# AMATH 350 Final Exam Review

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## Solving Linear DEs

Separable equations: $\frac{dy}{dx} = g(x)h(y) \implies \int \frac{1}{h(y)}dy = \int g(x)dx$

Integrating Factor: $\frac{dy}{dx} + p(x)y = q(x) \implies \frac{d}{dx}[\mu(x)y(x)] = \mu(x)q(x)$ where $\mu(x) = e^{\int p(x)dx}$

Homogeneous Equations (Char. Eqns.)

1. Distinct roots $m = r_1, r_2 \implies y = c_1e^{r_1x} + c_2e^{r_2x}$
2. Complex conjugates $m = \alpha \pm i\beta$ (use 'complete the square') $\implies y = e^{\alpha x}[A\cos\beta x + B\sin\beta x]$
3. Repeated roots $m = r \implies y = c_1e^{rx} + c_2xe^{rx}$

Inhomogeneous Equations (Method of Undetermined Eqns)

| Forcing Term | Trial Function |
|---|---|
| $e^{kx}$ | $Ae^{kx}$ |
| $\sin kx$,$\cos kx$ | $A\cos kx + B\sin kx$ |
| $x^n$ | $\sum_{k=0}^{n} A_kx^k$ |
| $xe^x$ | $(Ax+B)e^x$ |

Inhomogeneous Equations (Variation of Params)

- In the 1st order equation $y' + p(x)y = F(x)$, suppose that we have the homogeneous solution $y_h = c_1y_1$ then we try $y_p = u_1y_1$ and substitute $y = y_p$ and $y' = y_p'$ in the original DE and solve for $u$. Include the coefficient of integration to get the general solution.
- In the 2nd order equation $y'' + p(x)y' + q(x)y = F(x)$, suppose that we have the homogeneous solution $y_h = c_1y_1 + c_2y_1$ then we try $y_p = u_1y_1 + u_2y_2$ and we need (1) $u_1'y_1 + u_2'y_2 = 0$, (2) $u_1'y_1' + u_2'y_2' = F(x)$.

Inhomogeneous Equations (Reduction in Order)

Suppose that you have found a solution $y = y_1$ to the original DE. Then one can guess another solution $y_2 = uy_1$ to the DE and take derivatives up to the original order of the DE. One can then solve for $u$ when you plug these values back into the DE.

## Special Substitutions

1. The form $y' = f(ax+by) \implies$ replace $y(x)$ with $u(x)$ where $u = ax + by$
2. The form $y' = f\left(\frac{y}{x}\right)$ or $y' = f\left(\frac{x}{y}\right) \implies$ use $u = \frac{y}{x}$ where $\frac{dy}{dx} = x\frac{du}{dx} + u$
3. The form $\frac{dy}{dx} + p(x)y = q(x)y^n \implies$ multiply the original by $y^{-(n-1)}$ and use $v = y^{1-n} = y^{-(n-1)}$ where $\frac{dv}{dx} = -(n-1)y^{-n}\frac{dy}{dx}$

## Boundary Value Problems

- These are problems where we want to know what values of $k$ a certain homogeneous DE, like $y'' + ky = 0, y(0) = 0, y(1) = 0$ has solutions
- To do this, you examine the characteristic equation as a function of $k$ and check "interesting" cases for $k$
- We then calculate the eigenvalues (valid $k$ values) and eigenfunctions (valid functions implied by the eigenvalues)

## Graphing Solutions

- Solve DE if possible
- Identify any **exceptional solutions** which behave differently from the rest (usually set $C = 0$)
- Consider the behaviour of the other solutions as $x \to \pm\infty$ or near vertical asymptotes
- Set $\frac{dy}{dx} = 0$ in the DE to find the **horizontal isocline**
- Determine how $\frac{dy}{dx}$ behaves outside of the horizontal isocline

## Models

Malthusian Model: $\frac{dP}{dt} = rP$

Logistic Model: $\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$

# Vector DEs

Homogeneous Vector DEs

These are solved by finding the eigenvalues $\lambda_i$ and (realized) eigenvectors $v_i$ of $A$ in $\frac{dx}{dt} - Ax = 0$. We then break the process into cases.

