ACTSC 446 Final Exam Review

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Review of Derivatives

Forwards

- Prepaid Forward: $F_{0,T}^P = S_0 - PV(\text{All dividends paid over } (0,T))$
- Standard Forward Contract: $F_{0,T} = FV(F_{0,T}^P) = F_{0,T}^P e^{rT}$
- Synthetic Forward (example):

Cash Flows		
	Time 0	Time T
(i) Buy $e^{-\delta T}$ shares of stock	$-S_0 e^{-\delta T}$	S_T
(ii) Borrow $S_0 e^{-\delta T}$ at the risk-free rate	$S_0 e^{-\delta T}$	$S_0 e^{(r-\delta)T}$
Net	0	$S_T - S_0 e^{(r-\delta)T}$

Put-Call Parity

- Basic form: Call Price − Put Price $= PV_{0,T}$(Forward Price − Strike Price)
- Mathematical expression: $C(K,T) - P(K,T) = PV_{0,T}(F_{0,T} - K) = F_{0,T}^P - Ke^{-rT} = S_0 - PV_{0,T}(\text{Dividends} + K)$
- European case: $C_{Euro}(K,T) - P_{Euro}(K,T) = S_t - Ke^{-r(T-t)}$
- Concave Up/ Convexity Condition: $\frac{K_3-K_2}{K_3-K_1}V(K_1) + \frac{K_2-K_1}{K_3-K_1}V(K_3) \geq V(K_2)$ where the coefficients are weights to build an arbitrage

Swaps

- For cash flows $\{c_i\}_{i=1}^n$, the swap price is at level R where R satisfies $\sum_{t=1}^N \frac{c_t}{(1+i_t)^t} = \sum_{t=1}^N \frac{R}{(1+i_t)^t}$.

Discrete Time Securities Market

Binomial Lattice (Replicating Portfolio)

- By definition,

$$\triangle = \frac{C_u - C_d}{S_u - S_d}, B = e^{-rh} \cdot \frac{uC_d - dC_u}{u-d}, q = \frac{e^{(r-\delta)h} - d}{u-d}$$

- Alternatively, we compute Δ and set

$$B = \underbrace{e^{-rh}\left[C_u - \triangle e^{\delta h} \cdot S_u\right]}_{\text{continuous dividends}} \equiv \underbrace{e^{-rh}\left[C_u - \triangle(S_u + D)\right]}_{\text{discrete dividends}}$$

Binomial Lattice (Risk Neutral Pricing)

- In the American case

$$\begin{aligned} H_\alpha &= e^{-(r-\delta)h}\left[qC[P]_{\alpha u} + (1-q)C[P]_{\alpha d}\right] \\ C[P]_\alpha &= \max(H_\alpha, E_\alpha) \end{aligned}$$

and in the European case, we replace H_α with $C[P]_\alpha$

Stock Price Drift Models

- Lognormal Model: $S_t = S_0 u^{N_u} d^{N_d}$, $u = u = e^{(r-\delta)h+\sigma\sqrt{h}}$
- CRR Model: $S_t = S_0 u^{N_u} d^{N_d}$, $u = u = e^{\sigma\sqrt{h}}$

State Price Models (Single Period)

- If an **arbitrage** opportunity exists in the market, then there exists a trading strategy θ such that

$$S(0)\theta \leq 0 \text{ and } S_1(1,\Omega) > 0$$

- A **state price vector** $\psi = \frac{Q(\omega)}{1+i}$ is a strictly positive vector such that $S(0) = \psi S(1,\Omega)$.
  - When $S(1,\Omega)$ is invertible, then it is obvious that $\psi = S(0)S^{-1}(1,\Omega)$.
  - We can define $Q(\omega) = \psi(1+i)$ where $\sum_{\omega\in\Omega} Q(\omega) = 1$ and $0 \leq Q(\omega) \leq 1 \implies 0 \leq \psi(\omega) \leq 1+i$.
- The single period securities market model is arbitrage free (and complete) if and only if there exists a (unique) state price vector.

