Calculus IV Course Review (MIT)

(Almost equivalent to AMATH 231 (Calculus IV) at the University of Waterloo)

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These notes are based on the electronic lectures (from 19 onwards) located here. The lectures are taught at the Massachusetts Institute of Technology by Prof. Denis Auroux. Some of the content was also taken from Paul's Online Calculus Notes, here. Vector fields such as F, B and E usually have arrows on top of them but they have been intentionally left out so to preserve the idea that they are more like mathematical objects (vector functions rather than just vectors).

Note that these set of notes are still just a work in progress and may contain errors or missing sections.

1 Vector Calculus in $\mathbb{R}^2$

Here we will be discussing all the major components of basic modern day vector calculus. Some general notions will be introduced but the focus will be primarily in $\mathbb{R}^2$.

1.1 Line Integrals

Definition 1.1. A **vector field** in $\mathbb{R}^3$ is an equation of the form $F : \mathbb{R}^3 \mapsto \mathbb{R}^3$ where $F(x, y, z) = M\vec{i}+N\vec{j}+O\vec{k}$ where $M, N, O : \mathbb{R}^3 \mapsto \mathbb{R}$. Here $\vec{i} = (1, 0, 0) = e_1$, $\vec{j} = (0, 1, 0) = e_2$ and $\vec{k} = (0, 0, 1) = e_3$. Extension into $\mathbb{R}^n$ is trivial so I will not write this up.

Example 1.1. A **velocity field** is a vector field that assigns a velocity to each point in the domain. The same idea goes for **force fields** which assign a force at each point in the domain.

Notation 1. The video uses $d\vec{r} = \langle dx_1, ..., dx_n\rangle = \hat{T} \cdot ds$ where $d\vec{r}$ is the instantaneous direction of the particle, $\hat{T}$, called the trajectory, is the unit tangent vector and ds is the instantaneous arclength of the parametrized curve. Note that $r : \mathbb{R} \to \mathbb{R}^n$ is actually the equation of the curve for the single parameter t and image in the image of a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$.

Definition 1.2. The **line integral** of a parametrized curve C (i.e. $g(t) = \vec{x}(t)$ for $\vec{x} = (x_1, ..., x_n) \in \mathbb{R}^n$) in $\mathbb{R}^n$ is defined as $\int_C f(\vec{x})\, ds$ where $ds = \left\|\left(\frac{dx_1}{dt}, ..., \frac{dx_n}{dt}\right)\right\|_2 dt$[1] for scalar function f. The arclength of the curve C is when $f(\vec{x}) = 1$. Sometimes the notation $\oint_C$ is used in place of $\int_C$ but only for closed curves. Another way of writing a line integral is $\int_C F(\vec{x}) \cdot d\vec{r}$ which is used for vector valued functions F (note the dot product between F and the differential $d\vec{r}$)[2]. Note also that the notations are not equivalent.

Remark 1.1. Note that if $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ with $F(\vec{x}) = \sum_{i=1}^n M_i e_i = (M_1, ..., M_n)$ where e_i is the i^{th} elementary vector and $M : \mathbb{R}^n \to \mathbb{R}$, then $\int_C F(\vec{x})\, ds = \int_C \sum_{i=1}^n (M_i\, dx_i)$.

Remark 1.2. In physics, $\int_C F(\vec{x}) \cdot d\vec{r}$ describes the total work done by a particle on a curve C when F is a force field that describes the force that is being applied to the particle and $d\vec{r}$ is the instantaneous direction of the particle.

Fact 1.1. $\int_C f(\vec{x})\, ds = \int_{-C} f(\vec{x})\, ds$. *That is, the direction of the curve does not matter when using ds. However, note that $\int_C f(\vec{x})\, d\vec{r} = -\int_{-C} f(\vec{x}) \cdot d\vec{r}$.*

[1] This can be thought of as the velocity or direction of a particle at time t multiplied by dt which gives you a change in distance.
[2] Usually, we use ds for the one dimensional case.

