**Continuity from Below**

Let $(X, \mathcal{A}, \mu)$ be a measure space with $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$. IF $E_i \subseteq E_{i+1}$ for each $i \in \mathbb{N}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n)$$

*Proof.* First observe that if $\mu(E_n) = \infty$ for some $n \in \mathbb{N}$ then monotonicity shows that

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \infty = \lim_{n\to\infty} \mu(E_n).$$

As such, we may assume that $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$. Let $F_1 := E_1$ and for each $n \geq 2$ let $F_n = E_n \setminus E_{n-1}$. Then $\{F_n\}_{n=1}^{\infty}$ is a pairwise disjoint sequence with $E_n = \bigcup_{i=1}^{n} F_i$. Moreover,

$$\mu(F_n) = \mu(E_n) - \mu(E_{n-1})$$

for $n > 1$. It follows that

$$\sum_{n=1}^{m} \mu(F_n) = \mu(E_m).$$

Finally, we have

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} \mu(F_n) = \lim_{m\to\infty} \sum_{n=1}^{m} \mu(F_n) = \lim_{m\to\infty} \mu(E_m).$$

□

**Continuity from Above (Statement only)**

Let $(X, \mathcal{A}, \mu)$ be a measure space with $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$. If $\mu(E_1) < \infty$ and if $E_{i+1} \subseteq E_i$ for each $i \in \mathbb{N}$, then

$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n)$$

**Caratheodory's Theorem (Statement only)**

Let $\mu^*$ be an outer measure on $X$. The set $\mathcal{B}$ of $\mu^*$-measurable sets in $\mathcal{P}(X)$ is a $\sigma-$algebra and if $\mu = \mu^*\Big|_{\mathcal{B}}$, then $\mu$ is a complete measure on $\mathcal{B}$.

**Caratheodory Extension Theorem**

Let $\mu$ be a measure on an algebra $\mathcal{A} \subseteq \mathcal{P}(X)$. Let $\mu^*$ be the outer measure generated by $\mu$. Let $A^*$ be the $\sigma-$algebra of $\mu^*$ measurable sets. Then $\mathcal{A} \subseteq A^*$ and $\mu$ extends to a measure $\bar{\mu}$ on $\mathcal{A}^*$.

*Proof.* We only need to show that $\mathcal{A} \subseteq \mathcal{A}^*$. Let $E \in \mathcal{A}$ and let $A \subseteq X$. As before, we can assume that $\mu^*(A) < \infty$ and that we need only show that

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

Let $\epsilon > 0$ and choose $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that

$$A \subseteq \bigcup_{n=1}^{\infty} F_n \quad \text{and} \quad \sum_{n=1}^{\infty} \mu(F_n) \leq \mu^*(A) + \epsilon.$$

Note that

$$A \cap E \subseteq \bigcup_{n=1}^{\infty} E \cap F_n \quad \text{and} \quad A \cap E^c \subseteq \bigcup_{n=1}^{\infty} E^c \cap F_n.$$

It follows that

$$\mu^*(A \cap E) \leq \sum_{n=1}^{\infty} \mu(E \cap F_n) \quad \text{and} \quad \mu^*(A \cap E^c) \leq \sum_{n=1}^{\infty} \mu(E^c \cap F_n)$$

Hence

$$\begin{aligned} \mu^*(A \cap E) + \mu^*(A \cap E^c) &\leq \sum_{n=1}^{\infty} \mu(E \cap F_n) + \sum_{n=1}^{\infty} \mu(E^c \cap F_n) \\ &= \sum_{n=1}^{\infty} \mu(F_n) \leq \mu^*(A) + \epsilon \end{aligned}$$

Since $\epsilon$ was arbitrary, we have $\mu^*(A \cap E) + \mu^*(A \cap E) \leq \mu^*(A)$. The remainder of the theorem follows from Caratheodory's Theorem. □

**Hahn Extension Theorem (Statement only)**

Suppose that $\mu$ is a $\sigma-$finite measure on an algebra $\mathcal{A}$. Then there is a unique extension $\bar{\mu}$ to a measure on $\mathcal{A}^*$, the $\sigma-$algebra of all $\mu^*-$measurable sets.

