ACTSC 445 Final Exam Summary
Asset and Liability Management

1 Unit 5 - Interest Rate Risk (References Only)

Dollar Value of a Basis Point (DV01):

- Given by the absolute change in the price of a bond for a 1 basis point (0.01%) change in the yield.

Forward Rates:

- $(1+s_k)^k(1+f_k) = (1+s_{k+1})^{k+1}$

Duration:

- $D_m = -\frac{A'(y^*)}{A(y^*)} = \sum_{t>0} t \cdot \frac{A_t(1+y^*)^{-t-1}}{A^*}$ (modified)
- $D = \sum_{t>0} t \cdot \frac{A_t(1+y^*)^{-t}}{A^*} = (1+y^*)D_m$ (regular)
- $D_p = \sum_{i=1}^k w_i D_i, w_i = \frac{n_i A_i}{A^*}$ (portfolio)
- $D_{FW} = -\frac{1}{A^*}\sum_{i=1}^n \frac{\partial A}{\partial s_i}\Big|_{s_i=s_i^*}$ (Fisher-Weil; alt. below)
- $D_{FW} = \frac{1}{A^*}\sum_{t=1}^n tA_t e^{-t\cdot s_t^*}$
- $D_Q = \frac{1}{A^*}\sum_{t>0} tA_t(1+s_t^*)^{-t-1}$(quasi-modified)
- $D_{m,t} = -\frac{1}{A^*}\frac{\partial A}{\partial s_t}\Big|_{s_t=s_t^*}$ (partial; alt. below)
- $D_{m,t} = \begin{cases} \frac{1}{A^*}tA_t(1+s_t^*)^{-t-1} & \text{discrete case} \\ \frac{1}{A^*}tA_t e^{-ts_t^*} & \text{continuous case} \end{cases}$
- $D_m^e = \frac{A(y^*-\triangle y)-A(y^*+\triangle y)}{2A^*\triangle y}$ (effective)
- $\tilde{D}_{m,1} = \frac{-(\tilde{A}_1 - A^*)}{\triangle A^*}$ (key rate), where
- $s_t = \begin{cases} \tilde{s}_{t_1} & t < t_1 \\ \frac{t_{k+1}-t}{t_{k+1}-t_1}\tilde{s}_{t_k} + \frac{t-t_k}{t_{k+1}-t_1}\tilde{s}_{t_{k+1}} & t_{k-1} < t < t_{k+1} \\ \tilde{s}_{t_{\tilde{n}}} & t > t_{\tilde{n}} \end{cases}$

Convexity:

- $C = \frac{\sum_{t\geq 0} t(t+1)A_t(1+y^*)^{-t-2}}{A(y^*)}$ (standard)
- $C_{FW} = \frac{1}{A^*}\sum_{i=1}^n \frac{\partial^2 A}{\partial s_i^2}\Big|_{s_i=s_i^*}$ (Fisher-Weil; alt. below)
- $C_{FW} = \frac{1}{A^*}\sum_{t=1}^n t^2 A_t e^{-t\cdot s_t^*}$
- $C_m^e = \frac{A(y^*-\triangle y)-2A^*+A(y^*+\triangle y)}{A^*(\triangle y)^2}$ (effective)

Change estimation:

- $A(y^*+\triangle y) - A(y*) \approx -D_m \cdot A(y^*)\cdot\triangle y + C\cdot\frac{(\triangle y)^2}{2}\cdot A(y^*)$ ($|\cdot|$ change)
- $\frac{A(y^*+\triangle y)-A(y*)}{A(y*)} \approx -D_{FW}\triangle s + \frac{1}{2}C_{FW}(\triangle s)^2$ (% change)
- $\frac{\triangle A}{A^*} \approx \sum_{k=1}^n -D_{m,k}\triangle s_k$ (% change)
- $A_{New} = A^*(1+\sum -D_{m,k}\triangle s_k)$ (new value of A)

Remarks:

- The Macaulay duration of a zero-coupon bond is equal to its maturity.

2 Unit 6 - Immunization

Target Date Immunization:

- Let $V_k(y)$ be the value of a portfolio of securities at time k (measured in years) for a given ytm y (assume annual effective rate).
- In the target date immunization scenario, we want to match the target date of the portfolio with the duration of the portfolio since

$$V_D(\hat{y}) \geq V_D(y^*)$$

for any $\hat{y}$.