Remark 1.3. To evaluate $\int_C f(\vec{x})\,ds$ for scalar function f, convert the function into a function of t, ds into $\left\|\left(\frac{dx_1}{dt}, ..., \frac{dx_n}{dt}\right)\right\|_2 dt$ and integrate as usual. For vector function F, it is a similar process. Convert the $M_i's$ to functions of t and convert the $dx_i's$ into $dt's$ (i.e. $\frac{dx_i}{dt} = g(\vec{x}) \implies dx_i = g(\vec{x})\,dt$) and integrate the sum as usual.

Remark 1.4. When $F = -\nabla f(\vec{x})$, called the **potential field** for physicists, for some locally injective curve f, then $\int_C F(\vec{x})\,ds$ describes the work done by the motion of a particle in circular motion. When $F = \nabla f(\vec{x})$, we have what is called a **gradient field** or also a potential field for mathematicians.

Proposition 1.1. $\int_{-C} f(\vec{x})\,dx_i = -\int_C f(\vec{x})\,dx_i$ *for all* $i \in \{1, ..., n\}$.

Remark 1.5. $\sum_{i=1}^{n} f_{x_i} dx_i = f(\vec{x})\,d\vec{r}$ which is analogous to summing up the instantaneous/infinitesimal components of f for a small change $d\vec{r}$. Note that in here, we have $f : \mathbb{R}^n \mapsto \mathbb{R}$.

Theorem 1.1. Fundamental Theorem of Line Integrals

If a curve C is parametrized in terms of t where $t_0 \leq t \leq t_1$, then for a scalar function f, we have $\int_C \nabla f(\vec{x}) \cdot d\vec{r} = f(r(t_1)) - f(r(t_0))$.

Definition 1.3. We call C a **closed curve** if the parametrized function $r(t)$ contains equal starting and ending points.

Definition 1.4. A vector field F is **conservative** if $\int_C F(\vec{x}) \cdot d\vec{r} = 0$ for all simple closed curves, C.

Definition 1.5. We say a path is **simple** is it does not cross itself and a region D is called **simply-connected** if contains no holes or more formally, the interior of any closed curve in D is also contained in D. Connected is defined in the same way in MATH247 or you can think of it as a region where you can connect two points in D with a path that is entirely in D.

Consequences of the Fundamental Theorem of Line Integrals

Let $f(\vec{x})$ be some scalar valued function, which can be parametrized as a function $r(t)$ on a curve C, where $t \in [t_0, t_1]$.

1. $\int_C \nabla f(\vec{x}) \cdot d\vec{r}$ is path invariant (or path independent). That is, for any $r(t)$, which are similar in their endpoints, $\int_C \nabla f(\vec{x}) \cdot d\vec{r}$ will always have the same value.
2. $\int_C \nabla f(\vec{x}) \cdot d\vec{r}$ is conservative.
3. If F is a continuous vector field on an open connected region D and $\int_C F(\vec{x}) \cdot d\vec{r}$ is also path invariant for a segment C, then F is a conservative vector field on D.
4. $\int_C F(\vec{x}) \cdot d\vec{r}$ is independent of path (path invariant) *if and only if* $\int_C F(\vec{x}) \cdot d\vec{r} = 0$ is conservative *if and only if* F is a gradient field *if and only if* $F \cdot d\vec{r} = \sum_{i=1}^{n} M_i dx_i = df$ is an **exact differential** (the sum can be expressed in terms of the differentials of a single function f via the Chain Rule) for some function f with $M_i = f_{x_i}$.

Note that for a differential $F \cdot d\vec{r} = \sum_{i=1}^{n} A_i dx_i$, the condition that $(A_i)_j = (A_j)_i$ (i.e. $f_{ij} = f_{ji}$ if it is equal to df) for all $i \neq j$, on a simply connected region, is enough to satisfy condition 4 above. It is also one of the main methods to check whether or not F is a gradient field. Make sure to take Clairaut theorem of mixed partials into consideration (second derivatives of f are continuous) and ensure that F is defined and differentiable everywhere.

