ACTSC 372 Final Exam Summary
Corporate Finance II

1 Competitive Equilibrium

Competitive Equilibrium (Assumptions):

1. People live for only **two periods**, t and $t+1$
2. People are faced with a **time constraint** $h = l + N^s$ where N^s the **inelastic labour supply** and l is leisure. y_t, the endowment, is given by $y_t = w_t(h - l_t)$

Competitive Equilibrium (Goal):
Given (t_0, t_1, r), the competitive equilibrium is a set of decision rules and prices such that:
(1) The values (c_0^A, c_1^A) and (c_0^B, c_1^B) for agents A and B respectively are chosen such that their respective utilities are maximized subject to the initial budget constraints (IBC). In other words, the pairs are the solutions to the following programs for $i = A, B$:

$$\max_{\{c_0^i, c_1^i\}} \quad U^i(c_0^i, c_1^i)$$

$$\text{s.to} \quad (y_0^i - t_0^i) + \frac{y_1^i - t_1^i}{1+r} = c_0^i + \frac{c_1^i}{1+r}$$

(2) The government satisfies its intertemporal budget constraint: $G_0 + \frac{G_1}{1+r} = T_0 + \frac{T_1}{1+r}$
(3) All markets clear:
(i) Goods market: $C_i + G_i = Y_i$ for $i = 0, 1$
(ii) Credit market: $S_i^P + S_i^G = 0$ for $i = 0, 1$ where $S_i^G = T_i - G_i$
Solution:
This involves solving the two conditions simultaneously for r:

1. *Tangency condition:* $\frac{MU_0}{MU_1} = (1+r)$
2. *IBC condition:* $c_0 + \frac{c_1}{1+r} = w$

Borrowers vs. Lenders:
Graph c_0 (x-axis) vs. c_1 (y-axis) and draw a line from $W = y_0 - t_0 + \frac{y_1 - t_1}{1+r}$ on the x-axis to $W(1+r)$ on the y-axis. Let $E = (y_0 - t_0, y_1 - t_1)$ be the **endowment point**. Any agent to the right of E is a **borrower** and any agent to the left of E is a **lender**.

2 Markowitz Analysis

Markowitz Variables and Equations:

$$a = \mu^t \Omega^{-1} \mu, b = \mu^t \Omega^{-1} 1, b = 1^t \Omega^{-1} 1, c = ac - b^2$$

$$\Phi = \frac{a\Omega^{-1}1 - b\Omega^{-1}\mu}{d}, \Theta = \frac{c\Omega^{-1}\mu - b\Omega^{-1}1}{d}$$

$$\hat{w} = \Phi + \Theta\mu_p, \sigma_p^2 = \frac{c}{d}\left(\mu_p - \frac{b}{c}\right)^2 + \frac{1}{c}$$

CML line:

$$E[\tilde{r}_p] = rf + \sigma_p \left[\frac{E[\tilde{r}_M] - rf}{\sigma_M}\right]$$

for efficient porfolios $\tilde{r}_p$.
Markowitz vs. CAPM

CAPM (SML)	Markowitz (CML)
$\mu - \beta$ Space	$\mu - \sigma$ Space
Inefficient individual assets	Efficient portfolios
Risk is β	Risk is σ
SML = Security Market Line	CML = Capital Market Line
Slope = $E[\tilde{r}_M] - r_f$	Slope = $\frac{\mu_M - r_f}{\sigma_M}$ where M (point of tangency) depends on r_f

3 CAPM

Overarching Question: What is the required rate of return on asset j, $E[\tilde{r}_j]$, that an investor would require before investing in that asset?
Assumptions:

1. The market is in **equilibrium**
2. Investors are **risk averse**, living in a **two period world**, and want to **maximize their end of period 1 wealth** $E[U(\tilde{y}_1)]$
3. All investors can **borrow and lend at the risk free rate** rf
4. There is a **frictionless market** with no transaction costs
5. Investor's beliefs regarding the asset rate of returns and joint probability distributions are **homogeneous**