Single Liability Immunization:

- $\sum_{t>0} A_t(1+y^*)^{-t} = L(1+y^*)^{-k}$,
- $\sum_{t>0} tA_t(1+y^*)^{-t} = kL_k(1+y^*)^{-k}$
- If the assets are are symmetric about the time of the liability, put half of the PV in the first asset and half in the second asset

Multiple Liability Immunization:

(Redington's Basic Conditions; RBCs)

Let $S(y) = A(y) - L(y)$. Then the immunization conditions are:

1. $S(y) = A(y) - L(y)$, (i) $S(y^*) = 0$ [match PV]
2. $S'(y^*) = 0$ [match duration]
3. $S''(y^*) > 0$ [dispersion / convexity condition]

Immunization Strategies:

- Bracketing Strategy: If we have liability cash flows $t_1^L < t_2^L < ... < t_n^L$ and asset cash flows at $t^- < t_1^L$ and $t^+ > t_n^L$ then if RBC (i) + (ii) is satisfied then so is (iii).
  - If

- M^2 Strategy: Let $M_A^2 = \sum_{t>0} w_t^A (t - D_A)^2, w_t = \frac{A_t(1+y^*)^{-t}}{A(y^*)}$, then if $M_A^2 \geq M_L^2$ and RBC (i) + (ii) hold we have RBC (iii) holding.
  - Consider the probability measure $P(T = t) = \frac{A_t e^{-r_t t}}{A^*}$. Then for a portfolio, we have $D = E[T]$, $C = E[T^2]$, $M^2 = Var[T] = C - D^2$

Generalized Redington Theory:

- If $N_t = A_t - L_t$, let the current surplus be denoted by $S = \sum_{t>0} N_t P(0,t)$ and a shocked surplus (caused by interest rate changes) be denoted by $\hat{S} = \sum_{t>0} N_t \hat{P}(0,t)$
- Let $g(t) = \frac{\hat{P}(0,t)}{P(0,t)} - 1$ and $n_t = N_t P(0,t)$ which will imply that

$$\hat{S} - S = \sum_{t>0} n_t g(t)$$

- If $\sum_{t>0} n_t = 0$, $\sum_{t>0} t n_t = 0$, $\{n_k\}_{k>0}$ undergoes a $+,-,+$ sequence, and $g(t)$ is convex, then $\hat{S} - S \geq 0$.

3 Unit 8 - Interest Rate Models

General risk-neutral equation:

- For a payoff of V_T at time T, the value at time 0 is

$$V_0 = E\left[V_T e^{-\int_0^T r(t)dt}\right]$$

- The mean return of a stock is used in assessing the probabilities associated with threshold default models, whereas the risk neutral rate is used in pricing (in the Black Scholes and Merton models)

Properties of the Continuous Time Models:

- The Rendleman-Barter (lognormal) does not capture mean-reversion, but disallows negative interest rates; it is an **equilibrium** model
- The Vasicek model captures mean-reversion but does allows negative interest rates; it is an **equilibrium** model
- The Cox-Ingersoll-Ross model is an improvement on the Vasicek model since it captures mean reversion while disallowing negative interest rates; it is an **equilibrium** model

Monte-Carlo Simulation:

- Monte Carlo is a method to estimate $E[X]$ for a statistic X using the estimator $\frac{1}{n}\sum_{i=1}^n x_i$
- The exact steps are:
  1. We simulate a set of discount factors $\{v_1, v_2, ..., v_N\}$
  2. Find the simulated price $\sum_{i=1}^N c_t v_t$
  3. Repeat steps 1 and 2, n times where n is large; at the end of the process, we have n simulated prices $c_0^1, .., c_0^n$
  4. The estimated price of the security is given by $\frac{1}{n}\sum_{i=1}^n c_0^i$
- For simulation of the discount factors, write

$$v_t = e^{-\int_0^t r(s)ds} \approx e^{-(r_0+r_1+...+r_{t-1})t}$$

  and then simulate the sample path $\{r_1, ..., r_n\}$
- It is generally used when a pricing problem is too difficult to solve analytically

Discrete Binomial Trees and Embedded Options:

- Using backwards recursion, the general formula is:

$$V(t,n) = \frac{q(t,n) \cdot V(t+1,n+1) + [1-q(t,n)] \cdot V(t+1,n)}{1+i(t,n)}$$

- There are two approaches to pricing bonds with embedded options: (1) price the bond directly (2) price the components (option-free and option)
  - We always start with $V(T,k) = F$ for $k = 1, ..., n$
  - For a callable bond (option part) the option payoff at node (t,n) is $E(t,n) = \max(0, B(t,n) - K)$ and $V(t,n) = \max(E(t,n), H(t,n))$ where $H(t,n)$ depends on the previous $V(t+1,n+1)$ and $V(t+1,n)$ results and $B(t,n)$ is the price of the option-free component
  - For a putable bond, the algorithm is the same except now $E(t,n) = \max(0, K - B(t,n))$