Definition 1.6. A **potential function** is of the form $f(\vec{x}) = \int_C F(\vec{x}) \cdot d\vec{r} + K$ for some constant K, which is usually chosen to be $f(\vec{0})$, and arbitrary curve C. This is another consequence of Thm 1.1.

Remark 1.6. For a point $\vec{x}_0$, $\int_C F(\vec{x}) \cdot d\vec{r}$ is most easily calculated when the curve is split up into $C_1, ..., C_n$, $C = \sum_{i=1}^n C_i$, where each C_i is parallel to a unique axis with length equal to $|x_i|$. Sometimes it is easier to go along radially in polar or spherical coordinates. It all depends on the context.

Remark 1.7. Another method to calculate $\int_C F(\vec{x}) \cdot d\vec{r}$ is to find a function f such that $\nabla f = F$ by solving a system of equations. This system is solved, usually, by anti-derivatives. For any f_{x_i}, compute $\int f_{x_i}\, dx_i = g(\vec{x}) + h(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) = f$. Then, notice that $f_{x_j} = \left(\int f_{x_i}\, dx_i\right)_{x_j}$ for $i \neq j$. Using this method, one can eventually get the value for h and thus the value of f, since g is merely the anti-derivative of f_{x_i} with respect to x_i.

1.2 Curls, Flux, Divergence and Green's Theorem

Notation 2. From here on out, we will use the notation $\oint_C$ to denote the line integral over the curve C. Formally, this notation is usually used in only contour integrals (which operate in the complex plane) or closed curves.

Definition 1.7. For vector fields $F = \sum_{i=1}^n M_i e_i$ in $\mathbb{R}^3$, we define the **curl** of F, $\nabla\times$ or $curl(F)$, with $\nabla \times F = \det A$ where $A_{1i} = e_i$, $A_{2i} = \frac{\partial}{\partial x_i}$ and $A_{3i} = M_i$. If is obvious that if F is conservative, then $curl(F) = 0$. For $\mathbb{R}^2$, $curl(F) = (M_2)_{x_1} - (M_1)_{x_2}$. For a visual depiction of A, see

$$A = \begin{bmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ M_1 & M_2 & M_3 \end{bmatrix}$$

Remark 1.8. In a velocity field, the curl measures **rotation**. For example, for $F(x, y) = (-y, x)$, which describes a counter-clockwise rotation at unit angular speed, the magnitude curl of F is 2. Thus, the magnitude of the curl measures *twice* the angular speed of the rotational component of a motion at any given point in $\mathbb{R}^2$. It can just be thought of as simply the intensity of rotational motion at any given point. For positive curl (magnitude) values, the motion is counterclockwise and vice versa for negative values (think right hand rule; into the page is the negative z-direction and out of the page is the positive z-direction).

The curl vector also measures the direction of the rotational axis according to the right-hand rule. For a velocity field, the curl of the field measures angular velocity, for an acceleration field, the curl measures angular acceleration, and for a force field, the curl measures the torque per unit moment of inertia.

In $\mathbb{R}^3$, the magnitude of the curl measures the **torque** exerted on a test object where torque divided by moment of inertia is the derivative of angular velocity (angular acceleration) with respect to time. For a point-mass, moment of inertia of an object r distance away from the point of rotation is $I = r^2 m$ where m is the mass of the object. For several point masses, this is the sum of each individual point mass' moment of inertia (with varying values for m and r). Finally, for a general object the moment of inertia around an axis of rotation is $I = \int_V \rho(\mathbf{r})\, r_\perp^2\, dV$ where $\mathbf{r}$ is the vector from the center of the mass to a point in the body of the object, $\rho(\mathbf{r})$ is the density of the object at the point $\mathbf{r}$, $r_\perp$ is the perpendicular distance (shortest distance) to the axis of rotation, and V is the volume.

Alternatively, torque τ for an object can be thought of as $\vec{\tau} = \vec{r} \times F \triangle m$ where F is a force field, m is the mass of the object and $\vec{r}$ the position vector of the object.