**Monotone Convergence Theorem (only the 1st version)**

If $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{M}^+(X, \mathcal{A})$ is such that $f_n \leq f_{n+1}$ for all $n \geq 1$ and $\lim_{n\to\infty} f_n(x) = f(x)$ then

$$\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu$$

*Proof.* We know that $f \in \mathcal{M}^+(X, \mathcal{A})$ and

$$\int f_n \, d\mu \leq \int f_{n+1} d\mu \leq \int f \, d\mu$$

Hence $\lim_{n\to\infty} \int f_n \, d\mu \leq \int f \, d\mu$. Conversely, let $0 < \alpha < 1$ and let $\varphi \in \mathcal{M}^+(X, \mathcal{A})$ be simple with $0 \leq \varphi \leq f$. For each $n \in \mathbb{N}$, let $A_n = \{x \in X | f_n(x) \geq \alpha\varphi(x)\}$. Then $A_n \in \mathcal{A}$, $A_n \subseteq A_{n+1}$ and $X = \bigcup_{n=1}^{\infty} A_n$. We have

$$\int_{A_n} \alpha\varphi \, d\mu \leq \int_{A_n} f_n \, d\mu \leq \int f_n \, d\mu$$

By the Monotone Convergence Theorem for measures, $\int \varphi \, d\mu = \lim_{n\to\infty} \int_{A_n} \varphi d\mu$ since $\lambda(E) = \int_E \varphi \, d\mu$ is a measure. Therefore

$$\alpha \int \varphi \, d\mu = \lim_{n\to\infty} \int_{A_n} \alpha\varphi \, d\mu \leq \lim_{n\to\infty} \int f_n \, d\mu$$

Since $0 \leq \alpha < 1$ was arbitrary, we can take $\alpha \to 1^-$ and conclude

$$\int \varphi \, d\mu \leq \lim_{n\to\infty} \int f_n \, d\mu$$

Thus

$$\int f \, d\mu = \sup_{0\leq\varphi\leq f} \int \varphi \, d\mu \leq \lim_{n\to\infty} \int f_n \, d\mu$$

and the result is established. □

**Fatou's Lemma (Statement only)**

If $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{M}^+(X, \mathcal{A})$ then

$$\int \liminf_{n\to\infty} f_n \; d\mu \leq \liminf_{n\to\infty} \int f_n \; d\mu$$

**Lebesgue Dominated Convergence Theorem**

Let $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(X, \mathcal{A}, \mu)$. Assume that $f = \lim_{n\to\infty} f_n$ $\mu-$a.e. If there exists an integrable function $g \in \mathcal{L}(X, \mathcal{A}, \mu)$ such that $|f_n| \leq g$ for all $n \in \mathbb{N}$, then $f$ is integrable and

$$\int f \; d\mu = \lim_{n\to\infty} \int f_n \; d\mu$$

*Proof.* By redefining $f_n$, $f$ if necessary, we may assume that $f = \lim_{n\to\infty} f_n$ everywhere. This shows that $f$ is measurable. We have $|f| \leq g$, so $|f|$ is integrable. Hence $f$ is also integrable.

Notice that $g + f_n \geq 0$. By Fatou's Lemma,

$$\begin{aligned}
\int g \; d\mu + \int f \; d\mu &= \int (g+f) d\mu \\
&= \liminf_{n\to\infty} \int (g+f_n) d\mu \\
&\leq \int \liminf_{n\to\infty} (g+f_n) d\mu \\
&= \int g \; d\mu + \liminf_{n\to\infty} \int f_n \; d\mu
\end{aligned}$$

It follows that

$$\int f \; d\mu \leq \liminf_{n\to\infty} \int f_n d\mu.$$

On the other hand, $g - f_n \geq 0$, so by arguing as above, we see that

$$-\int f \; d\mu \leq \liminf_{n\to\infty} \int -f_n d\mu = -\limsup_{n\to\infty} \int f_n \; d\mu$$

and as such

$$\limsup_{n\to\infty} \int f_n \; d\mu \leq \int f \; d\mu.$$

Therefore $\int f \; d\mu = \lim_{n\to\infty} \int f_n \; d\mu$. □

**Completeness for $L_p$ when $1 \leq p < \infty$**

Let $1 \leq p < \infty$. Then $(L_p(X, \mathcal{A}, \mu), \|\cdot\|_p)$ is a Banach space.

*Proof.* Let $\{f_n\}_{n=1}^{\infty} \subseteq L_p(X, \mathcal{A}, \mu)$ be a Cauchy sequence. We can find a subsequence $\{g_k\}$ of $\{f_n\}$ such that $\|g_{k+1} - g_k\|_p \leq 1/2^k$. Define, for all $x \in X$,

$$g(x) = |g_1(x)| + \sum_{k=1}^{\infty} |g_{k+1}(x) - g_k(x)|$$

Then $g \in \mathcal{M}^+(X, \mathcal{A})$, and by Fatou's Lemma,

$$\int |g|^p \, d\mu \leq \liminf_{n\to\infty} \int \left( |g_1| + \sum_{k=1}^{n} |g_{k+1} - g_k| \right)^p d\mu.$$

Next, by Minkowski's Inequality, we get

$$\left( \int |g|^p \, d\mu \right)^{1/p} \leq \liminf_{n\to\infty} \|g_1\|_p + \sum_{k=1}^{n} \|g_{k+1} - g_k\|_p \leq \|g_1\|_p + 1 < \infty$$

