AMATH 350 Final Exam Review

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Black-Scholes PDE

The following are answers to the questions provided by Dr. Harmsworth regarding the Black-Scholes PDE.

(1) The behaviour of S, or rather the change in price, is determined by a deterministic growth or decay and a random change due to external factors.

Here, S is the stock price, μ is the drift or growth rate, t is the time index and σ a volatility coefficient which represents how large are the random changes usually. $\Delta W(t)$ here also represents a random variable with mean 0 and variance Δt.

(2) We have

$$\begin{aligned}
\Delta F &= F_S\left[\mu S\Delta t + \sigma S\Delta W(t)\right] + F_t\Delta t + \frac{1}{2}F_{SS}\left[\mu S\Delta t + \sigma S\Delta W(t)\right]^2 + F_{St}\left[\mu S\Delta t + \sigma S\Delta W(t)\right]\Delta t \\
&= F_S\mu S\Delta t + F_S\sigma S\Delta W(t) + F_t\Delta t + \frac{1}{2}F_{SS}\mu^2 S^2\Delta t^2 + F_{SS}\mu\sigma S^2\Delta t\Delta W(t) + \\
&\quad + \frac{1}{2}F_{SS}\sigma^2 S^2\left[\Delta W(t)\right]^2 + F_{St}\mu S(\Delta t)^2 + F_{St}\sigma S\Delta W(t)\Delta t + \frac{1}{2}F_{tt}(\Delta t)^2 + ...
\end{aligned}$$

From here, we neglect all the terms of order $(\Delta t)^2$ and $\Delta t\Delta W(t)$ since $\Delta W(t)$ is small. This leaves

$$\Delta F = \mu S F_S\Delta t + \sigma S F_S\Delta W(t) + F_t\Delta t + \frac{1}{2}\sigma^2 S^2 F_{SS}\Delta W(t) + ...$$

Finally, since $[\Delta W(t)]^2$ has mean Δt and small variance, we can approximate if by Δt to get

$$\Delta F = \sigma S F_S\Delta W(t) + \left[\mu S F_S + F_t + \frac{1}{2}\sigma^2 S^2 F_{SS}\right]\Delta t$$

and hence the stochastic equation is

$$dF = \sigma S F_S dW(t) + \left[\mu S F_S + F_t + \frac{1}{2}\sigma^2 S^2 F_{SS}\right]dt$$

(3) Construct a portfolio $\Pi(t) = F - \epsilon S$ under the assumption of short selling. Then $d\Pi = dF - \epsilon dS$ under the assumption of divisible stocks. Hence, by substitution,

$$d\Pi = \sigma S F_S dW(t) + \left[\mu S F_S + F_t + \frac{1}{2}\sigma^2 S^2 F_{SS}\right]dt - \epsilon(\mu S dt + \sigma S dW(t))$$

Set $\epsilon = F_S$ to get $\frac{d\Pi}{dt} = F_t + \frac{1}{2}\sigma^2 S^2 F_{SS}$. Assuming that arbitrage cannot exist, Π must have the same value as an investment at an interest rate r and so we also must have that $\frac{d\Pi}{dt} = r\Pi$. So,

$$\begin{aligned}
r\Pi = F_t + \frac{1}{2}\sigma^2 S^2 F_{SS} \quad &\Longrightarrow \quad r(F - SF_s) = F_t + \frac{1}{2}\sigma^2 S^2 F_{SS} \\
&\Longrightarrow \quad rF = \frac{1}{2}\sigma^2 S^2 F_{SS} + rSF_S + F_t
\end{aligned}$$

(4) The ICs and BCs are:

$$F(S,T) = (S-K)^+, F(0,t) = 0, F \to S \text{ as } S \to \infty$$

(5) We make these substitutions so:

- We can normalize the coefficient of S^2F_{SS}.
- We reverse time by letting $\tau = T - t$ which turns our final condition into a true initial condition.
- Replacing F with $v = \frac{F}{K}$ and S with $\frac{S}{K}$, we'll eliminate K from the problem.
- Replacing $\frac{S}{K}$ with e^x $(S = Ke^x)$ turns our PDE into a constant coefficient equation

This reduces our equation into the form

$$v_\tau = v_{xx} + \left(\frac{2r}{\sigma^2} - 1\right) v_x - \frac{2r}{\sigma^2} v$$

where a final substitution will turn this equation into the Heat Equation.