- If $\lambda_{1,2}$ real distinct, then
$$x = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$
- If $\lambda_{1,2} = a \pm ib$ complex conjugate distinct, then
$$x = e^a \left(c_1 \text{cis}(b) v_1 + c_2 \text{cis}(b) v_2\right)$$
where $v_2 = \bar{v}_1$.
- If $\lambda_1$ identical, then
$$x = c_1 v_1 e^{\lambda_1 t} + c_2 \left(v_1 t e^{\lambda_1 t} + w_1 e^{\lambda_1 t}\right)$$
where $w_1$ is such that $(A - \lambda_1 I)v_1 = w_1$. In the 3 by 3 case,

# Famous PDEs

Heat equation: $u_t = \gamma u_{xx}$
Wave equation: $u_{tt} = \alpha^2 u_{xx}$
Laplace's equation: $u_{xx} + u_{yy} = 0$

# Solving Linear PDEs

First Order PDEs (Method of Characteristics)

In the general form
$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y)$$
we want to solve $\frac{dy}{dx} = \frac{b}{a}$ for the constant of integration $\phi(x,y)$. We then set $\xi = x$ and $\eta = \phi(x,y)$, otherwise $\eta = y$ and $\xi = C(x,y)$. The new equation, under the change of basis becomes
$$\left[a\xi_x + b\xi_y\right]\hat{u}_\xi + \left[a\eta_x + b\eta_y\right]\hat{u}_\eta + c\hat{u} = f$$
under the chain rule $u_x = \hat{u}_\xi \xi_x$. This comes from the Lemma that says:

**Lemma.** *Consider the ODE* $\frac{dy}{dx} = f(x,y)$. *If its general solution can be written in implicit form as* $\phi(x,y) = K$ *then*
$$\frac{\phi_x}{\phi_y} = -\frac{dy}{dx} \implies \frac{\phi_x}{\phi_y} = -f(x,y)$$

Second Order PDEs (Method of Characteristics

A 2nd-order linear PDE is 2 variables has the form
$$\begin{aligned} a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} & \\ +d(x,y)u_x + e(x,y)u_y + f(x,y)u &= g(x,y) \end{aligned}$$
Analogous to the above method, we want to solve
$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

- If $b^2 - 4ac > 0$ (**hyperbolic equation**) then we may choose $\xi = \phi_1$ and $\eta = \phi_2$ where $\phi_1 = K_1$ and $\phi_2 = K_2$ are the general solutions to
$$\frac{dy}{dx} = \frac{b - \sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{dy}{dx} = \frac{b + \sqrt{b^2 - 4ac}}{2a}$$
- If $b^2 - 4ac = 0$ (**parabolic equation**) then we can set $\xi = \phi(x,y)$ where $\phi = K$ is the general solution to $\frac{dy}{dx} = \frac{b}{2a}$ and this will eliminate $\hat{u}_{\xi\xi}$.
- If $b^2 - 4ac < 0$ (**elliptic equation**) then we cannot eliminate $\hat{u}_{\xi\xi}$ or $\hat{u}_{\eta\eta}$.
- Here, we have:
$$\begin{aligned} A &= a\left(\xi_x\right)^2 + b\xi_x\xi_y + c\left(\xi_y\right)^2 \\ B &= 2a\xi_x\eta_x + b\left[\xi_x\eta_y + \eta_y\xi_x\right] + 2c\xi_y\eta_y \\ C &= a\left(\eta_x\right)^2 + b\eta_x\eta_y + c\left(\eta_y\right)^2 \end{aligned}$$

Separation of Variables

We *assume* that $u(x,t)$ can be expressed as $F(x)G(t)$. Then, we plug in the values for $u_{[x][t]}$ for any combination of $[x][t]'s$ and using equality, we should have something similar of the form
$$\frac{G^{(n)}(t)}{G(t)} = \frac{F^{(m)}(x)}{F(x)}$$
where both sides must equal a constant since this expression must hold for all $x$ and $t$. We then get a system of equations where we can then solve for $F$ and $G$ using any initial conditions that we may have.

Fourier Transform

We can take the Fourier transform $\mathcal{F}\{u\} = \int_{-\infty}^{\infty} ue^{-i\omega x}dx$, as well as use the inverse transform $\mathcal{F}^{-1}\{u\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} ue^{i\omega x}d\omega$, of any PDE and use the following properties to make our lives easier:

1. $\mathcal{F}\left\{f^{(n)}(x)\right\} = (i\omega)^n \hat{f}(\omega)$
2. $\mathcal{F}\{u_t\} = \hat{u}_t$
3. $\mathcal{F}\{f(x-a)\} = e^{-i\omega a}\hat{f}(\omega)$

The Fourier transform should reduce the system into some ODE which we can solve. After solving, we then convert $\hat{f}$ back using the inverse transform.