Torque is very much like the analog of **force** in rotational motion (rather than linear) where the moment of inertia is the analog of mass and $\frac{d}{dt}$ of angular velocity is like acceleration.

Theorem 1.2. Green's Theorem [Tangental Form]

If C is a simply closed, piecewise smooth, oriented and counterclockwise curve in $\mathbb{R}^2$, enclosing a region R, and $F : \mathbb{R}^2 \mapsto \mathbb{R}^2$, a vector field, is differentiable and defined in an open region containing R with the $M_i's$ having continuous partial derivatives, then

$$\oint_C F \cdot d\vec{r} = \iint_R curl(F)\, dA$$

or alternatively,

$$\oint_C F\, d\vec{r} = \oint_C (M_1 dx_1 + M_2 dx_2) = \iint_R \left((M_2)_{x_1} - (M_1)_{x_2}\right) dA$$

where the $\oint_C$ notation indicates the counterclockwise orientation. Note that we can equivalently write $\oint_C$ as $\oint_C$ or even $\int_{\partial R}$.

Fact 1.2. *$\iint_R x\, dA = Area(R) \times \bar{x}$ where $\bar{x}$ the x-coordinate of the center of mass in the region R since $\bar{x} = \frac{1}{mass} \iint_R \rho(x,y)\, x\, dA$ for density ρ. A similar definition follows for $\bar{y}$.*

Proposition 1.2. *If $curl(F) = 0$ everywhere, then F is conservative (Hint: Use Green's Theorem).*

Definition 1.8. We define the **flux** as the following integral: $flux(F, C) = \oint_C F(\vec{x}) \cdot \hat{n}\, ds = \oint_C F(\vec{x}) \cdot d\vec{C}$ where $\hat{n}$ is the normal unit vector to the curve and points to the "right" of the direction of $d\vec{r}$ (which points "outwards" if C encloses a region). Notation wise, $d\vec{C} = \hat{n} \cdot ds$. Formally, $\hat{n}$ is $-\frac{\nabla g(x,y)}{\|\nabla g(x,y)\|}$ where $g(x,y) = 0$ is the implicit parametrization of the curve C.

Remark 1.9. In physics, when F describes a velocity field, like in a fluid, then the flux, $\oint_C F(\vec{x}) \cdot \hat{n}\, ds$, measures how much fluid passes through a curve C, left to right, per unit time. When fluid is passing right to left, it is counted negatively.

Proposition 1.3. *$\hat{n}$ is just $\hat{T}$ in $\hat{T}\, ds = d\vec{r} = (dx_1, ..., dx_n)$, rotated 90 degrees clockwise in $\mathbb{R}^2$. That is, $\hat{n} = (dy, -dx)$.*

Corollary 1.1. *We may write $\oint_C F(\vec{x}) \cdot \hat{n}\, ds$ as $\oint_C (-M_2\, dx + M_1\, dy)$ for $F(\vec{x}) = (M_1, M_2)$.*

Theorem 1.3. Green's Theorem for Flux [Normal Form]

If C encloses a region R counterclockwise (in $\mathbb{R}^2$) and $F(\vec{x}) = (M_1, M_2)$ is defined and differentiable in R, then

$$\oint_C F \cdot \hat{n}\, ds = \iint_R div(F)\, dA$$

*where $div(F) = (M_1)_x + (M_2)_y$, called the **divergence** of F.*

Proposition 1.4. Extended Green's Theorem

Suppose we have a vector valued function that is defined and differentiable everywhere except at a single point $\vec{x_0}$. For two counterclockwise curves C_1 and C_2 enclosing $\vec{x_0}$, with the two curves enclosing regions R_1 and R_2 respectively, suppose that $R_2 \subset R_1$. Then, the Extended Green's Theorem states that

$$\oint_{C_1} F \cdot d\vec{r} - \oint_{C_2} F \cdot d\vec{r} = \iint_{R_1 \setminus R_2} curl(F)\, dA$$

Lemma 1.1. *For a function F where $curl(F) = 0$ everywhere, then any closed counterclockwise curve C_1 around the origin will always have the same work as any other curve C_2 that also moves counterclockwise around the origin.*

Remark 1.10. In physics, the interpretation of the divergence of F, $div(F)$, is that it measures how much a flow is "expanding" (positive divergence implies outward flow and vice versa for negative divergence). It can also be interpreted as the "source rate", the amount of fluid added to the system per unit time and per unit area.