State Price Models (Multiple Period)

- We say that X is measurable with respect to P_k if $X(w)$ is constant within each partition of P_k.
- A stochastic process $\psi = \{\psi(k), k = 0, 1, ..., T\}$ is said to be a state price process if the following hold
  - $\sum_{w\in\Omega} \psi(0,w) = 1$
  - ψ is adapted and strictly positive
  - For each $k = 0, 1, ..., T-1$, each $j = 1, 2, ..., N$ and each $H \in \mathcal{P}_k$

$$\sum_{w\in H} \psi(k,w) S_j(k,w) = \sum_{w\in H} \psi(k+1,w) S_j(k+1,w)$$

- An arbitrage opportunity exists if there is a self-financing strategy θ such that

$$V^\theta(0) = S(0)\theta(0) \leq 0 \text{ and } V^\theta(T) = S(T)\theta(T-1) > 0$$

- In the single period model we had $\psi = \frac{Q(w)}{1+i}$. We can find a unique parametrization in the multiperiod case. We have

$$\psi(k,w) = \frac{Q(H)}{|H| \cdot S_1(k,w)}$$

- The single period securities market model is arbitrage free (and complete) if and only if there exists a (unique) risk free measure Q.

Continuous Time Securities Market

Brownian Motion

- A **standard Brownian motion** (BM) is a stochastic process $W = \{W_t, t \geq 0\}$ such that the following hold:
  - $W_0 = 0$
  - The process has stationary and normally independent and identically distributed increments where $W_{t_2} - W_{t_1} \sim N(0, t_2 - t_1)$.
  - It has continuous sample paths
- Additionally $Cov(W_{t_i}, W_{t_j}) = \min(t_i, t_j)$
- A linear transformation of a Brownian motion process $\tilde{W}_t = \mu t + \sigma W_t$ is called a Brownian motion with drift μ and diffusion coefficient σ.

Ito Calculus

- Suppose that X and Y are random variables and consider $\sigma-$fields $\mathcal{F}_s$ and $\mathcal{F}_t$ for $s \leq t$. We have the following properties of conditional expectation:
  - $E[aX + bY|\mathcal{F}_t] = aE[X|\mathcal{F}_t] + bE[Y|\mathcal{F}_t]$
  - $E(E(X|\mathcal{F}_t)) = E(X)$
  - If X is $\mathcal{F}_t$ measurable, then $E[X|\mathcal{F}_t] = X$
  - If Y is $\mathcal{F}_t$ measurable, then $E[XY|\mathcal{F}_t] = YE[X|\mathcal{F}_t]$
  - $E(E(X|\mathcal{F}_s)|\mathcal{F}_t) = E(E(X|\mathcal{F}_t)|\mathcal{F}_s) = E(X|\mathcal{F}_s)$
- If (1) $E[|M_t|] < \infty$ for all t and (2) $E(M_t|\mathcal{F}_s) = M_s$ for all $s < t$ then M is a continuous **martingale** with respect to $\{\mathcal{F}_t, t \geq 0\}$.
- Some properties of the Ito integral $I(T) = \int_0^T \delta(t)dB(t)$ include
  - (Adaptedness) $I(T)$ is $\mathcal{F}_T$-measurable for all $T \geq 0$.
  - (Linearity) We have

$$I(T) = \int_0^T \delta(t)dB(t) \text{ and } J(T) = \int_0^T \gamma(t)dB(t) \implies c_1 I(T) \pm c_2 J(T) = \int_0^T (c_1\delta(t) + c_2\gamma(t))dB(t)$$

  - (Martingale) $I(T)$ is a martingale with respect to the filtration $\{\mathcal{F}_t, t \geq 0\}$ generated by B:

$$E\left[\int_0^T \delta(t)dB(t)\Big|\mathcal{F}_s\right] = \int_0^s \delta(t)B(t)$$

  - (Ito Isometry) If $\delta(t)$ is deterministic, then

$$E[I^2(t)] = E\left[\int_0^T \delta^2(t)dt\right]$$

  - (Normality) If δ is a deterministic function, then $I(T)$ is normally distributed.
  - Remark that $I(T)$ is a zero mean continuous time martingale.

- (Ito's Lemma) Succinctly,

$$X_t = X_0 + \int_0^t \delta_1(t, W_t)dt + \int_0^t \delta_2(t, W_t)dW_t \implies dX_t = \delta_1(t, W_t) + \delta_2(t, W_t)dW_t$$

Equivalently, if $f(t, x) \in C^2$ has the same dynamics as X_t and $Y_t = f(t, X_t)$ then

$$dY_t = f_t dt + f_{X_t} dX_t + \frac{1}{2} f_{X_t X_t} (dX_t)^2$$

with the rules (1) $dt \cdot dt = dt \cdot dW_t = dW_t \cdot dt = 0$, (2) $dW_t \cdot dW_t = dt$.