Interest Rate Caps and Floors:

- Let L be the notional amount of the loan
- *Caps* are used to protect the borrower of a loan from increases in the interest rate. It is formed by a series of "caplets". At time t, the payoff from a caplet is
  - $L(i_{t-1} - K)^+$ if settled in arrears
  - $L(i_t - K)^+$ if settled in advance
- *Floors* are used to protect the lender of a loan from decreases in the interest rate. It is formed by a series of "floorlets". At time t, the payoff from a floorlet is
  - $L(K - i_{t-1})^+$ if settled in arrears
  - $L(K - i_t)^+$ if settled in advance

Black-Derman-Toy Model:

- In this model, $q(t,n) = \frac{1}{2}$, and the interest node relationship is given as $i(t,n+1) = i(t,n)e^{2\sigma(t)}$ or equivalently

$$i(t,n) = i(t,0)e^{2\sigma(t)\cdot n}$$

- To calibrate with $s_t' s$ and $\sigma_t' s$ we use:
  - $r_{00} = s_1$
  - Solve r_{t0} with
  $$\frac{1}{(1+s_{t+1})^{t+1}} = \sum_{k=0}^{t} \frac{A(t,n)}{1+i(t,0)e^{2k\sigma(t)}}$$
  - This model is an **arbitrage-free** model

Option Adjusted Spread:

- Reasons for the spread:
  - Compared to option-free bonds, bonds with embedded options come with repayment/reinvestment risk.
  - Using the calibrated model if we compute the price of such a bond, we will have the theoretical price, this may differ from the actual market price.
  - The OAS is a fixed/flat spread over the rates of the calibrated free that gives the theoretical price is equal to market price.
  - Prepayment/reinvestment risk for a callable bond can be defined as the risk that the principal with be repaid before maturity, and that the proceeds will have to be invested at a lower interest rate.
- OAS is the rate such that the binomial interest rate lattice shifted by the OAS equates the new theoretical price with the market price (uniform shift)
- The OAS of an option free bond is 0
- Here are the steps to compute V_+/V_-:
  1. Given the security's market price, find the OAS.
  2. Shift the spot-rate curve by a small quantity y.
  3. Compute a binomial interest-rate lattice based on the shifted curve obtained in Step 2.
  4. Shift the binomial interest-rate lattice obtained in Step 2 by the OAS.
  5. Compute V_+/V_- based on the lattice obtained in Step 4.
- The V_+/V_- values are used in the calculation of effective duration and convexity through the formulas:
$$D_m^e = \frac{V_- - V_+}{2V_0\triangle y}, C_m^e = \frac{V_+ - 2V_0 + V_-}{V_0(\triangle y)^2}$$

4 Unit 9 - Value-at-Risk (VaR)

Standard Definition of VaR:

- The formal definition for VaR is implicitly defined throughIf we have a non-negative surplus and matched duration, then the portfolio of assets and liabilities will have $V_D(\hat{y}) \geq V_D(y^*), D_A = D_L$ where y^* is the current ytm and $\hat{y}$ is a shift in the ytm, then the realized rate of return can never fall below its initial yield.
$$P(L_n > VaR_{\alpha,n}) = 1 - F_{L_n}(VaR_{\alpha,n}) = 1 - \alpha$$
where L_n is the loss random variable.
- It is also equivalent to
$$\begin{aligned} VaR_{\alpha,n} &= \inf\{l \in \mathbb{R} | F_{L_n}(l) \geq \alpha\} \\ &= \inf\{l \in \mathbb{R} | P(L_n > l) \leq 1-\alpha\} \end{aligned}$$
for general distributions (i.e. discrete, continuous, and mixed)
- Alternatively, VaR can be interpreted as the change in portfolio value $\triangle V = V_n - V_0 = -L_n$ since $VaR_{\alpha,n}$ is such that
$$P(L_n \geq VaR_{\alpha,n}) = 1-\alpha \implies P(\triangle V \leq -VaR_{\alpha,n}) = 1-\alpha$$
- Remark that VaR is, in general, never sub-additive

Conditional Tail Expectation:

- This is the average loss that can occur if loss exceeds $VaR_{\alpha,n}$. For a loss distribution L_n and confidence α this is
$$\begin{aligned} CTE_{\alpha,n} &= E[L_n | L_n \geq VaR_{\alpha,n}] \\ &= \frac{\sum_{\text{all } l \text{ w/ } L \geq VaR_{\alpha,n}} l \cdot Pr(L_n = l)}{\sum_{\text{all } l \text{ w/ } L \geq VaR_{\alpha,n}} Pr(L_n = l)} \end{aligned}$$
- In general CTE is sub-additive for continuous distributions and not sub-additive for discrete distributions