Proposition 1.5. *If $curl(F) = 0$ for some function F and its domain is simply connected, then F is conservative and is a gradient.*

1.3 Spherical Surface Areas

Recall the change of variables from Euclidean space to Spherical space through the parameters θ, ϕ and r, where $z = r\cos\phi$, $x = r\cos\theta\sin\phi$ and $y = r\sin\theta\cos\phi$. Here, $\phi \in [0, \pi]$ represents the angle in the vertical direction, $\theta \in [0, 2\pi]$ is in the horizontal direction and r is the radius of the sphere. Be sure to take note of the conversion in the reverse direction. That is, $r = \sqrt{x^2 + y^2 + z^2}$, $\phi = \arccos\left(\frac{z}{r}\right)$ and $\theta = \arctan\left(\frac{y}{x}\right)$.

Remark 1.11. It is easy to see that the differential for the surface area of a sphere of radius r can be approximated by $dS = (r\sin\phi\, d\theta)(r\, d\phi) = r^2 \sin\phi\, d\phi\, d\theta$. Adding a thickness function of $\rho(\phi, \theta)$, we get the same multiplier seen in the change of variables section of Calculus III. That is, $dV = d\rho\, dS = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$. Note that the the previous multiplier is valuable for computing the surface area of a sphere.

1.4 Cylindrical Surface Areas

Similar to the above section, a cylindrical change of coordinates uses the variables θ, r and z where $x = r\cos\theta$, $y = r\sin\theta$ and $z = z$ for $\theta \in [0, 2\pi]$. The reverse direction will be left as an exercise.

Remark 1.12. Like the above, it is easy to see that the differential for surface area for the curved section of a cylinder is $dS = (dz)(r\, d\theta) = r\, d\theta\, dz$. The multiplier for volume will be left as an exercise.

2 Vector Calculus in $\mathbb{R}^3$

In this section, we will try to generalize some notions into $\mathbb{R}^3$, taking a look at concepts such as surface integrals and Stokes' Theorem.

2.1 Flux, Divergence and Surface Integrals in $\mathbb{R}^3$

Definition 2.1. In $\mathbb{R}^3$, we define a surface integral for some vector field F and some surface S in space as $\oiint_S F \cdot d\vec{S} = \oiint_S F \cdot \hat{T}\, dS$ (for closed surfaces) or $\iint_S F \cdot d\vec{S} = \iint_S F \cdot \hat{T}\, dS$. We will be using the latter notation for this section.

Definition 2.2. In $\mathbb{R}^3$, the definition of flux is $flux(F, S) = \iint_S F \cdot \hat{n}\, dS = \iint_S F \cdot d\vec{S}$ for some surface and $S = \{(x, y, z), z = f(x, y)\}$ (for some function $f(x, y)$) and vector field F. Similar to $\mathbb{R}^2$, $d\vec{S} = \hat{n} \cdot dS$. Note that $\hat{n}$ is still a unit vector that now points outward and away from the volume enclosed by the surface S (if it is closed). Similar to $\mathbb{R}^2$, $\hat{n}$ can be formally written as $\hat{n} = -\frac{\nabla g(x,y,z)}{\|\nabla g(x,y,z)\|}$ where $g(x, y, z) = z - f(x, y) = 0$ defines the surface S. The interpretation of flux is the same as in $\mathbb{R}^2$ and instead of left to right, we think in to out or upwards for a surface.