Standard Version (Ex-Ante Form) ; SML Line:

$$\begin{aligned} E[\tilde{r}_j] &= rf + \beta_{j,M}\sigma_M \left[\frac{E[\tilde{r}_M] - rf}{\sigma_M}\right] \\ &= rf + \rho_{j,M}\sigma_j \left[\frac{E[\tilde{r}_M] - rf}{\sigma_M}\right] \\ &= rf + \beta_{j,M}\left[E[\tilde{r}_M] - rf\right] \end{aligned}$$

where the systematic risk = $\beta_{i,M} = \frac{Cov(\tilde{r}_j, \tilde{r}_M)}{Var(\tilde{r}_M)} = \frac{Cov(\tilde{r}_j, \tilde{r}_M)}{\sigma_M^2}$, price of risk = $\frac{E[\tilde{r}_M] - rf}{\sigma_M}$, Sharpe ratio = $\frac{E[\tilde{r}_p] - rf}{\sigma_p}$, risk due to asset $j = \beta_{j,M}\sigma_M = \rho_{j,M}\sigma_j = \frac{d[\sigma_M]}{dw_j}$, required rate of return $= rr_p = rf + \beta_p\left(E[\tilde{r}_M] - rf\right)$, performance alpha $= \alpha_p = E[\tilde{r}_p] - rr_p$
Fair Game Equation:

$$\tilde{r}_{it} = E[\tilde{r}_{it}] + \beta_i (\tilde{r}_{Mt} - E[\tilde{r}_{it}])$$

Time Series Version (Ex-Post Form):
Substitute the fair game equation into the Ex-Ante CAPM (last version) to get

$$\tilde{r}_{it} - rf_t = +\beta_i (\tilde{r}_{Mt} - rf_t)$$

Here, $\tilde{r}_{it} = \frac{\tilde{P}_{it} - \tilde{P}_{i(t-1)} + \tilde{D}_{it}}{\tilde{P}_{i(t-1)}}$.
Stochastic LS Version:

$$(\tilde{r}_{it} - rf_t) = \alpha_i + \beta_{i,M} (\tilde{r}_{Mt} - rf_t) + \tilde{\epsilon}_{it}$$

Notes:

- Risk is measured by β as opposed to σ in Markowitz
- $\beta_p = \sum_{i=1}^n w_i \beta_i$, $\mu_p = \sum_{i=1}^n w_i \mu_i$,
- $\sigma_p = \sqrt{\sum_{i=1}^n \sigma_i^2 w_i^2 + 2 \sum_{i \neq j} \rho_{ij} \sigma_i \sigma_j w_i w_j}$
- The CAPM is not a good approximation for volatility sorted portfolios

Applications:

1. Allows for the modeling of portfolios of assets
2. Can be used to assess the performance of a portfolio via its alpha
3. β can be used in the calculation of the cost of capital in a firm

Variance:

$$Var(\tilde{r}_p) = \underbrace{\beta_p^2 \sigma_M^2}_{\textbf{systematic risk}} + \underbrace{Var\left[\sum_{i=1}^n w_i \tilde{\varepsilon}_{it}\right]}_{\textbf{non-systematic risk}}$$

where

$$w_i \approx \frac{1}{n} \implies Var\left[\sum_{i=1}^n w_i \tilde{\varepsilon}_{it}\right] = \frac{1}{n^2} Var[\tilde{\varepsilon}_{it}] = \frac{1}{n} \bar{\sigma}_{\tilde{\varepsilon}} \to 0$$

4 APT and Multifactor Models

Assumptions:

1. The market is in **equilibrium**
2. Investor's beliefs regarding the asset rate of returns and joint probability distributions are **homogeneous**, and they accept the K-factor model described below
3. The **market is rich** in the sense that there is a large number of assets traded with minimum trading restrictions
4. Portfolios in this market have **certain properties** (see below)

