is impossible and we are done here. We will show that either (i) or (ii) must be true. Choose a linearly independent set of vectors $a_{i_1}, ..., a_{i_n}$ from $a^1, ..., a^M$. Let $B = \{a_{i_1}, ..., a_{i_n}\}$. We apply the following (**simplex**) algorithm.

1. Write $b = \lambda_{i_1} a_{i_1} + ... + \lambda_{i_n} a_{i_n}$. If $\lambda_{i_1}, ..., \lambda_{i_n} \geq 0$ then (1) holds and we *stop.*

2. Choose the smallest index h among $i_1, ..., i_n$ having $\lambda_h < 0$. Let $\{x : C^T x = 0\}$ be the hyperplane spanned by $B \backslash \{a_h\}$. Scale C so that $C^T a_h = 1$. Note that this means

$$\begin{aligned} c^T b = c^T(\lambda_{i_1} a_{i_1} + ... + \lambda_{i_n} a_{i_n}) \quad &= \quad \lambda_{i_1} C^T a_{i_1} + ... + \lambda_{i_n} C^T a_{i_n} \\ &= \quad \lambda_h C^T a_h = \lambda_h < 0 \end{aligned}$$

3. If $C^T a^1 \geq 0, ..., C^T a^M \geq 0$ then (2) holds and we *stop.*

4. Choose the smallest s such that $C^T a_s < 0$. Replace B by removing a_h and adding a_s. That is, $B \mapsto (B \backslash \{a_h\}) \cup \{a_s\}$.

5. Go to step 1.

To prove the theorem, we only need to show that the algorithm terminates. Let B_k denote the set B in the k^{th} iteration. If the algorithm does not terminate, then must have $B_k = B_l$ for some $k < l$ (since there are only finitely many choices for the set B). Let r be the highest index for which a_r has been removed from B at the end of one of the iterations $k, ..., l-1$ which we will say, it is p. Since $B_k = B_l$, we must have that a_r is added to B, say in iteration $q < p$. Note that

$$B_p \cap \{a_{r+1}, ..., a_m\} = B_q \cap \{a_{r+1}, ..., a_m\}$$

Let $B_p \equiv \{a_{i_1}, ..., a_{i_n}\}$ and $b = \lambda_{i_1} a_{i_1} + ... + \lambda_{i_n} a_{i_n}$. Let C' be the vector C found in step 2 of iteration q. We have the contradiction

$$(*)\ 0 > C'^T b = C'^T(\lambda_{i_1} a_{i_1} + ... + \lambda_{i_n} a_{i_n}) = \lambda_{i_1} C'^T a_{i_1} + ... + \lambda_{i_n} C'^T a_{i_n} > 0\ (**)$$

where (*) is noted in step (2) of the simplex algorithm and (**) is done as follows. If $i_j > r$ then $C'^T a_{i_j} = 0$ which follows from the choice of C'. If $i_j = r$ then $\lambda i_j < 0$ because r was chosen in step (2) of iteration p and $C' a_{i_j} < 0$ because r was chosen in step (4) of iteration q. If $i_j < r$ then $\lambda_{i_j} \geq 0$ since r was the smallest index with $\lambda_{i_j} < 0$ in iteration p and $C'^T a_{i_j} \geq 0$ since r was the smallest index with $C^{'} a_{i_j} < 0$ in iteration q. ☐

Summary 2. Given $a_1, ..., a_m \in \mathbb{R}^n$ with rank t, only one of the two must be true [Robert Bland, 1979]:

(1) b is a non-negative combination of linearly independent vectors from $a_1, ..., a_m$

(2) There exists a hyperplane $\{x : C^T x = 0\}$ containing at least $(t-1)$ linear independent vectors from $a_1, ..., a_m$ such that $C^T a_i \geq 0$, $i = 1, ..., m$ and $C^T b < 0$.