Definition 2.3. Recall that the definition of the **cross product** of two vectors a and b in $\mathbb{R}^3$ is

$$a \times b = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Proposition 2.1. *Using the definition of the cross product, one may come up with the following useful substitution (in $\mathbb{R}^3$): $\hat{n}\, dS = (-f_x, -f_y, 1)\, dy\, dx$. However, do note that $\hat{n} \neq (-f_x, -f_y, 1)$ and $dS \neq dy\, dx$.*

Proposition 2.2. *For a parametrization of a surface given by*

$$S = \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$$

with $\vec{r}(u,v) = (x,y,z)$. Using intuition gathered by examining differentials, one can come up with another useful substitution (in $\mathbb{R}^3$): $\hat{n}\,dS = -\left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right) du\,dv$ where $\times$ is the cross product. A positive value of $\left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right)$ may also be possible based on the convention used.

Proposition 2.3. *For a surface that is a plane in $\mathbb{R}^3$, uniquely determined by the normal to the plane, $\vec{N} = (a,b,c)$, using the notions of angles, specifically the angle the plane makes with the xy plane or equivalently, its normal to the vector $\hat{k} = (0,0,k)$, we can deduce the following substitution (in $\mathbb{R}^3$): $\hat{n}\,dS = \frac{\vec{N}}{\vec{N}\cdot\hat{k}}\,dx\,dy$. Again, the sign depends on the convention being used. Note that this is just a specific case of Prop. 2.1.*

Theorem 2.1. Gauss-Green / Ostrogradsky / Divergence Theorem

For a closed surface S in $\mathbb{R}^3$ that encloses a volume D, oriented with $\hat{n}$ pointing outwards and $F(\vec{x}) = (M_1, M_2, M_3)$ defined and differentiable everywhere in D, then

$$\oiint_S F\,d\vec{S} = \iiint_D div(F)\,dV$$

where $div(F) = (M_1)_x + (M_2)_y + (M_3)_z$.

Remark 2.1. Here, divergence has similar notation as noted in $\mathbb{R}^2$; it is the "source rate" or the amount of flux generated per unit volume. If F is a velocity field, we note that applying the above theorem for a region D, then flux is the amount of fluid passing through the region D. Divergence measures the amount of sources or sinks in a given region.

Notation 3. Another notation for $div(F)$ is $\nabla \cdot F$.

Definition 2.4. We define the **Laplacian** differential operator on some function $f : \mathbb{R}^n \to \mathbb{R}$ as

$$\nabla^2 f = \triangle f = \nabla \cdot \nabla f = div(\nabla f) = \sum_{i=1}^{n} \frac{d^2 f}{dx_i^2}$$

Example 2.1. Let $u(x,y,z,t)$ be the concentration of a fluid at a given point. The **diffusion equation** (also known as the **heat equation**) governs the motion of a fluid in immovable air, and is given by

$$\frac{\partial u}{\partial t} = k\nabla^2 u = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

for some constant k. The motivation for this equation is as follows.

Let F, a vector field, be the flow of the smoke. From physics (and common sense), we know that smoke flows from high concentration to low concentration. So F is directed along $-\nabla u$. In fact $F = -k\nabla u$ for a constant k.

To relate F and $\frac{\partial u}{\partial t}$, we use the divergence theorem. Consider the amount of smoke through a closed surface S, enclosing a region D, per unit time (or the flux out of D through S), given by $\oiint_S F \cdot \hat{n}\,dS$ which is equivalent to $-\frac{d}{dt}\left(\iiint_D u\,dV\right)$ (amount of smoke that is being lost per unit time).

Applying the divergence theorem,

$$\oiint_S F \cdot \hat{n}\,dS = \iiint_D div(F)\,dV = -\frac{d}{dt}\left(\iiint_D u\,dV\right) = -\iiint_D \frac{du}{dt}\,dV.$$

The last equality is a bit hand-wavy, but we will assume that it is true. So $\frac{1}{vol(D)}\iiint_D div(F)\,dV = -\frac{1}{vol(D)}\iiint_D \frac{\partial u}{dt}\,dV$ and this weakly implies that the average of $div(F)$ in D is equal to the average of $-\frac{\partial u}{\partial t}$ in D (any region D). This gives the interpretation $div(F) = -\frac{\partial u}{\partial t}$ and $div(F) = div(-k\nabla u) \implies \frac{\partial u}{\partial t} = k\nabla^2 u$ which is our diffusion equation.