Let $E = \{x \in X : g(x) < \infty\}$. Then it follows from the above calculations that $\mu(X \backslash E) = 0$. Hence

$$|g_1(x)| + \sum_{k=1}^{\infty} |g_{k+1}(x) - g_k(x)|$$

converges to a finite number $\mu-$a.e. Let

$$f(x) = \begin{cases} g_1(x) + \sum_{k=1}^{\infty} g_{k+1}(x) - g_k(x) & x \in E \\ 0 & x \notin E \end{cases}$$

Note that since the series is telescoping, this actually shows that $g_k \to f$ $\mu-$a.e. We also notice that $|g_k(x)| \leq g(x)$ for all $x \in X$. The Lebesgue Dominated Convergence Theorem (LDCT) shows us that

$$\int |f|^p \, d\mu = \lim_{k\to\infty} \int |g_k|^p d\mu \leq \int |g|^p d\mu < \infty$$

Therefore $f \in L_p(X, \mathcal{A}, \mu)$. Since $|f| \leq g$, we have $|f - g_k| \leq 2|g|$ and again by the LDCT, $0 = \lim_{k\to\infty} \int |f - g_k|^p d\mu$ since $g_k \to f$ a.e. Therefore, the subsequence $\{g_k\}$ converges to $f$ in $L_p(X, \mathcal{A}, \mu)$. It follows that $\{f_n\}$ converges to $f$ in $L_p(X, \mathcal{A}, \mu)$. □

**Egoroff's Theorem (Statement Only)**

Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $\{f_n\}$ be a sequence of measurable real-valued functions which converge almost everywhere to a real-valued measurable function $f$. Then $f_n \to f$ almost uniformly.

**Hahn Decomposition Theorem (Statement Only)**

Let $\mu$ be a signed measure on $(X, \mathcal{A})$. Then there is a positive set $A \in \mathcal{A}$ and a negative set $B \in \mathcal{A}$ so that $X = A \cup B$ and $A \cap B = \emptyset$.

**Jordan Decomposition Theorem for Signed Measures (Statement only)**

Let $\mu$ be a signed measure on $(X, \mathcal{A})$. Then there exist two mutually singular positive measures $\mu^+$ and $\mu^-$ such that

$$\mu = \mu^+ - \mu^-$$

Furthermore, if $\lambda$ and $\nu$ are two positive measures with

$$\mu = \lambda - \nu$$

the for each $E \in \mathcal{A}$ we have

$$\lambda(E) \geq \mu^+(E) \quad \text{and} \quad \nu(E) \geq \mu^-(E)$$

Finally, if $\lambda \perp \nu$, then $\lambda = \mu^+$ and $\nu = \mu^-$.

**Radon-Nikodym Theorem (Statement Only)**

Let $\lambda$ and $\mu$ be $\sigma-$finite measures on $(X, \mathcal{A})$. Suppose that $\lambda$ is absolutely continuous with respect to $\mu$. Then there exists $f \in \mathcal{M}^+(X, \mathcal{A})$ such that

$$\lambda(E) = \int_E f \, d\mu$$

for every $E \in \mathcal{A}$. Moreover $f$ is uniquely determined $\mu-$almost everywhere.

**Lebesgue Decomposition Theorem (Proof of Existence Only)**

Let $\lambda$ and $\mu$ be $\sigma-$finite measures on $(X, \mathcal{A})$. Then there exists two measures $\lambda_1$ and $\lambda_2$ on $(X, \mathcal{A})$ such that $\lambda = \lambda_1 + \lambda_2$, $\lambda_1 \perp \mu$ and $\lambda_2 \ll \mu$. Moreover, these measures are unique.

*Proof.* Let $\nu = \lambda + \mu$. Then clearly $\nu$ is $\sigma-$finite, $\lambda \ll \nu$ and $\mu \ll \nu$. It follows that there are functions $f, g \in \mathcal{M}^+(X, \mathcal{A})$ such that

$$\lambda(E) = \int_E f \, d\nu \quad \text{and} \quad \mu(E) = \int_E g \, d\nu$$

for every $E \in \mathcal{A}$. Let

$$A = \{x \in X : g(x) = 0\} \quad \text{and} \quad B = \{x \in X : g(x) > 0\}.$$