(6) We then have

$$\begin{aligned}
v &= e^{\left[\frac{1-\delta}{2}\right]x - \left[\frac{(1-\delta)^2}{4}+\delta\right]y}[u] \\
v_x &= e^{\left[\frac{1-\delta}{2}\right]x - \left[\frac{(1-\delta)^2}{4}+\delta\right]y}\left[\left(\frac{1-\delta}{2}\right)u + u_x\right] \\
v_y &= e^{\left[\frac{1-\delta}{2}\right]x - \left[\frac{(1-\delta)^2}{4}+\delta\right]y}\left[\left(-\frac{(1-\delta)^2}{4} - \delta\right)u + u_y\right] \\
v_{xx} &= e^{\left[\frac{1-\delta}{2}\right]x - \left[\frac{(1-\delta)^2}{4}+\delta\right]y}\left[\left(\frac{1-\delta}{2}\right)^2 u + (1-\delta)u_x + u_{xx}\right]
\end{aligned}$$

Let $K = e^{\left[\frac{1-\delta}{2}\right]x - \left[\frac{(1-\delta)^2}{4}+\delta\right]y}$. The DE can then be written equivalently as

$$\underbrace{\left(\frac{1-\delta}{2}\right)^2 u + (1-\delta)u_x + u_{xx}}_{K^{-1}v_{xx}} \underbrace{-\frac{(1-\delta)^2}{2}u - (1-\delta)u_x}_{+K^{-1}(\delta-1)v_x} + \underbrace{\left(\frac{(1-\delta)^2}{4} + \delta\right)u - u_y}_{-K^{-1}v_y} - \underbrace{\delta u}_{K^{-1}\delta v} = 0$$

and in reduced form, this is

$$u_{xx} = u_y \iff u_{xx} - u_y = 0$$

which is the heat equation with coefficient $\alpha = 1$ and what we expected.

(7) Given that

$$u_t = \gamma u_{xx}, u(x,0) = f(x)$$

We'll use the FT method. Let $\hat{u}(\omega, t) = \mathcal{F}\{u(x,t)\}$. Then $u_t \mapsto \hat{u}_t$ and $u_{xx} \mapsto (i\omega)^2\hat{u} = -\omega^2\hat{u}$. Also, $u(x,0) = f(x) \mapsto \hat{u}(\omega,0) = \hat{f}(\omega)$. The BVP then becomes

$$\hat{u}_t = -\gamma\omega^2\hat{u}, \hat{u}(\omega,0) = \hat{f}(\omega)$$

Solving this IVP gives us

$$\begin{aligned}
\hat{u}(\omega,t) = H(\omega)e^{-\gamma\omega^2 t}, H(\omega) = \hat{f}(\omega) \quad &\Longrightarrow \quad \hat{u}(\omega,t) = \hat{f}(\omega)e^{-\gamma\omega^2 t} \\
&\Longrightarrow \quad u(x,t) = \mathcal{F}^{-1}\{\hat{f}(\omega)e^{-\gamma\omega^2 t}\} = f(x) * \mathcal{F}^{-1}\{e^{-\gamma\omega^2 t}\}
\end{aligned}$$

We still have to calculate

$$\begin{aligned}
\mathcal{F}^{-1}\{e^{-\gamma\omega^2 t}\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\gamma\omega^2 t}e^{i\omega x}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(\gamma\omega^2 t - i\omega x)}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\gamma t\left(\left[\omega - \frac{ix\omega}{2\gamma t}\right]^2 + \frac{x^2}{4\gamma^2 t^2}\right)}d\omega \\
&= \frac{1}{2\pi}e^{-\frac{x^2}{4\gamma t}}\int_{-\infty}^{\infty} e^{-\gamma t\left[\omega - \frac{ix}{2\gamma t}\right]^2}d\omega
\end{aligned}$$