2.2 Line Integrals in $\mathbb{R}^3$

Line integrals in $\mathbb{R}^2$ are done in the same way as explained in the first section. That is, for a vector field $F(\vec{x}) = (M_1, ..., M_n) = \sum_{i=1}^{n} M_i e_i$ and curve C, the line integral is given by $\int_C \sum_{i=1}^{n} (M_i \, dx_i)$. Intuition for the other concepts relating to line integrals in $\mathbb{R}^3$ that were found in section 1 will be left as an exercise for the reader to try and figure out.

2.3 Curls in $\mathbb{R}^3$

This is just a review of the concepts mentioned in section 1.

Definition 2.5. If $F = \sum_{i=1}^{3} M_i e_i$, then the curl of Fin $\mathbb{R}^3$ is defined as

$$curl(F) = [(M_3)_y - (M_2)_z] \, e_1 + [(M_1)_z - (M_3)_x] \, e_2 + [(M_2)_x - (M_1)_y] \, e_3$$

which the the expansion of the formula given in the first section.

Remark 2.2. Recall that a vector field is conservative if and only if $curl(F) = 0$.

2.4 Stokes' Theorem

Theorem 2.2. Kelvin-Stokes' Theorem

A generalization of Green's Theorem, Stokes' Theorem states that for some closed, counterclockwise curve C in $\mathbb{R}^3$, S any simply connected surface bounded (in terms of x and y coordinates) by C, and vector field F, where F is defined and differentiable on S,

$$\oint_C F \cdot d\vec{r} = \iint_S (\nabla \times F) \cdot \hat{n} \, dS = \iint_S curl(F) \cdot \hat{n} \, dS$$

Note here that if one were to walk along C, one would see S to the "left" and the vector $\hat{n}$ pointing "upwards" and vice versa for the counterclockwise orientation.

Remark 2.3. (Right hand rule) For Stokes' Theorem, above, with your right hand, point your thumb in the direction of the curve C and your index finger pointing to the interior of the surface S. Raising your middle finger, the middle finger should point in the general direction of the vector $\hat{n}$.

The only unique case is that of a cylinder (curves C on the top and C' on the bottom) where we want the surface S (the curved side of the cylinder) to have $\hat{n}$ pointing outside the cylinder. C should be clockwise and C' should be counterclockwise.

Proposition 2.4. *Consider comparing Green's Theorem with Stokes' Theorem. Green's Theorem is just a special case of Stokes' Theorem when the surface lies on the xy plane and the curve as well (Stokes' Theorem uses $\hat{n} = \hat{k}$ where $\hat{k}$ is the normal to the xy plane).*

Example 2.2. So what are some examples of surfaces where Stokes' Theorem obviously breaks down?

1. The surface of a torus (it is not simply connected; you can make connected curves around the torus and over it)
2. A mobius strip (you cannot orient the $\hat{n}$ vector consistently so flux cannot be defined)

Remark 2.4. Why is Stokes' Theorem surface independent (invariant)? Consider two closed surfaces S_1, S_2, vector field F defined and differentiable everywhere on the surface, and the integral $\iint_{S_1} (\nabla \times F) \cdot \hat{n}\, dS - \iint_{S_2} (\nabla \times F) \cdot \hat{n}\, dS$. This describes the flux integral of a surface S that combines S_1 and the reverse orientation of S_2. By the divergence theorem, since S is a closed surface, we have

$$\begin{aligned} \iint_{S_1} (\nabla \times F) \cdot \hat{n}\, dS - \iint_{S_2} (\nabla \times F) \cdot \hat{n}\, dS &= \iint_{S=S_1-S_2} (\nabla \times F) \cdot \hat{n}\, dS \\ &= \iiint_D div(\nabla \times F)\, dV \end{aligned}$$