Then $\{A, B\}$ is a partition of $X$. Let

$$\lambda_1(E) = \lambda(E \cap A) \quad \text{and} \quad \lambda_2(E) = \lambda(E \cap B)$$

for every single $E \in \mathcal{A}$. Clearly $\lambda = \lambda_1 + \lambda_2$. Since

$$\mu(A) = \int_A g \, d\nu = \int_A 0 \, d\nu$$

we have $\lambda_1 \perp \mu$. If $\mu(E) = 0$ then $\int_E g \, d\nu = 0$ so $g(x) = 0$ for $\nu-$almost everywhere in $E$. It follows that $\nu(E \cap B) = 0$ and hence that

$$\lambda_2(E) = \lambda(E \cap B) = 0$$

since $\lambda \ll \nu$. That is $\lambda_2 \ll \mu$. □

**Lebesgue's Differentiation Theorem (Statement Only)**

Let $f : [a, b] \mapsto \mathbb{R}$ be increasing. Then $f$ is differentiable almost everywhere on $[a, b]$, $f'$ is measurable, integrable and

$$\int_{[a,b]} f' \, dm \leq f(b) - f(a)$$

**FTC for Absolutely Continuous Functions (Statement Only)**

A function $F : [a, b] \mapsto \mathbb{R}$ is of the form

$$F(x) = F(a) + \int_a^x g(t) \, dt$$

for some integrable function $g$ if and only if $F$ is absolutely continuous. Moreover, in this case $F'(x) = g(x)$ a.e.

**Riesz Representation Theorem I for $L_p(X, \mu)^*$ where $1 \leq p < \infty$ (Statement only)**

Let $\Gamma \in L_p(X, \mu)^*$ where $1 \leq p < \infty$ and $\mu$ *is $\sigma-$finite*. Then if $\frac{1}{p} + \frac{1}{q} = 1$, there exists a unique $g \in L_q(X, \mu)^*$ such that

$$\Gamma(f) = \int_X fg \, d\mu = \phi_g(f)$$

Moreover, $\|\Gamma\| = \|g\|_q$.

**Riesz Representation Theorem II for $L_p(X,\mu)^*$ where $1 < p < \infty$ (Statement only)**

Let $\Gamma \in L_p(X,\mu)^*$ where $1 < p < \infty$. Then if $\frac{1}{p} + \frac{1}{q} = 1$, there exists a unique $g \in L_q(X,\mu)^*$ such that

$$\Gamma(f) = \int_X fg \; d\mu$$

*for all* $f \in L_p(X,\mu)$. Moreover, $\|\Gamma\| = \|g\|_q$.

**Riesz Representation Theorem for $C([a,b])^*$ (Statement only)**

Let $\Gamma \in C([a,b])^*$. Then there exists a unique finite signed measure $\mu$ on the Borel subsets of $[a,b]$ such that

$$\Gamma(f) = \int_{[a,b]} f \; d\mu$$

for each $f \in C([a,b])$. Moreover, $\|\Gamma\| = |\mu|([a,b])$.

**Product Measure Theorem (Statement only)**

Let $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\lambda)$ be measure spaces. Then there exists a measure $\pi$ on $(X \times Y, \mathcal{A} \times \mathcal{B})$ such that $\pi(A \times B) = \mu(A)\lambda(B)$. Moreover, if $\mu$ and $\lambda$ are $\sigma-$finite, then $\pi$ is unique and $\sigma-$finite.

In the case where $\mu$ and $\lambda$ are $\sigma$-finite, we denote the uniquely obtained measure as

$$\pi = \mu \times \lambda$$

and call the measure the product of $\mu$ and $\lambda$.

**Tonelli's Theorem (Statement Only)**

Let $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\lambda)$ be $\sigma-$finite measure spaces. Let $F : Z = X \times Y \mapsto [0,\infty]$ be measurable. Then the functions defined by $f(x) = \int_Y F_x \; d\lambda$ and $g(y) = \int_X F^y \; d\mu$ are measurable and $\int_X f \; d\mu = \int_Z F \; d\pi = \int_Y g \; d\lambda$ where $\pi = \mu \times \lambda$. That is to say,

$$\int_X \left( \int_Y F(x,y) d\lambda(y) \right) d\mu(x) = \int_Z F \; d\pi = \int_Y \left( \int_X F(x,y) d\mu(x) \right) d\lambda(y)$$

**Fubini's Theorem (Statement Only)**

Let $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\lambda)$ be $\sigma-$finite measure spaces and let $\pi = \mu \times \lambda$. If $F$ is integrable with respect to $\pi$ on $Z = X \times Y$, then the extended real valued functions defined almost everywhere by $f(x) = \int_Y F_x \; d\lambda$ and $g(y) = \int_X F^y \; d\mu$ have finite integrals and $\int_X f \; d\mu = \int_Z F \; d\pi = \int_Y g \; d\lambda$. That is to say,

$$\int_X \left( \int_Y F(x,y) d\lambda(y) \right) d\mu(x) = \int_Z F \; d\pi = \int_Y \left( \int_X F(x,y) d\mu(x) \right) d\lambda(y)$$