Theorem 2.3. *A cone is polyhedral if and only if it is finitely generated (previously stated in a previous lecture).*

Proof. ($\Longleftarrow$) [A] Let $x_1, ..., x_m \in \mathbb{R}^n$ and assume $x_1, ..., x_m$ span $\mathbb{R}^n$. Otherwise, we can work in a subspace of $\mathbb{R}^n$. Consider all linear hyperplanes $\{x : C^T x = 0\}$ that are spanned by $(n-1)$ linearly independent vectors from $x_1, ..., x_m$ and have the property $C^T x_1 \geq 0, ..., C^T x_m \geq 0$. There are only finitely many such C. Call them $C^1, ..., C^l$. If $^- \in cone\{\bar{x}_1, ..., x_m\}$, then $C^{iT}\bar{x} \geq 0, \forall i = 1, ..., l$. On the other hand, if $\bar{x} \notin cone\{x_1, ..., x_m\}$, then by the fundamental theorem, there must be some $i \in \{1, ..., l\}$ such that $C^{iT}\bar{x} < 0$. Thus,

$$cone\{x_1, ..., x_m\} = \{x : C^{iT}x \geq 0, ..., C^{lT}x \geq 0\}$$

($\implies$) [B] Let $C = \{x : a_1^T x \leq 0, ..., a_m^T x \leq 0\}$. By [A], there exists vectors $b_1, ..., b_t$ such that

$$(*)\ cone\{a_1, ..., a_m\} = \{x : b_1^T x \leq 0, ..., b_t^T x \leq 0\}$$

We will show that $C = cone\{b_1, ..., b_t\}$. To do this, we first show that $cone\{b_1, ..., b_t\} \subseteq C$. This is clear because $b_i \in C$ since $b_i^T a_j \leq 0$ for all $j = 1, ..., m$ by the definition of a cone and (*).

Conversely, to show that $C \subseteq cone\{b_1, ..., b_t\}$, let $\bar{y} \in C$ and suppose $\bar{y} \notin cone\{b_1, ..., b_t\}$. By [A], $cone\{b_1, ..., b_t\}$ is polyhedral. So

$$cone\{b_1, ..., b_t\} = \{y : w^{iT}y \leq 0, ..., w^{kT}y \leq 0\}$$

for some vectors $w^1, ..., w^k$. Thus, for some i, we must have $w^{iT}\bar{y} > 0$. Note that $w^{iT}b_j \leq 0$ for all j. By (*), $w^i \in cone\{a_1, ..., a_m\}$ and thus

$$w^i = \lambda_1 a_1 + ... + \lambda_m a_m$$

where $\lambda_1 \geq 0, ..., \lambda_m \geq 0$. Hence, for each $x \in C$ we have

$$\begin{aligned} w^{iT}x &= (\lambda_1 a_1 + ... + \lambda_m a_m)^T x \\ &= \lambda_1 a_1^T x + ... + \lambda_m a_m^T x \leq 0 \end{aligned}$$

This is a contradiction since $\bar{y} \in C$ and $w^{iT}\bar{y} > 0$. ☐

Theorem 2.4. *(Caratheodory's Theorem) Let $x_1, ..., x_m \in \mathbb{R}^n$ and suppose $x \in cone\{x_1, ..., x_m\}$. Then, x can be written as a non-negative linear combination of linearly independent vectors from $x_1, ..., x_m$.*

Proof. Fundamental Theorem. (Exercise: Fill in the blanks) ☐

Lemma 2.1. *Let S be a convex set with $x_1, ..., x_m \in S$. Let $\lambda_1, ..., \lambda_m \geq 0$ with $\sum \lambda_i = 1$. Then $\sum_{i=1}^m \lambda_i x_i \in S$.*

Proof. By definition, $1 - \lambda_1 = \sum_{j=2}^m \lambda_j$ and hence

$$v = \frac{1}{1-\lambda_1}\left(\sum_{j=2}^m \lambda_j x_j\right) \in S$$

by induction. This implies $\sum_{i=1}^m \lambda_i x_i = \lambda_1 x_1 + (1-\lambda_1)v \in S$ by convexity. ☐

Corollary 2.1. *By the lemma above,*

$$Convex\_Hull(X) = \left\{\sum_{i=1}^t \lambda_i x_i, t \geq 0, x_j \in X, \lambda_j \geq 0, j \in \{1, ..., t\}, \sum_{k=1}^t \lambda_k = 1\right\}$$