Alternate Definition (One Factor):

- We can re-write VaR as
$$VaR_{\alpha,n} = V_0(\sigma_1 z_\alpha \sqrt{n} - n\mu_1) = V_0(\sigma_n z_\alpha - \mu_n)$$
where $z_\alpha = \Phi^{-1}(\alpha)$ and $\Phi(\alpha) = P(\mathcal{N}(0,1) \leq \alpha)$
- If $\mu_1 = 0$ then $\sqrt{n} VaR_{\alpha,1} = VaR_{\alpha,n}$

Alternate Definition (Two Factor):

- We can re-write VaR as
$$VaR_{\alpha,n} = V_0(\sigma_V z_\alpha - \mu_V)$$
where the two factor representation is
$$\triangle V = V_n - V_0 = V_0(w_1(1+R_1) + w_2(1+R_2)) - V_0$$
and $R_V = \frac{\triangle V}{V_0} \sim \mathcal{N}(\mu_V, \sigma_V^2)$ with $\mu_V = w_1\mu_1 + w_2\mu_2$, $\sigma_V^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2\rho w_1 w_2 \sigma_1 \sigma_2$

Delta Normal Method:

- For a portfolio with multiple factors, we have through a first order Taylor expansion,

$$dV \approx \sum_{i=1}^{m} \frac{\partial V}{\partial f_i} df_i = \sum_{i=1}^{m} \triangle_i df_i = \sum_{i=1}^{m} f_i \triangle_i \frac{df_i}{f_i} = \sum_{i=1}^{m} f_i \triangle_i R_i$$

where $\triangle_i = \partial V / \partial f$

- We can then compute

$$Var(dV) = \sigma_V^2 = \sum_{i=1}^{m} (f_i \triangle_i)^2 Var(R_i) + 2 \sum_{i \neq j} f_i f_j \triangle_i \triangle_j Cov(R_i, R_j)$$

and assuming that μ_V, we can approximate VaR as

$$VaR_{\alpha,n} \approx \sigma_V z_\alpha$$

- For the special case of options,

$$dV = \triangle dS = S_0 \triangle \frac{dS}{S_0} = S_0 \triangle R_S$$

where $\triangle$ is the delta of the option. Thus we can use the approximation

$$\sqrt{Var(dV)} = S_0 |\triangle| \sigma_S = \sigma_V \implies VaR_{\alpha,1} = \sigma_V z_\alpha$$

5 Unit 10 - Credit Risk

- Remark that in computing probabilities, we tend to use the Black-Scholes formula that involves μ_V (Merton's model), but in pricing, we use the formula that involves the risk-free rate r (options pricing)

Types of models:

- *Static v. Dynamic*: static models are for credit risk management while dynamic models are for pricing risky securities
- *Structural and Threshold v. Reduced-form*: Threshold models are when default occurs when a selected random process falls under a threshold; reduced form models are when the time to default is modeled as a non-negative random variable whose distribution depends on a set of economic variables

Challenges of Credit Risk Management:

- *Lack of public information and data*; interpreted as-is
- *Skewed loss distributions*; problems of frequent small profits and occasional large losses
- *Dependence modeling*; defaults tend to happen simultaneously and this impacts the credit loss distribution

Structural Models of Default:

- Let S_t, B_t be the equity and debt values and of a firm at time t respectively; these are modeled as stochastic processes
- Denote $V_t = S_t + B_t$ where V_t is the firm's value
- Assume that no dividends are paid and a payment B is paid at time T from the firm issuing a bond
- At time T we have

$$S_T = \max(0, V_T - B)$$

$$B_T = \min(V_T, B) = B - \max(0, B - V_T)$$

and so V_T is the payoff of a call option S_T of strike B, B units of a T year ZCB

  - This is because at time T, if $V_T < B$, the whole firm liquidates its assets to debtholders since it has defaulted and missed a payment
  - In the former case, since shareholders are paid last, they get nothing
  - Thus default occurs when $V_T < B$

Merton's Model:

- Merton's model assumes V_t behaves as Brownian motion and implies

$$\begin{aligned} dV_t &= \mu_V V_t dt + \sigma_V V_t dB_t \\ \implies V_t &= V_0 e^{(\mu_V - \sigma_V/2)^2 + \sigma B_t} \end{aligned}$$

where $B_t \sim N(0, t)$.