Standard Version:

$$\tilde{r}_i = E[\tilde{r}_i] + \sum_{j=1}^k b_{i1} \tilde{F}_1 + \tilde{\varepsilon}_i$$

where $\tilde{F}_j$ is a zero mean factor (sometimes denoted by $\tilde{F}_j = \tilde{f} - E[\tilde{f}]$), $\tilde{\varepsilon}_i$ is such that $E[\tilde{\varepsilon}_i] = 0$, $Cov(\tilde{\varepsilon}_i, \tilde{\varepsilon}_j) = 0$ for $i \neq j$, $Cov(\tilde{\varepsilon}_i, \tilde{F}_i) = 0$ for all i, $b_{ik} = \frac{dE[\tilde{r}_i]}{dE[\tilde{r}_{p_k}]} = \frac{dE[\tilde{r}_i]}{dE[\tilde{F}_k]}$
Assumption 4 (about portfolios):

- **Self financed**: $w^t 1 = 0$
- **Well-diversified**: Selected portfolio weights are small and the number of assets are big so that any idiosyncratic risk component is zero-valued
- **Zero-sensitivity** to common factors: $w^t b_j = 0$ for any j
- The market (including portfolios) is **free from arbitrage**

These results imply that $\tilde{r}_p = 0 = E[\tilde{r}_p] \implies w^t \mu = 0$. For an appropriate number of assets, this would imply that $\mu \in \text{span}\{1, b_1, ..., b_k\}$ which gives us our lambda model below.
Lambda Version:

$$E[\tilde{r}_i] - rf = \sum_{j=1}^k \lambda_j b_{ij} = \sum_{j=1}^k (E[\tilde{r}_{p_j}] - rf) b_{ij}$$

where $\lambda_j = E[\tilde{r}_{p_j}] - rf$ and p_j is the **pure factor portfolio** with unit factor sensitivity to the common factor j.
Why is APT more robust than CAPM?

1. APT makes no assumption about the distribution of asset returns
2. APT only requires risk aversion and makes no strong assumptions about individuals' utility functions
3. APT asserts that the rate of return is based on many factors rather than 1
4. The market portfolio is not special as it is CAPM (it is efficient in CAPM)
5. The APT can be extended to a multi-period framework

5 AD-Pricing

Let $A_{n \times m}$ be a matrix that represents the return of m assets over n states of the economy. That is, the columns are representative of assets and the rows are states of the economy. If

$n = m$ then $X_{m \times n} = A^{-1}$ is the matrix of weights where the columns are representative of the states of the economy and the rows correspond to assets.

Competitive Equilibrium in an A-D Economy (Assumptions):

- Agents will live for only 2 periods
- We have a **complete market** (the number of states of nature is equal to the number of linearly independent assets)
- The set of states of nature are **collectively exhaustive** and **mutually exclusive**
- We assume only one **perishable** consumption good

Competitive Equilibrium in an A-D Economy (Goal):

Given $(\pi_1, ..., \pi_n, P_1, ..., P_n, \delta^i)$, the competitive equilibrium is a set of decision rules and prices such that:

(1) The values (c_0^A, c_1^A) and (c_0^B, c_1^B) for agents A and B respectively are chosen such that their respective utilities are maximized subject to the initial budget constraints (IBC). In other words, the pairs are the solutions to the following program for $i = 1, 2$:

$$\max_{\{c_0^i, c_1^i\}} \quad U(c_0^i, c_1^i) = U_0^i(c_0^i) + \delta^i \sum_{\theta=1}^{n} \pi_\theta U^i(c_\theta^i)$$

$$\text{s.to} \quad y_0^i + \sum_{\theta=1}^{n} P_\theta y_\theta^i \geq c_0^i + \sum_{\theta=1}^{n} P_\theta c_\theta^i$$

(2) All markets clear:

(i) $C_0 = Y_0$

(ii) $C_\theta = Y_\theta$ for $\theta = 1, ..., n$

Solution:

$$P_\theta = \frac{\delta^i \pi_\theta M U_\theta^i}{M U_0^i}$$