Theorem 2.5. *A set P is a polyhedron if and only if P is the sum of a polytope and a cone.*

Proof. ($\Longrightarrow$) Suppose that $P = \{x : Ax \le b\}$. We show $P = Q + C$ where Q is a polytope and C is a cone. Consider the polyhedral cone

$$T = \left\{ \begin{pmatrix} x \\ \lambda \end{pmatrix} : x \in \mathbb{R}^n, \lambda \in \mathbb{R}, \lambda \ge 0, Ax - \lambda b \le 0 \right\}$$

We know that T is finitely generated by vectors $\begin{pmatrix} x_1 \\ \lambda_1 \end{pmatrix}, ..., \begin{pmatrix} x_2 \\ \lambda_2 \end{pmatrix}$ and we may scale these vectors so that for each i, $\lambda_i = 0$ or $\lambda_i = 1$. Notice that $x \in P \iff \begin{pmatrix} x \\ 1 \end{pmatrix} \in T$. If $\begin{pmatrix} x \\ 1 \end{pmatrix} \in T$ and

$$\begin{pmatrix} x_1 \\ \lambda_1 \end{pmatrix} = \gamma_1 \begin{pmatrix} x_1 \\ \lambda_1 \end{pmatrix} + ... + \gamma_m \begin{pmatrix} x_m \\ \lambda_m \end{pmatrix}, \gamma_1 \ge 0, ..., \gamma_m \ge 0$$

then $\sum(\gamma_i : \lambda_i = 1) = 1$. So $\begin{pmatrix} x \\ 1 \end{pmatrix} \in T \iff x \in \sum(\gamma_i x_i : \lambda = 0) + \sum(\gamma_i x_i : \lambda = 1)$ with $\gamma_1, ..., \gamma_m \ge 0$ and $\sum(\gamma_i : \lambda = 1) = 1$. Thus, letting C be the cone generated by $\{x_i : \lambda_i = 0\}$ and letting Q be the convex hull of $\{x_i : \lambda_i = 1\}$ we have $P = Q + C$. ◻

($\Longleftarrow$) Now suppose that $P = Q + C$ for some polytope Q and polyhedral cone C. We must show that P is a polyhedron. Let $C = cone(y_1, ..., y_t)$ and $Q = Convex\_Hull(x_1, ..., x_m)$. So $\bar{x} \in P \iff \bar{x}$ can be written as

$$\lambda_1 y_1 + ... + \lambda_t y_t + \gamma_1 x_1 + ... + \gamma_m x_m$$

with $\lambda_i, \gamma_i \ge 0$ and $\sum \gamma_i = 1$. So $\bar{x} \iff$

$$\begin{pmatrix} \bar{x} \\ 1 \end{pmatrix} = \lambda_1 \begin{pmatrix} y_1 \\ 0 \end{pmatrix} + ... + \lambda_t \begin{pmatrix} y_t \\ 0 \end{pmatrix} + \gamma_1 \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + ... + \gamma_m \begin{pmatrix} x_m \\ 1 \end{pmatrix}, \gamma_i \ge 0, \lambda_i \ge 0$$

and $\iff$

$$\begin{pmatrix} \bar{x} \\ 1 \end{pmatrix} = cone\left(\begin{pmatrix} y_1 \\ 0 \end{pmatrix}, ..., \begin{pmatrix} y_t \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, ..., \begin{pmatrix} x_m \\ 1 \end{pmatrix} \right) = S$$

But S is a polyhedral cone $S = \left\{ \begin{pmatrix} x \\ \lambda \end{pmatrix} : Ax + \lambda b \le 0 \right\}$ for some A and b. Thus,

$$\bar{x} \in P \iff \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix} \in S \iff A\bar{x} + b \le 0 \iff A\bar{x} \le -b$$

and $P = \{x : Ax \le -b\}$ which is polyhedral.