Black-Scholes

- If $dS_t = \mu S_t dt + \sigma S_t dW_t$ then $S_t = S_0 \exp\left(\left[\mu - \frac{\sigma^2}{2}\right] t + \sigma W_t\right)$
- Self-financing condition: $dV_t = a_t dS_t + b_t d\beta_T = (\mu a_t S_t + r b_t \beta_t)dt + a_t \sigma S_t dW_t$
- Option pricing
  - Call and put respectively:

$$C(t, S_t) = S_t N(d_1) - K e^{-r(T-t)} N(d_2)$$

$$P(t, S_t) = K e^{-r(T-t)} N(-d_2) - S_t N(-d_1)$$

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, d_2 = d_1 - \sigma\sqrt{T-t}, N(x) = \mathbb{P}(N(0,1) \leq x)$$

- An application of a dividend only changes $S_t \mapsto S_t e^{-\delta(T-t)}$

Risk-Neutral Pricing

- A risk-neutral probability measure is a probability measure $\mathbb{Q}$ on Ω such that
  - $\mathbb{Q}$ is equivalent to $\mathbb{P}$ in the sense that for any $A \subseteq \Omega$, $\mathbb{Q}(A) = 0 \iff \mathbb{P}(A) = 0$.
  - $\frac{S}{\beta}$ is a martingale under $\mathbb{Q}$
- An Ito process is a martingale if and only if it has zero drift. See the notes for justification.
- An arbitrage opportunity exists if there exists a self-financing portfolio such that:
  - $V_0 \leq 0$
  - $\mathbb{P}(V_T \geq 0) = 1$ and $\mathbb{P}(V_T > 0) > 0$
- A model is arbitrage free if and only if there exists a risk-neutral probability measure.
- An arbitrage-free market is said to be **complete** if every adapted cash flow stream can be replicated by some trading strategy (not necessarily self-financing).
  - An arbitrage-free model is complete iff there exists a unique risk-neutral probability measure $\mathbb{Q}$.
- The **Radon-Nikodym** derivative of a probability measure $\mathbb{Q}$ with respect to $\mathbb{P}$ is a random variable $\frac{d\mathbb{Q}}{d\mathbb{P}}$ defined implicitly by

$$E^{\mathbb{Q}}(X) = E^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} X\right]$$

- (Cameron-Martin-Girsanov Theorem) Let $W = \{W_t, 0 \le t\}$ be a $\mathbb{P}$ standard Brownian motion and let θ_t be a (bounded) adapted process such that $E^{\mathbb{P}}\left[e^{\frac{1}{2}\int_0^T \theta_t^2 dt}\right] < \infty$. Then there exists a measure $\mathbb{Q}$ such that
  - $\mathbb{Q}$ is equivalent to $\mathbb{P}$
  - $\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \theta_t dW_t - \frac{1}{2}\int_0^T \theta_t^2 dt\right)$
  - $Z_t = W_t + \int_0^t \theta_s ds$ is a Brownian motion for $0 \le t \le T$
- Explicitly, in the risk-neutral probability measure $\mathbb{Q}$, we have $W_t^{\mathbb{Q}} = W_t^{\mathbb{P}} + \int_0^t \frac{\mu - r}{\sigma} ds$ where $W_t^{\mathbb{Q}}$ is a $\mathbb{Q}$ standard BM

<u>Exotic Option Pricing</u>

- Method 1 (Match Vanilla Options)
  1. Let Y_t be the function of the stock in the payoff. Use the fact that $S_t = S_0 \exp\left[\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right]$ to derive a similar expression $Y_t = S_0 f(r, \sigma, t, W_t)$
  2. Take the limit of the discretization and calculate the mean μ^* and variance $\sigma^* T$ of the limit of $\ln V_t$
  3. V_t has behaviour $Y_T = S_0 \exp\left[\mu^* T + \sigma^* W_T\right]$ and we want to find A such that $e^A S_T^* = Y_T \iff A + \left(r - \frac{\sigma^2}{2}\right)T = \mu^* T$
  4. If the payoff is $(Y_t - K)^+$, for example, this is equivalent to holding e^A units of a call option with volatility σ^*, which we can price
- Method 2 (Risk Neutral Pricing)
  - Basically, just use intuition and the fact that $\mathbb{Q}(S > K) = N(d_2)$.

<u>Compound Options</u>

- $\underbrace{\text{CallOnCall}}_{E\left[e^{-rT_1}\max(C-L,0)\right]} - \underbrace{\text{PutOnCall}}_{E\left[e^{-rT_1}\max(L-C,0)\right]} = + \underbrace{C(S_0, 0, T_2, K)}_{E\left[e^{-rT_2}\max(S_T-K,0)\right]} - Le^{-rT_1}, T_1 < T_2$
- Knock-out +Knock-In = Ordinary Option