Now let $v = \sqrt{\gamma t}\left(\omega - \frac{ix}{2\gamma t}\right) \implies dv = \sqrt{\gamma t}d\omega$. We get that

$$\mathcal{F}^{-1}\{e^{-\gamma\omega^2 t}\} = \frac{1}{2\pi}e^{-\frac{x^2}{4\gamma t}} \underbrace{\int_{-\infty}^{\infty} e^{-v^2}\frac{dv}{\sqrt{\gamma t}}}_{=\sqrt{\pi}/\sqrt{\gamma t}} = \frac{1}{2\sqrt{\pi\gamma t}}e^{-\frac{x^2}{4\gamma t}}$$

Call this $g(x) = \mathcal{F}^{-1}\{e^{-\gamma\omega^2 t}\}$. So

$$u(x,t) = f(x) * g(x) \quad = \quad \frac{1}{2\sqrt{\pi\gamma t}}\int_{-\infty}^{\infty} f(x-\tau)e^{-\frac{\tau^2}{4\gamma t}}d\tau$$

Let $y = -\frac{\tau}{2\sqrt{\gamma t}} \implies dy = -\frac{d\tau}{2\sqrt{\gamma t}}$ with $y \to \pm\infty \implies \tau \to \mp\infty$. So

$$\begin{aligned} u(x,y) &= \frac{1}{2\sqrt{\pi\gamma t}}\int_{\infty}^{-\infty} f(x+2y\sqrt{\gamma t})e^{-y^2}\left(-2\sqrt{\gamma t}dy\right) \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x+2y\sqrt{\gamma t})e^{-y^2}dy \end{aligned}$$

(8) Note that

$$e^{\beta(x+2\sqrt{\tau}y)} - e^{\alpha(x+2\sqrt{\tau}y)}, \beta > \alpha \iff x + 2\sqrt{\tau}y > 0$$

and hence the lower limit of the integral may be replaced with $-\frac{x}{2\sqrt{\tau}}$ and the integral

$$u(x,\tau) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-y^2}\left[e^{\beta(x+2\sqrt{\tau}y)} - e^{\alpha(x+2\sqrt{\tau}y)}\right]^+ dy$$

may be split into the two parts

$$u(x,\tau) = \frac{1}{\sqrt{\pi}}e^{\beta x}\int_{-\frac{x}{2\sqrt{\tau}}}^{\infty} e^{2\beta\sqrt{\tau}y-y^2}dy - \frac{1}{\sqrt{\pi}}e^{\alpha x}\int_{-\frac{x}{2\sqrt{\tau}}}^{\infty} e^{2\alpha\sqrt{\tau}y-y^2}dy$$

(9) Completing the square gives us

$$I = \frac{e^{\beta x+\beta^2\tau}}{\sqrt{\pi}}\int_{-\frac{x}{2\sqrt{\tau}}}^{\infty} e^{-(y-\beta\sqrt{\tau})^2}dy$$

We then make the substitution

$$z = -\sqrt{2}(y - \beta\sqrt{\tau}) = -\sqrt{2}y - \beta\sqrt{2\tau}) \implies dz = -\sqrt{2}dy$$

This gives us

$$I = \frac{e^{\beta x+\beta^2\tau}}{\sqrt{\pi}}\int_{\frac{x+2\beta\tau}{\sqrt{2\tau}}}^{-\infty} e^{-\frac{z^2}{2}}\left(-\frac{dz}{\sqrt{2}}\right) = e^{\beta x+\beta^2\tau}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{x+2\beta\tau}{\sqrt{2\tau}}} e^{-\frac{z^2}{2}}dz = e^{\beta x+\beta^2\tau}\Phi\left(\frac{x+2\beta\tau}{\sqrt{2\tau}}\right)$$

(10) The following are assumptions we made in deriving the Black-Scholes PDE:

- Continuous Trading (construction of the portfolio)
- Divisible Assets (behaviour of the stock)
- No Transaction Costs (construction of the portfolio)
- Short-Selling Permitted (construction of the portfolio)
- No Dividend (behaviour of the return of the stock)
- $S(t)$ is stochastic (behaviour of the return of the stock)
- Can always invest at constant interest rate r (removal of stochastic term)
- No Arbitrage (