3 Linear Optimization

Theorem 3.1. *(Weak Duality Theorem) If $\bar{x}$ satisfies $Ax \le b$ and $\bar{y}$ satisfies $\bar{y}^T A = c^T, y \ge 0$ then $c^T \bar{x} \le \bar{y}^T b$.*

Proof. We have $A\bar{x} \le b$. Multiplying by $\bar{y}$ we have $\bar{y}^T A \bar{x} \le \bar{y}^T b$. By $\bar{y}^T A = c^T$ we have

$$c^T \bar{x} = \bar{y}^T A \bar{x} \le \bar{y}^T b$$

◻

Theorem 3.2. *(Duality Theorem [Von Neumann 1947]) We have*

$$\underbrace{\max(c^T x : Ax \le b)}_{\textit{Primal Problem}} = \underbrace{\min(y^T b : y^T A = c^T, y \ge 0)}_{\textit{Dual Problem}}$$

provided each of the two LP models have feasible solutions.

Proof. By Weak Duality, we need to show there exists $\bar{x}$ and $\bar{y}$ such that $c^T\bar{x} \geq \bar{y}^T b$ (which implies $c^T\bar{x} = \bar{y}^T b$). Thus, we need to show there exists a solution to

$$Ax \leq b, y^T A = c^T, c^T x \geq y^T b, y \geq 0$$

Note that $y^T A = c^T \iff A^T y = c$. Writing as a matrix,

$$\begin{matrix} u \\ \lambda \\ v \\ w \end{matrix} \begin{bmatrix} A & 0 \\ -c^T & b^T \\ 0 & A^T \\ 0 & -A^T \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} b \\ 0 \\ c \\ -c \end{bmatrix}, y \geq 0$$

By Farkas, this system has a solution if and only if $u^T b + v^T c - w^T c \geq 0$ for all $u, \lambda, v, w \geq 0$ such that $u^T A - \lambda c^T = 0$ and $\lambda b^T + v^T A^T - w^T A^T = 0$. To prove this theorem, we show that this is true via considering cases.

Case I ($\lambda > 0$): We have

$$\begin{aligned} u^T b &= b^T u = \frac{1}{\lambda}\lambda b^T u \\ &\geq \frac{1}{\lambda}(w^T - v^T)A^T u \\ &= \frac{1}{\lambda}(w^T - v^T)\lambda c \\ &= (w^T - v^T)c \end{aligned}$$

and so $u^T b - (w^T - v^T)c \geq 0$ which is what we want.

Case 2 ($\lambda = 0$)*:* Let $\bar{x}, \bar{y}$ satisfy $A\bar{x} \leq b, \bar{y}^T A = c^T, y \geq 0$. Thus, $u^T b \geq u^T Ax = \lambda c^T \bar{x} = 0$ and

$$\begin{aligned} (w^T - v^T)c &= (w^T - v^T)A^T\bar{y} \\ &\leq \lambda b^T \bar{y} = 0 \end{aligned}$$

and hence $u^T b \geq (w^T - v^T)c$ which is what we want. ☐

Theorem 3.3. *If the primal LP* $\max(c^T x : Ax \leq b)$ *has an optimal solution, the dual LP* $\min(y^T b : y^T A = 0, y \geq 0)$ *also has an optimal solution and the Duality Theorem holds.*

Proof. It suffices to show that the dual LP has a feasible solution. Suppose that the dual LP has no solution, where $A^T y = c$ and $y \geq 0$. By Farkas, there exists a solution z such that $z^T c \leq -1$ and $z^T A^T \geq 0$. That is, $Az \geq 0$ and $c^T z \leq -1$. Let x^* be an optimal solution to the primal LP. But

$$A(x^* - z) = Ax^* - Az \leq b$$

$$c^T(x^* - z) = c^T x^* - c^T z > c^T x^*$$

This is a contradiction since x^* is an optimal solution. ☐

Theorem 3.4. *(Affine Farkas' Lemma) Suppose* $c^T x \leq \delta$ *for all* x *such that* $Ax \leq b$ *and suppose there exists a solution to* $Ax \leq b$*. Then for some* $\delta' \leq \delta$ *we have that* $c^T x \leq \delta'$ *is a non-negative linear combination of* $Ax \leq b$*.*