It is easy to check that $div(\nabla \times F) = 0$ always as follows. By definition,

$$\begin{aligned} div(\nabla \times F) &= [(M_3)_y - (M_2)_z]_x + [(M_1)_z - (M_3)_x]_y + [(M_2)_x - (M_1)_y]_z \\ &= (M_3)_{yx} - (M_2)_{zx} + (M_1)_{zy} - (M_3)_{xy} + (M_2)_{xz} - (M_1)_{yz} \\ &= 0 \end{aligned}$$

when each component of F is C^1. Another interpretation is using the fact that $u \cdot (u \times v) = 0$ for any vectors u and v. Thus, $div(\nabla \times F) = \nabla \cdot (\nabla \times F) = 0$ (this is really sketchy handwaving).

2.5 Maxwell's Equations

Definition 2.6. An **electric field**, E, tells how much force via charge will be exerted on a particle. These are very much analogous to force fields. This is the gradient of an electric potential, known as voltage. The implicit formula is $F = qE$ where q is the charge and F is the force caused by the electric field.

Definition 2.7. A **magnetic field**, B, tells how moving charged particles interact with each other and the environment via electrical forces (this field is much harder to describe). It is implicitly described by the formula $F = q\vec{v} \times B$ where q is again a charge, $\vec{v}$ is the velocity of the moving charged particle, and F is the force generated by the magnetic field.

Maxwell's equations, in particular, give information regarding the divergence and curl of the two above fields. The equations are Propositions 2.5., 2.6., 2.7. and 2.8. below. These are only brief sketches and their true derivations and intuitive interpretations should be obtained in a physics class.

Proposition 2.5. Gauss-Coulomb Law

For an electric field E, the Gauss-Coulomb Law states that $div(E) = \nabla \cdot E = \frac{\rho}{\epsilon_0}$ for some constant ϵ_0 and ρ is the electric charge density.

Remark 2.5. Consider the integral $flux(E, S) = \oiint_S E \cdot d\vec{S}$ where E is the electric field on the closed surface S enclosing a region D. By the divergence theorem and Prop 2.5.,

$$\oiint_S E \cdot d\vec{S} = \iiint_D div(E)\, dV = \frac{1}{\epsilon_0} \iiint_D \rho\, dV = \frac{Q}{\epsilon_0}$$

where Q is the electric charge in D.

Proposition 2.6. Faraday's Law

For an electric field E, Faraday's Law stats that $curl(E) = \nabla \times E = -\frac{\partial B}{\partial t}$ where B is a variable magnetic field. Note that $\frac{\partial B}{\partial t}$ is a column vector of partials.

Remark 2.6. Consider the example of a transformer which intertwines two looping wires together with AC current in one of the wires. The wire with the AC current generates a magnetic field over time because of the different currents generated. This creates a value for $-\frac{\partial B}{\partial t}$ and a value for the curl of the electric field which generates voltage between the two endpoints of the wire which does not have AC current.

A simple application of this would be power outlets. The transformer is located inside the outlet to provide electrical current for plugs that are connected to the outlet.

To find the actual voltage generated, we consider the closed loop of the wire that is not being powered by AC current (our plug in the example above). The voltage on a closed curve C enclosing a surface S is defined as $\oint_C E\, d\vec{r}$ for electric field E. By Stokes' Theorem and Faraday's Law, this is

$$\oint_C E\, d\vec{r} = \iint_S (\nabla \times E)\, d\vec{S} = \iint_S \left(-\frac{\partial B}{\partial t}\right)\, d\vec{S}.$$

Note that this is $-vol(S) \cdot \left(\frac{\partial B}{\partial t}\right)$ when the magnetic field is not moving (it is not a function of location).

Proposition 2.7. *For a magnetic field B, $\nabla \cdot B = 0$.*

Proposition 2.8. *For a magnetic field B, $\nabla \times B = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial E}{\partial t}$ where $\vec{J}$ is a vector called the vector current density.*