- This implies that V_t is lognormally distributed and compute quantities like

$$\begin{aligned} P(\text{default}) &= P(V_T \leq B) = P(\ln V_T \leq \ln B) \\ &= P\left(\mathcal{N}(0,1) \leq \frac{\ln B - \ln V_0 - \left(\mu_V - \sigma_V^2/2\right) T}{\sigma_V \sqrt{T}}\right) \end{aligned}$$

- Going back to the first point of this section, let r be the risk-free rate. If a security has a payoff of $h(V_T)$ at time T, then its price is

$$E_Q(e^{-rT} h(V_T))$$

where this expectation is done under the risk-neutral measure.

- This is equivalent to

$$V_t = V_0 e^{(r - \sigma_V^2/2)t + \sigma_V B_t}$$

which is the Black-Scholes framework under r

Threshold Models:

- Used to model default in the case of a portfolio of securities issued by a large number of obligors

- This is a generalization of Merton's model where firm i defaults if $V_{T,i} < B_i$
- In a general threshold model, firm i defaults if its associated "critical" random variable X_i falls below some threshold d_i

Threshold Model Notation:

- Let d_{ij} be the critical threshold of firm i at rating j (e.g. credit rating)
- Let $D = [d_{ij}]_{m \times n} \in \mathbb{R}^{m \times n}$ where $X_i < d_{i1}$ implies default
- Let S_i be the state of firm i with $S_i \in \{0, 1, ..., n\}$ and $S_i = j \iff d_{ij} < X_i \leq d_{i(j+1)}$ with $d_{i,0} = -\infty$, $d_{i(n+1)} = \infty$
- $S_i = 0$ is true iff there is default
- Let $Y_i = \chi_{X_i(T) < d_{i1}}$, the default indicator variable for X_i
- We denote the marginal cdf of X_i through the following equivalent forms:

$$\bar{p}_i = P(X_i \leq d_i) = F_{X_i}(d_i) = F_i(d_i) = P(Y_i = 1)$$

- $M = \sum_{i=1}^{m} Y_i$ is the number of obligors who have defaulted at time T
- $L = \sum_{i=1}^{m} \delta_i e_i Y_i$ is the overall loss of the portfolio where e_i is the exposure of firm i and δ_i is the fraction of money that is lost from default
- The default correlation is given as

$$\rho(Y_i, Y_j) = \frac{E(Y_i Y_j) - \bar{p}_i \bar{p}_j}{\sqrt{(\bar{p}_i - \bar{p}_i^2)(\bar{p}_j - \bar{p}_j^2)}}$$

Intro to Copulas:

- A copula is a joint distribution of uniform random variables such that

$$C(F_{X_1}(u_1), F_{X_2}(u_2)) = F_{X_1,X_2}(u_1, u_2)$$

  which implies that

$$C(u_1, u_2) = F_{X_1,X_2}(F_{X_1}^{-1}(u_1), F_{X_2}^{-1}(u_2))$$

- It has the property that
  - $C(u, 1) = C(1, u) = u$
  - $C(u, 0) = C(0, u) = 0$
  - C is increasing in u_1 and u_2

Special Copulas:

- Suppose that $U_1, U_2 \sim Unif(0, 1)$
  1. If $U_1 \perp U_2$ then $F(u_1, u_2) = F_{U_1}(u_1) F_{U_2}(u_2)$
  2. If $U_1 = 1 - U_2$ then $F(u_1, u_2) = P(1 - u_2 \leq U_1 \leq u_1)$
  3. If $U_1 = U_2$ then $F(u_1, u_2) = P(U_1 \leq \min(u_1, u_2))$
- These results are similar if $U_1, U_2 \sim \mathcal{N}(0, 1)$ and $U_1 = -U_2$ in the second case; this gives us some copulas:
  1. $C_{ind}(u_1, u_2) = u_1 u_2$
  2. $C_{neg}(u_1, u_2) = \max(u_1 + u_2 - 1, 0)$
  3. $C_{pos}(u_1, u_2) = \min(u_1, u_2)$
- Generalization is easily done for more than two variables with similar dependence structure
  - This can be seen in the *Gauss copula* of the form

$$C_\Sigma(u_1, ..., u_m) = \Phi_\Sigma(\phi^{-1}(u_1), ..., \phi^{-1}(u_m))$$

- Note that $C(u_1, u_2) = u_1 + u_2$ is not a copula

Applications of Copulas:

- They are mainly useful in calculating binary results for firms which are of the form

$$P(d_{Aj_1} < X_A < d_{Aj_2}, d_{Bj_1} < X_B < d_{Bj_2})$$

  which is usually calculated by drawing the encompassing region and re-writing the expression in terms of additions and subtractions of cdfs