Proof. Following the previous argument, there exists a solution to $A^T y = c, y \geq 0$. Thus, by the duality theorem, there is some $\bar{y}$ such that $\bar{y}$ is an optimal solution to

$$\min(y^T b : y^T A = c^T, y \geq 0) = \delta'$$

Thus, $\bar{y}$ gives the non-negative combinations of $Ax \leq b$ where

$$\bar{y}^T Ax \leq \bar{y}^T b \implies c^T x \leq \delta' \leq \delta$$

and $\bar{y}$ gives the non-negative combination of $Ax \leq b$. ☐

Proposition 3.1. *Suppose that $\bar{x}$ and $\bar{y}$ are feasible solutions to the primal and dual LPs respectively. Then the following are equivalent.*

1) $\bar{x}$ and $\bar{y}$ are optimal solutions

2) $c^T\bar{x} = \bar{y}^T b$

3) If a component of $\bar{y}$ is positive, then the corresponding inequality $Ax \leq b$ is satisfied by $\bar{x}$ as an equation. That is $\bar{y}^T(b - A\bar{x}) = 0$

*In (3), we can say that being an optimal solution is equivalent to the **complementary slackness conditions (CSC)** which are for each $j = 1, ..., m$ either $\bar{y}_j = 0$ OR $a_j^T\bar{x} = b_j$.*

Proof. (1) $\iff$ (2) Use the Duality Theorem.

(2) $\implies$ (3) We have

$$\begin{aligned} c^T x = y^T A\bar{x} \leq \bar{y}^T b \quad &\implies \quad c^T\bar{x} = y^T b \iff \bar{y}^T A\bar{x} = \bar{y}^T b \\ &\iff \quad \bar{y}^T A\bar{x} - \bar{y}^T b = 0 \\ &\iff \quad \bar{y}^T(A\bar{x} - b) = 0 \end{aligned}$$

(3) $\implies$ (2) Same proof. □

Theorem 3.5. *(Motzkin's Transposition Theorem) There exists a vector x with $Ax < b$, $Bx \leq c$ iff for all vectors $y \geq 0, z \geq 0$,*

(i) If $y^T A + z^T B = 0$ then $y^T b + z^T c \geq 0$.

(ii) If $y^T A + z^T B = 0, y \neq 0$, then $y^T b + z^T c > 0$

Proof. It is easy to see that the conditions (i) and (ii) are necessary ($\implies$ is done). Now suppose that (i) and (ii) hold. By Farkas, we know there exists a solution x to $Ax \leq b$ and $Bx \leq c$. Notice that (ii) implies that for each inequality $a_i^T x \leq b_i$ in $Ax \leq b$ there is no solution to

$$y \geq 0, z \geq 0, y^T A + z^T B = -a_i^T, y^T b + z^T c \leq -b_i$$

This implies that there exists a vector x^i with

$$Ax^i \leq b, Bx^i \leq c, a_i^T x^i < b_i$$

(See Assignment 2 for details). The **barycentre** $\bar{x} = \frac{1}{m}(x^1 + ... + x^m)$ satisfies

$$A\bar{x} < b, B\bar{x} \leq c$$

which is what we wanted. □

Lemma 3.1. *We have $y \in Char\_Cone(P) \iff \exists x \in P$ with $x \in \lambda y \in P$ for any $\lambda \geq 0$.*

Proof. Let $y \in Char\_Cone(P)$. Let $x \in P$. Thus, $x + ky \in P$ for all $k = 1, 2, ...$ Since P is convex, $x + ky \in P$ for all $\lambda \geq 0$. Let $x \in P$ and let y be a vector such that $x + \lambda y \in P$ for all $\lambda \geq 0$. Let $Ax \leq b$ be a system such that $P = \{x : Ax \leq b\}$. Then we must have $Ay \leq 0$. That is, if $a_i^T y > 0$ then for large enough λ we would have $a_i^T(x + \lambda y) > b_i$. Thus, for any $\bar{x} \in P$ we have $A(\bar{x} + \bar{y}) = A\bar{x} + A\bar{y} \leq b$. □

Lemma 3.2. *Let $x^1, ..., x^m \in \mathbb{R}^n$ and let $w \in \mathbb{R}^n$. If $x^1, ..., x^m$ are affinely independent then $x^1 - w, ..., x^m - w$ are affinely independent.*

Proof. Suppose

$$\begin{cases} \sum_{i=1}^m \lambda_i(x^i - w) & = 0 \\ \sum_{i=1}^m \lambda_i & = 0 \end{cases}$$

We have

$$\sum_{i=1}^m \lambda_i(x^i - w) = \sum_{i=1}^m \lambda_i x^i - w \underbrace{\left(\sum_{i=1}^m \lambda_i\right)}_{=0} = \sum_{i=1}^m \lambda_i x^i = 0$$

and hence $\lambda_1 = ... = \lambda_m = 0$. □

Lemma 3.3. *We have*

$$Affine\_Hull(P) = \{x : A^= x = b^=\} = \{x : A^= x \leq b^=\}$$

Proof. (1) $[Affine\_Hull(P) \subseteq \{x : A^= x = b^=\}]$ By definition $P \subseteq \{x : A^= x = b^=\}$. Suppose that $\bar{x} = \lambda_1 x^1 + ... + \lambda_t x^t$ with $x^1 + ... + x^m \in P$ and $\lambda_1 + ... + \lambda_t = 1$. Then,

$$A^= \bar{x} = \lambda_1 A^= x^1 + ... + \lambda_t A^= x^t = \lambda_1 b^= + ... + \lambda_t b^= = b^=$$

(2) $[\{x : A^= x = b^=\} \subseteq \{x : A^= x \leq b^=\}]$ Trivial by definition.

(3) $[\{x : A^= x \leq b^=\} \subseteq Affine\_Hull(P)\}]$ Let $\bar{x}$ satisfy $A^= \bar{x} \leq b^=$. Let $x' \in P$ be such that $A^= x' = b^=, A^+ x' < b$. If $\bar{x} = x'$ then $\bar{x} \in P \implies \bar{x} \in Affine\_Hull(P)$. If $\bar{x} \neq x'$, then the line segment connecting $\bar{x}$ and x' contains more that one point in P. Therefore, the affine hull of P contains the entire line through x' and $x \implies \bar{x} \in Affne\_Hull(P)$. □

Theorem 3.6. *F is a face of $P \iff F \neq \emptyset$ and $F = \{x \in P : A'x = b'\}$ for some subsystem $Ax' \leq b'$ of $Ax \leq b$.*

Proof. ($\implies$) Suppose $F = P \cap \{x : c^T x = \delta\}$. Consider the LP problem $\max(c^T x : Ax \leq b)$. Since $c^T x \leq \delta$ is valid, this LP has a finite optimal value. By the duality theorem, there exists an optimal solution to $\min(y^T b : y^T A = c^T, y \geq 0)$. Let y^* be an optimal solution. Let $I = \{i : y_i^* > 0\}$. By the CSC, a vector $\bar{x}$ is optimal for the primal LP $\iff a_i^T \bar{x} = b_i$ for all $i \in I$.

But F is the set of optimal solutions to the primal LP. Thus, $F = \{x \in P : A'x = b\}$ where $A'x = b'$ are the equations $a_i^T x = b_i$ for any $i \in I$.

($\Longleftarrow$) Suppose $F = \{x \in P : A'x = b'\}$. We want to construct c such that $\max(c^T x : Ax \leq b) = F$. Let c be the sum of the rows of A'. Then every optimal solution satisfies $Ax' = b$ (since every $x \in P$ satisfies $A'x \leq b'$). □

Algorithm 1. (Simplex Algorithm) The standard algorithm works with the standard form A (an $m \times n$ matrix) where we are solving the problem

$$\begin{array}{rcl} \min & & c^T x \\ Ax & = & b \\ x & \geq & 0 \end{array}$$

We will equivalently denote $x = X$. Let B be an ordered set of indices $\{B_1, ..., B_m\}$from $\{1, ..., n\}$. B is called a **basis header** and determines a basis $\mathbb{B} = A_B$ consisting of columns $A_{B_1}, ..., A_{B_m}$ if $\mathbb{B}$ is non-singular. N denotes the non-basic variables $\{1, ..., n\} \backslash B$. We then have the new algorithm

$$\begin{array}{rcl} \min & & C_B^T X_B + C_N^T X_N \\ A_B X_B + A_N X_N & = & b \\ X_B \geq 0 & & X_N \geq 0 \end{array}$$

B is **primal feasible** if $\mathbb{B}^{-1}b \geq 0$. In a general iteration of the (revised) primal simplex algorithm, we have a primal feasible B and vectors

$$X_B = \mathbb{B}^{-1}b \text{ and } D_N = C_N - A_N^T \left(\mathbb{B}^{-1}\right)^T C_B$$

The steps are the following.

(1) [Pricing] If $D_N \geq 0$ then B is optimal and you stop. Otherwise let

$$j = \text{argmin}(D_k : k \in N)$$

where variable X_j is the **entering variable**.

(2) [FTRAN] Solve $\mathbb{B}y = A_j$ (column of A)

(3) [Ratio Test] If $y \leq 0$ then the LP is unbounded and we stop. Otherwise, let

$$i = \text{argmin}(\left[X_B\right]_k / y_k : y_k > 0, k = 1, ..., m)$$

where the variable $\left[X_B\right]_i$ is the **leaving variable**.

(4) [BTRAN] Solve $\mathbb{B}^T z = e_i$ where e_i is the i^{th} unit vector.

(5) [Update] Compute $\alpha_N = -A_N^T z$. Set $B_i = j$. Update X_B (using y) and update D_N (using α_N).

4 Linear Integer Programming

Theorem 4.1. *(Meyer 1974) If P is a rational polyhedron, then P_I is a polyhedron.*

Proof. Write $P = Q + C$ with Q a polytope and C a cone. We have $C = \{\lambda_1 d_1 + ... + \lambda_s d_s \geq 0\}$ with $d_1, ..., d_s$ integer vectors. Let B be the bounded set

$$B = \{\lambda_1 d_1 + ... + \lambda_s d_s : 0 \leq \lambda_i \leq 1, i = 1, ..., s\}$$

We claim that $P_I = (Q + B)_I + C$. We are done because since $Q + B$ is bounded, $(Q + B)_I$ is a polytope, thus P_I is a polyhedron. To prove this claim, let $p \in P \cap \mathbb{Z}^n$. Then $p = q + c$ for some $q \in Q$ and $c \in C$. It follows that $c = b + c'$ with $b \in B$ and $c' \in C \cap \mathbb{Z}^n$. So $p = q + b + c'$ and $q + b$ is integral. This implies

$$p \in (Q + B)_I + C \implies P_I \subseteq (Q + B)_I + C$$

The other direction is

$$(Q + B)_I + C \subseteq P_I + C = P_I + C_I \subseteq (P + C)_I = P_I$$

□

Theorem 4.2. *(Schrijver) If P is rational, then P' is a rational polyhedron.*

Proof. (Sketch) Write $P = \{x : Ax \leq b\}$ with A and b integer valued. We obtain a C-G cut for each $y \geq 0$ such that $y^T A$ is integer valued, where

$$\begin{array}{rcl} a_1^T x & \leq & b_1 \\ a_2^T x & \leq & b_2 \\ & \vdots & \\ a_m^T x & \leq & b_m \end{array}$$

and

$$(a_1^T y_1 + a_2^T y_2 + ... + a_m^T y_m)x \leq b_1 y_1 + ... + b_m y_m$$

The C-G cut is

$$\underbrace{(a_1^T y_1 + a_2^T y_2 + ... + a_m^T y_m)}_{w^T} x \leq \underbrace{\lfloor b_1 y_1 + ... + b_m y_m \rfloor}_{t}$$

If $y_1 \geq 1$, look at the cut obtained by

$$\begin{aligned} y_1' &= y_1 - 1 \\ y_2' &= y_2 \\ &\vdots \\ y_m' &= y_m \end{aligned}$$

The new cut is

$$(w - a_1)^T x \leq t - b_1$$

but every $\bar{x} \in P$ that satisfies the new cut also satisfies $w^T x \leq t$, so we only need C-G cuts such that $0 \leq y \leq 1$ and $y^T A$ integer valued. There are only finitely many such vectors y so we only need finitely many C-G cuts. Hence P' is a polyhedron. □

Theorem 4.3. *(Chvatal's Theorem) If P is rational, then there exists k such that $P^{(k)} = P_I$.*

Proof. (Rough Sketch: RE-CHECK FOR FINAL) P_I is a polyhedron defined as $P_I = \{x : Mx \leq d\}$. Let $w^T x \leq t$ be an inequality in $Mx \leq d$. It suffices to show that for some k we have

$$P^{(k)} = (...((P')')...')' \subseteq \{x : w^T x \leq t\}$$

Now let $\delta = \max\{w^T x : x \in P\}$. Thus, $w^T x \leq \lfloor \delta \rfloor$ is a C-G cut. Suppose for large enough k we know that $w^T x \leq q$ is valid for $P^{(k)}$. It suffices to show that for some $k' > k$ we have $w^T x < q$ is valid for $P^{(k')} \implies w^T x \leq q - 1$ is valid for $P^{(k'+1)}$. Let $F = \{x \in P : w^T x = q\}$. By induction on the dimension of the polyhedron, we can assume there exists l such $F^{(l)} = \emptyset$. Applying these cutting planes to the polyhedron $P \cap \{x : w^T x \leq q\}$ we obtain a polyhedron such that $w^T x < q$ is valid. □

Theorem 4.4. *(Edmonds & Giles) Rational P is an integer polyhedron $\iff$ every supporting hyperplane of P contains integral vectors.*

Proof. ($\implies$) Easy, since intersection of a supporting hyperplane of P contains integral vectors.

($\impliedby$) Follows from Integer Farkas Lemma □

Theorem 4.5. *Rational (polyhedron) P is an integer polyhedron $\iff$ for each integral w such that $\max(w^T x : Ax \leq b)$ exists, the value $\max(w^T x : Ax \leq b)$ is an integer.*

Proof. ($\implies$) Easy, since x^* is integer and so $w^T x^*$ is integer.

($\impliedby$) Follows from above theorem and the fact that if w has relatively prime integer components, then $w^T x = \delta$ has an integer solution for any integer δ. □

Theorem 4.6. *$Ax \leq b$ is TDI $\iff$ $\forall$ faces $F = \{x : A^0 x = b^0, A' x \leq b'\}$ the rows of A^0 form a Hilbert basis (HB).*

Proof. Follows from complementary slackness conditions (CSS). □

Theorem 4.7. *If C is a rational cone, then $\exists$ an integral H.B. that generates C.*

Proof. Consider $C = Cone(d_1, ..., d_k)$ with $d_1, ..., d_k$ integral vectors. Let $H = \{a_1, ..., a_t\}$ be the set of integral vectors in the bounded set

$$\{\lambda_1 d_1 + ... + \lambda_k d_k : 0 \leq \lambda_i \leq 1, i = 1, ..., k\}$$

Note $H \subseteq C$ and $d_1, ..., d_k \in H$. So H generates C. Let $b \in C \cap \mathbb{Z}^n$. Then $b = \mu_1 d_1 + ... + \mu_k d_k$ for some $\mu_i \geq 0$. Write this as

$$\underbrace{b}_{\in \mathbb{Z}} = \underbrace{\lfloor \mu_1 \rfloor d_1 + ... + \lfloor \mu_k \rfloor d_k}_{\in \mathbb{Z}} + \underbrace{(\mu_1 - \lfloor \mu_1 \rfloor) d_1 + ... + (\mu_k - \lfloor \mu_k \rfloor) d_k}_{\in H}$$

Since b is a non-negative combination of vectors in H, H is a Hilbert basis. □